# On Flatness of Discrete-time Nonlinear Systems

**Arvo Kaldmäe Ülle Kotta**

*Institute of Cybernetics at TUT, Akadeemia tee 21, 12618, Tallinn, Estonia (e-mail: arvo@cc.ioc.ee, kotta@cc.ioc.ee)*

**Abstract:** The paper addresses the problem of dynamic feedback linearization of discrete-time nonlinear control systems. Analogously to the continuous-time case, necessary and sufficient conditions for flatness property are obtained and showed to be equivalent to previously known results on feedback linearizability by endogenous dynamic feedback. An example is added to illustrate the results.

## 1. INTRODUCTION

The concept of differential flatness for continuous-time nonlinear systems was introduced about 20 years ago, see Fliess et al. [1992, 1995], although the dynamic feedback linearization of state equations was addressed already in Isidori et al. [1986]. Since then it has been applied successfully to address many different control problems in various application areas. The reader may consult the books by Sira-Ramirez and Agrawal [2004], Lévine [2009] and the references therein. The concept of difference flatness for discrete-time systems, first mentioned in Sira-Ramirez and Agrawal [2004], was described analogously to its continuous-time counterpart. Namely, the flat system (in discrete-time) allows a complete parametrization of all system variables, including the control variables, in terms of a special set of independent fictitious variables, called the flat outputs, and their forward shifts. Flatness of discrete-time systems have been studied only in a few papers. The book Sira-Ramirez and Agrawal [2004] only addressed the linear systems. The paper by Fliess and Marquez [2000] proved that like for the continuous-time linear systems, in the discrete-time setting flatness is still equivalent to controllability. The problem of (dynamic) feedback linearization, intimately related to flatness, was studied in Aranda-Bricaire et al. [1996]. The necessary and sufficient linearizability conditions were suggested together with the procedure for finding the flat output. However, these conditions are not constructive since they depend on the existence of certain unimodular matrix. In principle, the results of Aranda-Bricaire et al. [1996] extend to the discrete-time case the results of Aranda-Bricaire et al. [1995]. The new aspect was pointed to in Aranda-Bricaire and Moog [2008], where it has been demonstrated that the linearizing outputs may depend, besides the state, input, and the forward shift of inputs also on their past values. The respective dynamic feedback was called exogenous. Note that in this paper, flatness is defined in such a manner that it corresponds to the dynamic endogenous feedback linearization. Finally, the paper by Pawluszewicz and Bartosiewicz [1998] that mostly focused on the concept of dynamic equivalence of nonlinear system to a controllable linear system in terms of free output universes, obtained as a by-product the results regarding the linearization outputs. Namely, the set of generators of output universe plays the role of linearizing outputs. The generators may depend on future or past values of system variables.

⋆ This work was supported by the European Union through the European Regional Development Fund and the target funding project SF0140018s08 of Estonian Ministry of Education and Research. A. Kaldmäe was additionally supported by the ETF grant nr. 8787 and Ü. Kotta by the ETF grant nr. 8365.

Constructing the flat outputs is, in general, an extremely difficult task, since no finite algorithm exists for their finding. By this reason, Lévine [2011] developed a 2-step procedure for computing the flat outputs. This procedure has been implemented in Maple by Antritter and Verhoeven [2010]. The goal of this paper is to find the relations (in the discrete-time) between the necessary and sufficient solvability conditions in Lévine [2011] and Aranda-Bricaire et al. [1996] as well as the one-forms they depend on. We will prove that the one-forms are equal, up to difference field isomorphism and then we show that the solvability conditions are also the same, again up to field isomorphism. Both the results, in Aranda-Bricaire et al. [1996] and Lévine [2011], rely on the existence of certain unimodular matrix. Note that we do not address the problem of computing these matrices in this paper.

## 2. PREVIOUS RESULTS

Consider the discrete-time control system, described by the state equations

$$x(t+1) = f(x(t), u(t)), \tag{1}$$

where $t$ is the time instant, $x(t) \in X \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $m \leq n$ and $f$ is nonlinear analytic function. Assume that $f(0,0) = 0$ and system (1) satisfies generically (i.e. everywhere except on a set of measure zero) the condition $\text{rank}[\partial f/\partial u(t)] = m$ and the so-called submersivity condition

$$\text{rank}\frac{\partial f}{\partial(x(t), u(t))} = n, \tag{2}$$

being necessary for system accessibility, see Grizzle [1993].

The following notations are used throughout the paper. Instead of $x(t+k)$, we use $x^{[k]}$ for $k \in \mathbb{Z}$. For $x^{[0]}$ we use just $x$ and for $x^{[1]}$ we sometimes use alternatively $x^+$. Similar notations are used for the other variables.

We briefly recall the algebraic formalism as well as the necessary and sufficient feedback linearizability conditions from Aranda-Bricaire et al. [1996]. Let us extend the map $f : (x, u) \mapsto x^+$ to the map $\bar{f} : (x, u) \mapsto (x^+, z)$, where $z = \chi(x, u)$, $z \in \mathbb{R}^m$ such that $\bar{f}^{-1}$ exists generically. This is possible under the assumption (2). Let $\widetilde{\mathcal{K}}$ be the set of meromorphic functions in a finite number of variables from $\{x, u^{[k]}, z^{[-l]}, k \geq 0; l \geq 1\}$. The forward and backward shift operators $\tilde{\delta} : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and $\tilde{\delta}^{-1} : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}$ are defined by

$$\begin{aligned}
&\tilde{\delta}\varphi(x, u, \ldots, u^{[k]}, z^{[-1]}, \ldots, z^{[-l]}) = \\
&\qquad \varphi(f(x, u), u^{[1]}, \ldots, u^{[k+1]}, z, \ldots, z^{[-l+1]}) \\
&\tilde{\delta}^{-1}\varphi(x, u, \ldots, u^{[k]}, z^{[-1]}, \ldots, z^{[-l]}) = \\
&\qquad \varphi(\bar{f}^{-1}(x, z^{[-1]}), u^{[-1]}, \ldots, u^{[k-1]}, z^{[-2]}, \ldots, z^{[-l-1]}).
\end{aligned}$$

Since $\tilde{\delta}$ is an automorphism of $\widetilde{\mathcal{K}}$, the pair $(\widetilde{\mathcal{K}}, \tilde{\delta})$ is an inversive difference field. We use sometimes abridged notations $\varphi^+ := \tilde{\delta}\varphi$ and $\varphi^- := \tilde{\delta}^{-1}\varphi$ for $\varphi \in \widetilde{\mathcal{K}}$. Let $\widetilde{\mathcal{E}} = \mathrm{span}_{\widetilde{\mathcal{K}}}\{\mathrm{d}\varphi \mid \varphi \in \widetilde{\mathcal{K}}\}$ be the vector space of one-forms. The operators $\tilde{\delta}$ and $\tilde{\delta}^{-1}$ are extended to $\widetilde{\mathcal{E}}$ by the rules $\tilde{\delta}(\sum_i a_i \mathrm{d}\varphi_i) = \sum_i a_i^+ \mathrm{d}\varphi^+$ and $\tilde{\delta}^{-1}(\sum_i a_i \mathrm{d}\varphi_i) = \sum_i a_i^- \mathrm{d}\varphi^-$. Again, we sometimes use the notations $\omega^+ = \tilde{\delta}\omega$ and $\omega^- = \tilde{\delta}^{-1}\omega$ for $\omega \in \widetilde{\mathcal{E}}$. The relative degree $r$ of a one-form $\omega$ is defined by $r = \min\{k \in \mathbb{N} \mid \tilde{\delta}^k\omega \notin \mathrm{span}_{\widetilde{\mathcal{K}}}\{\mathrm{d}x\}\}$. If there does not exist such integer, then set $r := \infty$. The one-forms, with infinite relative degree are called autonomous one-forms.

Define the non-increasing sequence of subspaces $\mathcal{H}_k$ of $\widetilde{\mathcal{E}}$ by $\mathcal{H}_1 = \mathrm{span}_{\widetilde{\mathcal{K}}}\{\mathrm{d}x\}$, $\mathcal{H}_k = \mathrm{span}_{\widetilde{\mathcal{K}}}\{\omega \in \mathcal{H}_{k-1} \mid \omega^+ \in \mathcal{H}_{k-1}\}$, for $k \geq 2$. The sequence converges. Let $k^*$ be such that $\mathcal{H}_{k^*-1} \neq \mathcal{H}_{k^*}$, but $\mathcal{H}_{k^*+1} = \mathcal{H}_{k^*}$ and define $\mathcal{H}_\infty := \mathcal{H}_{k^*}$. Note that $\mathcal{H}_\infty$ is exactly the set of autonomous one-forms Halas et al. [2009]. From now on, we assume that $\mathcal{H}_\infty = \{0\}$.

*Theorem 1.* Suppose that for system (1) $\mathcal{H}_\infty = \{0\}$. Then there exist one-forms $\hat{\omega}_1, \ldots, \hat{\omega}_m \in \mathrm{span}_{\widetilde{\mathcal{K}}}\{\mathrm{d}x\}$ with relative degrees $r_1, \ldots, r_m$ respectively, such that

(i) $\mathrm{span}_{\widetilde{\mathcal{K}}}\{\tilde{\delta}^k\hat{\omega}_i, i = 1, \ldots, m; k = 0, \ldots, r_i - 1\} = \mathrm{span}_{\widetilde{\mathcal{K}}}\{\mathrm{d}x\}$

(ii) $\mathrm{span}_{\widetilde{\mathcal{K}}}\{\tilde{\delta}^k\hat{\omega}_i, \ i = 1, \ldots, m; \ k = 0, \ldots, r_i\} = \mathrm{span}_{\widetilde{\mathcal{K}}}\{\mathrm{d}x, \mathrm{d}u\}$

(iii) the one-forms $\{\tilde{\delta}^k\hat{\omega}_i, i = 1, \ldots, m; k \geq 0\}$ are linearly independent and $\sum_i r_i = n$.

A function $y = h(x, u, \ldots, u^\mu)$, $y \in \mathbb{R}^m$ is said to be an endogenous linearizing output of system (1) if any variable of system (1) can be expressed as a function of $y$ and a finite number of its forward-shifts.

Let $\widetilde{\mathcal{K}}[\delta]$ be the non-commutative polynomial ring with coefficients in $\widetilde{\mathcal{K}}$, where multiplication is defined by the rule $\delta \cdot a = a^+\delta$, where $a \in \widetilde{\mathcal{K}}$. The ring of $p \times q$ matrices over $\widetilde{\mathcal{K}}[\delta]$ is denoted by $\widetilde{\mathcal{K}}[\delta]^{p\times q}$. A matrix $U \in \widetilde{\mathcal{K}}[\delta]^{p\times p}$ is called unimodular if there exists a matrix $U^{-1} \in \widetilde{\mathcal{K}}[\delta]^{p\times p}$ such that $UU^{-1} = U^{-1}U = I_p$.

*Theorem 2.* Let $\mathcal{H}_\infty = \{0\}$ and $\hat{\omega} = (\hat{\omega}_1, \ldots, \hat{\omega}_m)^T$ be the one-forms defined in Theorem 1. Then, for system (1) there exists an endogenous linearizing output $y$ iff there exists an unimodular matrix $M \in \widetilde{\mathcal{K}}[\delta]^{p\times p}$ such that $\mathrm{d}(M\hat{\omega}) = 0$.

## 3. ALTERNATIVE APPROACH

In this section we define, following Lévine [2011], flatness of implicit discrete-time systems, obtained from equations (1) by eliminating the control variables $u$, and show that this concept is equivalent to the existence of endogenous linearizing outputs, defined in Aranda-Bricaire et al. [1996].

### *3.1 Algebraic formalism for implicit systems*

Consider the implicit representation

$$F(x(t), x(t+1)) = 0 \tag{3}$$

of system (1), where

$$\mathrm{rank}\frac{\partial F(\cdot)}{\partial x(t+1)} = n - m. \tag{4}$$

Representation (3) can be obtained [1] from (1) by eliminating the control variables $u$. Reorder, if necessary, the components of the vector function $f = (f_1, \ldots, f_n)$ in (1) such that $\mathrm{rank}\frac{\partial(f_{n-m+1},\ldots,f_n)}{\partial u} = m$. Then from the last $m$ equations of (1) one obtains

$$u = \phi(x, x^+_{n-m+1}, \ldots, x^+_n), \tag{5}$$

where by $x_i$ is denoted the $i$th component of $x$. Substituting $u$ from (5) into the first $n - m$ equations of (1), one gets $x_i^+ = f_i(x, \phi(\cdot))$, $i = 1, \ldots, n - m$ and thus

$$F_i(x, x^+) := x_i^+ - f_i(x, \phi(\cdot)) = 0, \quad i = 1, \ldots, n - m. \tag{6}$$

Note, that condition (4) is satisfied globally, since $\frac{\partial F}{\partial x^+} = (I_{n-m}, G)$, where the matrix $G$ does not depend on $x_1^+, \ldots, x_{n-m}^+$.

Next, we define an another field $\mathcal{K}_x$, associated with the representation (3), by transforming the variables of the field $\widetilde{\mathcal{K}}$ into the variables of the field $\mathcal{K}_x$ according to the rules

$$\begin{aligned}
x = x, \ \ u^{[k]} &= \phi(x^{[k]}, x^{[k+1]}_{n-m+1}, \ldots, x^{[k+1]}_n) \\
z^{[-l]} &= \chi(x^{[-l]}, \phi(x^{[-l]}, x^{[-l+1]}_{n-m+1}, \ldots, x^{[-l+1]}_n)),
\end{aligned} \tag{7}$$

where $k \geq 0$, $l \geq 1$ and $\phi$ is defined by (5). Also, it is possible to transform the variables of the field $\mathcal{K}_x$ into those of the field $\widetilde{\mathcal{K}}$ by

$$\begin{aligned}
x = x, \ x^{[k]} &= f(\cdot, u^{[k-1]}) \circ f(\cdot, u^{[k-2]}) \circ \cdots \circ f(x, u) \\
x^{[-k]} &= \bar{f}^{-1}(\cdot, z^{[-k]}) \circ \bar{f}^{-1}(\cdot, z^{[-k+1]}) \circ \cdots \circ \bar{f}^{-1}(x, z^{[-1]}),
\end{aligned} \tag{8}$$

where $k \geq 0$. Thus, we have that the fields $\widetilde{\mathcal{K}}$ and $\mathcal{K}_x$ are isomorphic. Note that in the field $\mathcal{K}_x$, $F_i(x, x^+) = 0$, for $i = 1, \ldots, n - m$. Really, by (1) $x_i^+ - f_i(x, u) = 0$, and so by (6)

$$F_i(x, x^+) = x_i^+ - f_i(x, \phi(\cdot)) = 0. \tag{9}$$

A forward shift operator $\delta_x : \mathcal{K}_x \to \mathcal{K}_x$, applied on a function $\varphi \in \mathcal{K}_x$ is defined by shifting the arguments of the function $\varphi$ according to the rule

[1] We follow the procedure, given in Lévine [2011] for the continuous-time case.

$$\begin{aligned}\delta_x x^{[k]} &= \tilde{\delta} x^{[k]} = f(x^{[k]}, u^{[k]}) \\ &= f(x^{[k]}, \phi(x^{[k]}, x^{[k+1]}_{n-m+1}, \ldots, x^{[k+1]}_n)) = x^{[k+1]},\end{aligned}$$

where $k \in \mathbb{Z}$; (see (7) and (9)). To resume, in order to shift a function $\varphi \in \mathcal{K}_x$, we first transform it by field isomorphism into the element of the field $\widetilde{\mathcal{K}}$, shift it in $\widetilde{\mathcal{K}}$ and finally transform the shifted element back to the field $\mathcal{K}_x$. This is done to obtain isomorphism between difference fields. The inverse operator of $\delta_x$, i.e. $\delta_x^{-1} : \mathcal{K}_x \to \mathcal{K}_x$, called backward shift, is defined in a similar manner by shifting the arguments of the function $\varphi \in \mathcal{K}_x$ backward according to the rule

$$\begin{aligned}\delta_x^{-1} x^{[k]} &= \tilde{\delta}^{-1} x^{[k]} = \bar{f}^{-1}(x^{[k]}, z^{[k-1]}) \\ &= \bar{f}^{-1}(x^{[k]}, \chi(x^{[k-1]}, \phi(x^{[k-1]}, x^{[k]}_{n-m+1}, \ldots, x^{[k]}_n))) \\ &= x^{[k-1]}\end{aligned}$$

where $k \in \mathbb{Z}$. Here, we have used the fact that $x^{[-1]} - \bar{f}^{-1}(x, z^{[-1]}) = 0$ to get the last equality. Because the operator $\delta_x$ is an automorphism of the field $\mathcal{K}_x$, the pair $(\mathcal{K}_x, \delta_x)$ is an inversive difference field, which is isomorphic to difference field $(\widetilde{\mathcal{K}}, \tilde{\delta})$.

The elements of the vector space $\mathcal{E}_x = \mathrm{span}_{\mathcal{K}_x}\{\mathrm{d}x_i^{[j]}, i = 1, \ldots, n; j \in \mathbb{Z}\}$ are the one-forms

$$\omega = \sum_{j \in \mathbb{Z}} \sum_{i=1}^{n} \omega_{i,j} \mathrm{d}x_i^{[j]} \tag{10}$$

where only a finite number of coefficients $\omega_{i,j} \in \mathcal{K}_x$ are non-zero. Define the forward shift operator $\delta_x$ on $\mathcal{E}_x$ as

$$\delta_x \omega := \sum_{j \in \mathbb{Z}} \sum_{i=1}^{n} \delta_x(\omega_{i,j}) \mathrm{d}\big[\delta_x(x_i^{[j]})\big]. \tag{11}$$

Note that there exists an isomorphism between the vector-spaces $\widetilde{\mathcal{E}}$ and $\mathcal{E}_x$ induced by (7) and (8).

Denote $\mathbb{R}^n_\infty := \mathbb{R}^n \times \mathbb{R}^n \times \ldots$ and let $\bar{x} = (x, x^1, \ldots, x^k, \ldots)$ be the coordinates of $\mathbb{R}^n_\infty$. Define an operator $\bar{\delta} : \mathbb{R}^n_\infty \to \mathbb{R}^n_\infty$ as $\bar{\delta}\bar{x} = (x^1, x^2, \ldots)$. Denote by $\tilde{x} := (x, x^+, \ldots)$ the trajectory of system (1) (or (3)). Define $\delta_x \tilde{x} := (\delta_x x, \delta_x x^+, \ldots)$. Note that, in general, $\bar{x} \neq \tilde{x}$ and thus $\bar{\delta}\bar{x} \neq \delta_x \tilde{x}$.

*Definition 1.* A pair $(\mathbb{R}^n_\infty, F)$ satisfying the condition $\mathrm{rank}\frac{\partial F}{\partial x^+} = n - m$ is called an implicit control system.

*Example 1.* (Aranda-Bricaire et al. [1996]) Consider the system

$$\begin{aligned} x_1^+ &= x_2 + u_1 \quad & x_2^+ &= x_3 u_1 \\ x_3^+ &= x_3 u_2 \quad & x_4^+ &= x_4 + u_1. \end{aligned} \tag{12}$$

We extend the system (12) by $z_1 = x_1$ and $z_2 = x_3$ to get an invertible mapping $\bar{f}$, and find the implicit form (3) for the system (12). From the 1st and 3rd equations $u_1 = x_1^+ - x_2$, $u_2 = x_3^+/x_3$ and substituting those into the 2nd and 4th equations, we obtain

$$F(x, x^+) := \begin{pmatrix} x_2^+ - x_3 x_1^+ + x_3 x_2 \\ x_4^+ - x_4 - x_1^+ + x_2 \end{pmatrix}.$$

Take $\bar{x} = (x_1, \ldots, x_4, x_1^1, \ldots, x_4^1, \ldots)$. Then we have $\bar{\delta}\bar{x} = (x_1^1, \ldots, x_4^1, \ldots)$. The point $\bar{x} \in \mathbb{R}^n_\infty$ can be a trajectory of system (12), if $x^k = x^{[k]}$.

### 3.2 Definition of flatness

Consider a system of the form $(\mathbb{R}^m_\infty, 0)$ with coordinates $\bar{y} \in \mathbb{R}^m_\infty$, which is called a trivial system. In the similar manner as above, define the field $\mathcal{K}_y$, the forward shift operator $\delta_y : \mathcal{K}_y \to \mathcal{K}_y$ and the set of one-forms $\mathcal{E}_y$ for the trivial system. Set $Y_0 = \{\bar{y} \in \mathbb{R}^m_\infty \mid \bar{\delta}\bar{y} = \delta_y \tilde{y}\}$ and

$$X_0 = \{\bar{x} \in \mathbb{R}^n_\infty \mid F(x^k, x^{k+1}) = 0, \text{ and } \bar{\delta}\bar{x} = \delta_x \tilde{x}; k \geq 0\}.$$

The condition $\bar{\delta}\bar{x} = \delta_x \tilde{x}$ in the definition of $X_0$ relates the formal coordinates $\bar{x}$ with the trajectory of system (3), depending on discrete time $t$.

*Definition 2.* Implicit control system $(\mathbb{R}^n_\infty, F)$ is called flat if there exists an invertible meromorphic mapping $\Phi = (\varphi^0, \varphi^1, \ldots) : Y_0 \to X_0$ with the inverse $\Psi = (\psi^0, \psi^1, \ldots)$ that transforms the trajectories of the trivial system into the trajectories of a given system $(\mathbb{R}^n_\infty, F)$ and vice versa. The vector variable $y = \psi^0(\bar{x})$ is called a flat output of the system $(\mathbb{R}^n_\infty, F)$.

*Remark 1.* Assume that system $(\mathbb{R}^n_\infty, F)$ is flat and let $\bar{x} = \Phi(\bar{y})$. Then $x = \varphi^0(\bar{y})$ and $x^1 = \varphi^1(\bar{y})$. Because $\bar{x} \in X_0$, $x^1 = x^+$ and thus $\varphi^1 = \delta_y \varphi^0$. Therefore the mapping $\Phi$ is determined by $\varphi^0$. Analogously, the mapping $\Psi$ is determined by $\eta = \psi^0(\bar{x})$.

Let $\Phi$ be the mapping specified by Definition 2. Define the pull-back of a one-form $\omega(\bar{x}) = \sum_{j=0}^{K} \sum_{i=1}^{n} \omega_{i,j}(\bar{x}) \mathrm{d}x_i^{[j]} \in \mathcal{E}_x$ by $\Phi$ as follows, (see Weintraub [1997])

$$(\Phi^* \omega)(\bar{y}) = \sum_{j=0}^{K} \sum_{i=1}^{n} \omega_i^j(\Phi(\bar{y})) \mathrm{d}\varphi_i^j(\bar{y}), \tag{13}$$

where $\varphi_i^j$ is the component of $\Phi$, defined by $x_i^{[j]} = \varphi_i^j(\bar{y})$. From Remark 1, $\Phi$ is determined by a function $\varphi^0$ which depends on a finite number of variables, therefore the one-form $\Phi^*\omega(\bar{y}) \in \mathcal{E}_y$ has also a finite number of non-zero terms.

*Theorem 3.* The system $(\mathbb{R}^n_\infty, F)$ is flat iff there exists an invertible meromorphic mapping $\Phi : \mathbb{R}^m_\infty \to \mathbb{R}^n_\infty$ such that $\Phi(\bar{0}) = \bar{0}$ where $\bar{0} = (0, 0, \ldots)$, that satisfies $\bar{\delta}\bar{x} = \delta_y \Phi(\bar{y})$, $\bar{\delta}\bar{y} = \delta_x \Psi(\bar{x})$ and

$$\Phi^* \mathrm{d}F = 0. \tag{14}$$

*Proof: Necessity.* For flat systems there exists by Definition 2 an invertible mapping $\Phi : Y_0 \to X_0$. Let $\bar{x} = \Phi(\bar{y})$. Because $\bar{x} \in X_0$, $\bar{\delta}x = \delta_x x = \delta_y \varphi^0(\bar{y}) = \varphi^1(\bar{y})$. Continuing the same way, one gets $\bar{\delta}\bar{x} = \delta_y \Phi(\bar{y})$. In a similar manner we get $\bar{\delta}\bar{y} = \delta_x \Psi(\bar{x})$. It remains to show that (14) is satisfied. Since $\bar{x} \in X_0$, then $F(x, x^+) = 0$ and $F(\varphi^0(\bar{y}), \varphi^1(\bar{y})) = 0$, and obviously, $\Phi^* \mathrm{d}F = \mathrm{d}F(\varphi^0(\bar{y}), \varphi^1(\bar{y})) = 0$.

*Sufficiency.* By Definition 2, one has to prove that $\Phi$ transforms $Y_0$ into $X_0$ and vice versa. Note that the one-forms $\Phi^* \mathrm{d}F_i = \mathrm{d}(F_i(\Phi(\bar{y})))$, $i = 1, \ldots, n-m$ are exact, and thus, (14) implies [2] $F_i(\varphi^0(\bar{y}), \varphi^1(\bar{y})) = c_i$, where $c_i$ are arbitrary constants. From the assumption $f(0, 0) = 0$ and the construction of $F$ one concludes $F(0, 0) = 0$. Then obviously $\bar{0} \in X_0$ and $c_i = F_i(\varphi^0(\bar{0}), \varphi^1(\bar{0})) = F_i(0, 0) = 0$. When $\bar{x} = \Phi(\bar{y})$, then $F(x, x^1) = 0$ and $\bar{\delta}\bar{x} = \delta_y \Phi(\bar{y}) = \delta_x \bar{x}$. Thus $\bar{x} \in X_0$. Because of $\bar{\delta}\bar{y} = \delta_x \Psi(\bar{x}) = \delta_y \bar{y}$, function $\Psi$ transforms $X_0$ into $Y_0$. □

[2] Function $F$ in (3) depends only on $x$ and $x^+$.

### 3.3 Polynomial Matrices

In the similar manner as above, we denote by $\mathcal{K}_x[\delta]$ the non-commutative polynomial ring with coefficients in $\mathcal{K}_x$ and by $\mathcal{K}_x[\delta]^{p\times q}$ the ring of $p\times q$ matrices over $\mathcal{K}_x[\delta]$. The set of $p\times p$ unimodular matrices $U\in\mathcal{K}_x[\delta]^{p\times p}$ is denoted by $\mathcal{U}_p[\delta]$.

We investigate further the condition (14). From now on, we use the same notation $\delta$ for $\delta_x$ and $\delta_y$. Note that by (11), operators $\delta$ and d commute, so $\delta^j\mathrm{d}x=\mathrm{d}x^{[j]}$. From (3),

$$\mathrm{d}F=\frac{\partial F}{\partial x}\mathrm{d}x+\frac{\partial F}{\partial x^+}\mathrm{d}x^+.$$

Consider a mapping $\Phi=(\varphi^0,\varphi^1,\dots):\mathbb{R}^m_\infty\to\mathbb{R}^n_\infty$. From $x^{[j]}=\delta^j x$ and $x^{[j]}=\varphi^j(\bar y)$ one respectively obtains $\mathrm{d}x^{[j]}=\delta^j\mathrm{d}x=\delta^j\mathrm{d}\varphi^0(\bar y)$ and $\mathrm{d}x^{[j]}=\mathrm{d}\varphi^j(\bar y)$, yielding $\mathrm{d}\varphi^j(\bar y)=\delta^j\mathrm{d}\varphi^0(\bar y)$. The pull-back of one-form $\mathrm{d}F$ by $\Phi$, evaluated at the point $\bar y$, is

$$\Phi^*\mathrm{d}F_{|\bar y}=\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial x^+}\delta\right)_{|\bar x=\Phi(\bar y)}\mathrm{d}\varphi^0(\bar y). \tag{15}$$

Since the function $\varphi^0$ depends on a finite number of variables, there exists an integer $j^*$, such that for some $i\in\{1,\dots,m\}$, $\frac{\partial\varphi^0}{\partial y_i^{[j^*]}}\neq 0$, but $\frac{\partial\varphi^0}{\partial y_i^{[j]}}=0$ for every $i=1,\dots,m$ and $j>j^*$. The integer $j^*$ is the degree of the polynomial $\sum_{j\geq 0}\frac{\partial\varphi^0}{\partial y_i^{[j]}}\delta^j$. Define the matrices

$$P(F)=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial x^+}\delta,\qquad P(\varphi^0)=\sum_{j=0}^{j^*}\frac{\partial\varphi^0}{\partial y^{[j]}}\delta^j, \tag{16}$$

where $P(F)\in\mathcal{K}_x[\delta]^{(n-m)\times n}$ and $P(\varphi^0)\in\mathcal{K}_x[\delta]^{n\times m}$.

We say that $\sigma_i\in\mathcal{K}_x[\delta]$ is a divisor of $\sigma_j\in\mathcal{K}_x[\delta]$ iff there exists $\alpha\in\mathcal{K}_x[\delta]$ such that $\sigma_j=\alpha\sigma_i$.

*Theorem 4.* Cohn [1985] **(Jacobson decomposition)** For every $M\in\mathcal{K}_x[\delta]^{p\times q}$, there exist matrices $V\in\mathcal{U}_p[\delta]$ and $U\in\mathcal{U}_q[\delta]$ such that

$$VMU=\begin{cases}(\Delta_p,0_{p,q-p}), & \text{if } p\leq q;\\[2ex] \begin{pmatrix}\Delta_q\\ 0_{p-q,q}\end{pmatrix}, & \text{if } p\geq q,\end{cases} \tag{17}$$

where $0_{p,q-p}$ and $0_{p-q,q}$ are the matrices with zero entries, $\Delta_p$ and $\Delta_q$ are square diagonal matrices with elements $(\sigma_1,\dots,\sigma_s,0,\dots,0)$ such that $\sigma_i\in\mathcal{K}_x[\delta]$, for $i=1,\dots,s$, and $\sigma_i$ is a divisor of $\sigma_{i+1}$ for all $i$.

Note that $U$ and $V$ in (17) are not unique whereas $\Delta_p$ and $\Delta_q$ are.

*Definition 3.* Lévine [2011] Matrix $M\in\mathcal{K}_x[\delta]^{p\times q}$ is called hyper-regular iff $\Delta_p$ ($\Delta_q$) in its Jacobson decomposition is identity matrix.

*Lemma 1.* A square matrix $M\in\mathcal{K}_x[\delta]^{p\times p}$ is hyper-regular iff it is unimodular.

*Proof: Necessity.* If $M$ is hyper-regular, then there exist $V\in\mathcal{U}_p[\delta]$ and $U\in\mathcal{U}_p[\delta]$, such that $VMU=I_p$. Then $MU=V^{-1}$ and $MUV=I_p$. Thus $M^{-1}=UV$ and $M$ is unimodular.

*Sufficiency.* If $M$ is unimodular, then $MM^{-1}=I_p$ and thus, $M$ is hyper-regular. □

### 3.4 Necessary and sufficient condition

The matrix $P(F)\in\mathcal{K}_x[\delta]^{(n-m)\times n}$ in (16) admits a Jacobson decomposition $VP(F)U=(\Delta_{n-m},0_{n-m,m})$.

*Lemma 2.* If $\mathcal{H}_\infty=\{0\}$ for system (1), then matrix $P(F)$ for system (3) is hyper-regular.

*Proof:* The proof is by contradiction. Assume, without a loss of generality, that $P(F)$ is not hyper-regular, i.e the matrix $\Delta_{n-m}$ in Jacobson decomposition (17) has the form $\Delta=diag\{\sigma_1,\dots,\sigma_{n-m}\}$, where $\sigma_i\in\mathcal{K}_x[\delta]$ for every $i=1,\dots,n-m$ and $\deg\sigma_j=0$ for $j=1,\dots,n-m-1$ and $\deg\sigma_{n-m}=1$. We show that then there exists an autonomous one-form $\tau_{n-m}\in\mathcal{E}_x$. Let $\tau=(\tau_1,\dots,\tau_n)^T=U^{-1}\mathrm{d}x$. Then

$$\begin{aligned}VP(F)\mathrm{d}x&=(\Delta_{n-m},0_{n-m,m})\,U^{-1}\mathrm{d}x\\&=(\Delta_{n-m},0_{n-m,m})\,\tau=0.\end{aligned}$$

Thus $\sigma_{n-m}\tau_{n-m}=0$ is an autonomous one-form and $\mathcal{H}_\infty\neq\{0\}$, see Halas et al. [2009]. Really, $\tilde\tau_{n-m}\in\mathcal{H}_\infty$, where $\tilde\tau_{n-m}\in\widetilde{\mathcal{E}}$ is a one-form obtained by transforming $\tau_{n-m}$ by isomorphism between vector-spaces $\mathcal{E}_x$ and $\widetilde{\mathcal{E}}$. □

We assumed that $\mathcal{H}_\infty=\{0\}$ and thus $P(F)$ is hyper-regular, i.e. $VP(F)U=(I_{n-m},0_{n-m,m})$. By (14) and (15) one gets $P(F)P(\varphi^0)\mathrm{d}y=0$ yielding

$$P(F)P(\varphi^0)=0. \tag{18}$$

We characterize now the set of all matrices $P(\varphi^0)\in\mathcal{K}_x[\delta]^{n\times m}$ that satisfy the condition (18). First, solve the equation

$$P(F)\Theta=0, \tag{19}$$

with respect to $\Theta\in\mathcal{K}_x[\delta]^{n\times m}$. Denote by $\widehat U=U\begin{pmatrix}0_{n-m,m}\\ I_m\end{pmatrix}$.

*Lemma 3.* Every hyper-regular matrix $\Theta\in\mathcal{K}_x[\delta]^{n\times m}$ that satisfies (19) may be decomposed as

$$\Theta=\widehat UW, \tag{20}$$

with arbitrary $W\in\mathcal{U}_m[\delta]$.

*Proof:* First, we prove that the set of hyper-regular matrices $\Theta\in\mathcal{K}_x[\delta]^{n\times m}$ satisfying (19) is not empty. This can be done by showing that $\widehat U$ is hyper-regular and satisfies (19). Really, multiplying

$$U^{-1}\widehat U=\begin{pmatrix}0_{n-m,m}\\ I_m\end{pmatrix}.$$

from the right by a permutation matrix $I$, satisfying $\begin{pmatrix}0_{n-m,m}\\ I_m\end{pmatrix}I=\begin{pmatrix}I_m\\ 0_{n-m,m}\end{pmatrix}$, one proves that $\widehat U$ is hyper-regular. Since

$$\begin{aligned}&VP(F)U\begin{pmatrix}0_{n-m,m}\\ I_m\end{pmatrix}\\&\qquad=VP(F)\widehat U=(I_{n-m},0_{n-m,m})\begin{pmatrix}0_{n-m,m}\\ I_m\end{pmatrix}=0,\end{aligned}$$

$\widehat U$ is a solution of (19).

Suppose next that hyper-regular $\Theta$ satisfies (19) and we show that it yields (20). From

$$VP(F)\Theta=VP(F)UU^{-1}\Theta=(I_{n-m},0_{n-m,m})U^{-1}\Theta=0$$

we get $U^{-1}\Theta = \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W$ or $\Theta = \widehat{U}W$,where $W \in \mathcal{K}_x[\delta]^{m\times m}$ is arbitrary. Because $\Theta$ is hyper-regular, then $\widehat{U}W$ is hyper-regular and from that, $W$ is hyper-regular and thus also unimodular. □

Under assumptions of Lemma 3, $P(F)\Theta \mathrm{d}y = 0$. Then one can take $x = \varphi^0(\bar{y})$ where $\mathrm{d}\varphi^0(\bar{y}) = \Theta\mathrm{d}y$ iff one-forms $\Theta\mathrm{d}y$ are exact. Since by Lemma 3, $\Theta = \widehat{U}W$, where $W \in \mathcal{U}_m[\delta]$ is arbitrary, to get $\varphi^0$ one must find a matrix $W$ such that the one-forms $\widehat{U}W\mathrm{d}y$ are exact.

If there exists such $W$ that one-forms $\widehat{U}W\mathrm{d}y$ are exact, it remains to show that mapping $\Phi = (\varphi^0, \delta\varphi^0, \ldots)$, where $\varphi^0$ is defined by $\mathrm{d}\varphi^0 = \widehat{U}W\mathrm{d}y$, is invertible, i.e. there exists a matrix $H \in \mathcal{K}_x[\delta]^{m\times n}$ such that $H\widehat{U}W = I_m$, because then one can find $\mathrm{d}y = H\mathrm{d}x$.

*Lemma 4.* Let

$$Q_0 = \begin{pmatrix} 0_{m,n-m} & I_m \\ I_{n-m} & 0_{n-m,m} \end{pmatrix} U^{-1} \tag{21}$$

and

$$\widetilde{Q}_0 = (I_m, 0_{m,n-m})Q_0. \tag{22}$$

Then $W^{-1}\widetilde{Q}_0\widehat{U}W = I_m$

*Proof:* The result is obtained by direct computation. □

Thus, one may take $H = W^{-1}\widetilde{Q}_0$. Then $\mathrm{d}y = W^{-1}\widetilde{Q}_0\mathrm{d}x$. Note that one can also find matrix $W$ such that one-forms $W^{-1}\widetilde{Q}_0\mathrm{d}x$ are exact.

*Theorem 5.* The implicit control system $(\mathbb{R}^n_\infty, F)$ is flat iff there exists an unimodular matrix $W \in \mathcal{U}_m[\delta]$ such that the one-forms $W^{-1}\widetilde{Q}_0\mathrm{d}x$ are exact.

*Proof: Necessity.* If the system is flat, then there exists a function $\varphi^0$ such that $x = \varphi^0(\bar{y})$ and $F(x, x^+) = 0$. Thus $\mathrm{d}x = P(\varphi^0)\mathrm{d}y$. Because $F(x, x^+) = 0$, $P(F)\mathrm{d}x = 0$, and therefore $P(F)P(\varphi^0)\mathrm{d}y = 0$. The last equality is true iff $P(F)P(\varphi^0) = 0$. By Lemma 3 matrix $P(\varphi^0) = \widehat{U}W$ where $W \in \mathcal{U}_m[\delta]$. Since $P(\varphi^0)\mathrm{d}y$ is exact, $\widehat{U}W\mathrm{d}y$ is exact. Then, by Lemma 4, the one-forms $W^{-1}\widetilde{Q}_0\mathrm{d}x = \mathrm{d}y$ are exact.

*Sufficiency.* If $W \in \mathcal{U}_m[\delta]$ is such that the one-forms $W^{-1}\widetilde{Q}_0\mathrm{d}x$ are exact, then take the functions $\varphi^0$ and $\psi^0$ such that $\mathrm{d}\psi^0 = W^{-1}\widetilde{Q}_0\mathrm{d}x$ and $\mathrm{d}\varphi^0 = \widehat{U}W\mathrm{d}y$. Since $P(F)\widehat{U}W\mathrm{d}y = 0$, the conditions of Theorem 3 are satisfied for mapping $\Phi = (\varphi^0, \delta\varphi^0, \ldots)$ and the system is flat. □

To compute the flat outputs define the one-forms

$$\omega = (\omega_1, \ldots, \omega_m)^T = \widetilde{Q}_0\mathrm{d}x. \tag{23}$$

Then it remains to be found an unimodular matrix $W$ such that $\mathrm{d}(W^{-1}\omega) = 0$.

*Example 2.* (Continuation of Example 1) The matrix $P(F)$, defined by (16), is

$$P(F) = \begin{pmatrix} -x_3\delta & \delta + x_3 & -x_1^+ + x_2 & 0 \\ -\delta & 1 & 0 & \delta - 1 \end{pmatrix}$$

and its Jacobson decomposition is $VP(F)U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, where $V = I_2$ and $U$ is such that its inverse is

$$U^{-1} = \begin{pmatrix} -x_3\delta & x_3 + \delta & -x_1^+ + x_2 & 0 \\ -\delta & 1 & 0 & \delta - 1 \\ -x_3^- & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Compute the matrices $Q_0$ and $\widetilde{Q}_0$ from (21) and (22), respectively,

$$Q_0 = \begin{pmatrix} -x_3^- & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -x_3\delta & x_3 + \delta & -x_1^+ + x_2 & 0 \\ -\delta & 1 & 0 & \delta - 1 \end{pmatrix}$$

$$\widetilde{Q}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} Q_0 = \begin{pmatrix} -x_3^- & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

The one-forms (23) for system (12) are

$$\omega := \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \widetilde{Q}_0\mathrm{d}x = \begin{pmatrix} \mathrm{d}x_2 - x_3^-\mathrm{d}x_1 \\ \mathrm{d}x_4 - \mathrm{d}x_1 \end{pmatrix}. \tag{24}$$

Note that these one-forms satisfy the conditions of Theorem 1 and have been found already in Aranda-Bricaire et al. [1996]. To prove that the system (12) is flat by Theorem 5 it remains to be shown that there exists an unimodular matrix $W \in \mathcal{U}_2[\delta]$, such that the one-forms $W\omega$ are exact, where $\omega$ is defined by (24). Really, take

$$W = \begin{pmatrix} \dfrac{1}{1 - x_3^-} & \dfrac{x_3^-\delta - x_3^-}{1 - x_3^-} \\ 0 & 1 \end{pmatrix},$$

then $W\omega = (\mathrm{d}x_2, \mathrm{d}x_4 - \mathrm{d}x_1)^T$. Thus, one choice for the flat outputs of system (12) is $y_1 = x_2$ and $y_2 = x_4 - x_1$.

### *3.5 Comparison*

Note that Theorems 2 and 5 claim the same, if $\omega$ in (23) equals to $\hat{\omega}$, defined in Theorem 1.

*Theorem 6.* The one-forms $\hat{\omega}_i$, $i = 1, \ldots, m$, defined in Theorem 1, are equal to those in (23), up to the isomorphism of difference fields $\widetilde{\mathcal{K}}$ and $\mathcal{K}_x$.

*Proof:* By (*i*) of Theorem 1, the set $\{\tilde{\delta}^j\hat{\omega}_i, i = 1, \ldots, m; j = 0, \ldots, r_i - 1\}$ forms a basis of $\mathrm{span}_{\widetilde{\mathcal{K}}}\{\mathrm{d}x\}$. Denote $\hat{\omega}_{i,j} := \tilde{\delta}^{j-1}\hat{\omega}_i$, then,

$$\begin{pmatrix} \hat{\omega}_{1,1} \\ \vdots \\ \hat{\omega}_{m,r_m} \end{pmatrix} = \begin{pmatrix} \hat{a}_{1,1} & \cdots & \hat{a}_{1,n} \\ \vdots & \ldots & \vdots \\ \hat{a}_{n,1} & \cdots & \hat{a}_{n,n} \end{pmatrix} \mathrm{d}x =: \hat{A}\mathrm{d}x,$$

for some $\hat{a}_{k,l} \in \widetilde{\mathcal{K}}$. Note, that the one-forms $\hat{\omega}_{i,j}$ are defined over the field $\widetilde{\mathcal{K}}$. Using (7), we redefine them over the field $\mathcal{K}_x$ and denote by $\tilde{\omega}_{i,j}$. Moreover, let $a_{k,l} \in \mathcal{K}_x$ be the functions, obtained by transforming $\hat{a}_{k,l}$ according to (7). So,

$$\begin{pmatrix} \tilde{\omega}_{1,1} \\ \vdots \\ \tilde{\omega}_{m,r_m} \end{pmatrix} := \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ldots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \mathrm{d}x =: A\mathrm{d}x.$$

According to (*iii*) of Theorem 1, the one-forms $\hat{\omega}_{i,j}$ are linearly independent and thus the matrix $\hat{A}$ is invertible. Because we used an isomorphism to transform $\hat{A}$ into $A$, the matrix $A$ is also invertible. Let $A^{-1} = (\bar{a}_{k,l})$ be the inverse of $A$.

From the proof of Theorem 1, the global linearization of system (1) (variational system) can be given in the form

$$\hat{\omega}_{i,j}^{+} = \hat{\omega}_{i,j+1} \qquad j = 1, \ldots, r_i - 1; i = 1, \ldots, m \quad (25)$$
$$\hat{\omega}_{i,r_i}^{+} = \sum_{s=1}^{m} \sum_{j=1}^{r_s} c_{s,j}^{i} \hat{\omega}_{s,j} + \sum_{j=1}^{m} p_j^i \mathrm{d}u_j,$$

where $c_{s,j}^i, p_j^i \in \widetilde{\mathcal{K}}$ and matrix $P$ with elements $p_j^i$ is invertible. Using the one-forms $\tilde{\omega}_{i,j}$ instead of $\hat{\omega}_{i,j}$, we get the implicit representation of (25)

$$\mathrm{d}F = \begin{pmatrix} \tilde{\omega}_{1,1}^{+} - \tilde{\omega}_{1,2} \\ \vdots \\ \tilde{\omega}_{m,r_m-1}^{+} - \tilde{\omega}_{m,r_m} \end{pmatrix}. \quad (26)$$

We next compute the one-forms (23) for system (26). Consider first for simplicity the case $m = 1$. Then

$$\begin{aligned} \mathrm{d}F &= \begin{pmatrix} \tilde{\omega}_{1,1}^{+} - \tilde{\omega}_{1,2} \\ \vdots \\ \tilde{\omega}_{1,n-1}^{+} - \tilde{\omega}_{1,n} \end{pmatrix} \\ &= \begin{pmatrix} a_{1,1}^{+}\delta - a_{2,1} & \cdots & a_{1,n}^{+}\delta - a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1}^{+}\delta - a_{n,1} & \cdots & a_{n-1,n}^{+}\delta - a_{n,n} \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_n \end{pmatrix} \\ &=: P(F)\mathrm{d}x. \end{aligned}$$

Find the Jacobson decomposition of $P(F)$. Note that one may take $U := A^{-1}B$, where

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & \cdots & 0 & \delta \\ -\delta & -1 & \cdots & 0 & \delta^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -\delta^{n-2} & -\delta^{n-3} & \cdots & -1 & \delta^{n-1} \end{pmatrix}, \quad (27)$$

since then $P(F)U = P(F)A^{-1}B = (I_{n-1}, 0_{n-1,1})$. Since

$$B^{-1} = \begin{pmatrix} \delta & -1 & 0 & \cdots & 0 & 0 \\ 0 & \delta & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \delta & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

one gets

$$U^{-1} = \begin{pmatrix} a_{1,1}^{+}\delta - a_{2,1} & \cdots & a_{1,n}^{+}\delta - a_{2,n} \\ \vdots & \cdots & \vdots \\ a_{n-1,1}^{+}\delta - a_{n,1} & \cdots & a_{n-1,n}^{+}\delta - a_{n,n} \\ a_{1,1} & \cdots & a_{1,n} \end{pmatrix}.$$

Now, finding $Q_0$ from (21) and $\widetilde{Q}_0$ from (22), one obtains from (23) that $\omega_1 = \widetilde{Q}_0 \mathrm{d}x = \tilde{\omega}_{1,1}$. Because the one-forms $\tilde{\omega}_{1,1}$ and $\hat{\omega}_{1,1} = \hat{\omega}_1$ are equal up to the field transformation (7), Theorem 6 is proved for the case $m = 1$. In the general case the proof is similar. One may take $U = A^{-1}B\bar{I}$, where $B = blockdiag\{B_1, \ldots, B_m\}$, matrices $B_i \in \mathcal{U}_{r_i}[\delta]$ for $i = 1, \ldots, m$ are of the form (27) and $\bar{I}$ is a permutation matrix such that $P(F)A^{-1}B\bar{I} = (I_{n-m}, 0_{n-m,m})$. Computing then the one-forms (23), using (21) and (22), one gets $\tilde{\omega}_{i,1} = \hat{\omega}_i$, $i = 1, \ldots, m$, up to transformation (7). □

## 4. CONCLUSION

This paper addresses the property of flatness of the discrete-time nonlinear control system. Necessary and sufficient conditions analogous to those in Lévine [2011] were derived for checking the property and proved to be equivalent to the conditions in Aranda-Bricaire et al. [1996]. In the future, it is important to find what is the correct way to define flatness for discrete-time systems, is flatness equivalent to linearization by dynamic endogenous or exogenous feedback. Also, methods must be developed to compute matrix $M$ in Theorem 2 (or matrix $W$ in Theorem 5).

## REFERENCES


F. Antritter and G.G. Verhoeven. On symbolic computation of flat outputs for differentially flat systems. In *Preprints of the 8th IFAC Symposium on Nonlinear Control Systems*, pages 677–682. Bologna, Italy, 2010.

E. Aranda-Bricaire and C.H. Moog. Linearization of discrete-time systems by exogenous dynamic feedback. *Automatica*, 44:1707–1717, 2008.

E. Aranda-Bricaire, C.H. Moog, and J.-B. Pomet. A linear algebraic framework for dynamic feedback linearization. *IEEE Trans. on Automatic Control*, 40(1): 127–132, 1995.

E. Aranda-Bricaire, Ü. Kotta, and C.H. Moog. Linearization of discrete-time systems. *SIAM J. Control and Optimization*, 34(6):1999–2023, 1996.

P.M. Cohn. *Free rings and their relations*. Academic Press, London, UK, 1985.

M. Fliess and R. Marquez. Towards a module theoretic approach to discrete-time linear predictive control. *International Journal of Control*, 73:606–623, 2000.

M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *Comptes Rendus de l'Académie des Sciences*, 315:619–624, 1992.

M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *Int. J. of Control*, 61(6):1327–1361, 1995.

J.W. Grizzle. A linear algebraic framework for the analysis of discrete-time nonlinear systems. *SIAM Journal on Control and Optimization*, 31:1026–1044, 1993.

M. Halas, Ü. Kotta, Z. Li, H. Wang, and C. Yuan. Submersive rational difference systems and their accessibility. In *Proc. of the 2009 Int. Symp. on Symbolic and Algebraic Computation*, pages 175–182. Seoul, Korea, 2009.

A. Isidori, C.H. Moog, and A. De Luca. A sufficient condition for full linearization via dynamic state feedback. In *Proc. of the 25th IEEE Conference on Decision and Control*, pages 203–208. Athens, Greece, 1986.

J. Lévine. *Analysis and control of nonlinear systems: a flatness-based approach*. Springer, Berlin, 2009.

J. Lévine. On necessary and sufficient conditions for differential flatness. *Applicable Algebra in Engineering, Communication and Computing*, 22(1):47–90, 2011.

E. Pawluszewicz and Z. Bartosiewicz. External dynamic feedback equivalence of observable discrete-time control systems. *Differential Geometry and Control, Proceedings of Symposia in Pure Mathematics*, 64, 1998.

H. Sira-Ramirez and S.K. Agrawal. *Differentially flat systems*. CRC Press, New York, 2004.

S.H. Weintraub. *Differential forms: a complement to vector calculus*. Academic Press, San Diego, 1997.