Feedback Classification of Invariant Control Systems on Three-Dimensional Lie Groups

Rory Biggs * **Claudiu C. Remsing** **

* *Rhodes University, 6140 Grahamstown, South Africa (e-mail: rorybiggs@gmail.com).*
** *Rhodes University, 6140 Grahamstown, South Africa (e-mail: c.c.remsing@ru.ac.za)*

Abstract: We consider left-invariant control affine systems evolving on Lie groups. In this context, feedback equivalence specializes to detached feedback equivalence. We characterize (local) detached feedback equivalence in a simple algebraic manner. We then classify all (full-rank) systems evolving on three-dimensional Lie groups. A representative is identified for each equivalence class. Systems on the Heisenberg group, the Euclidean group, and the orthogonal group are treated in full, as typical examples. In these three cases, simple algebraic characterizations of the equivalence classes are also exhibited. A few remarks conclude the paper.

1. INTRODUCTION

Invariant control systems are smooth control systems evolving on (real, finite-dimensional) Lie groups, whose dynamics are invariant under translations.

A wide range of dynamical systems from fields as diverse as classical and quantum mechanics, elasticity, electrical networks, robotics, and molecular chemistry can be modeled by invariant control systems on (matrix) Lie groups. Many variational problems (with constraints) can be formulated in the geometric language of modern optimal control theory. Treatments of various invariant optimal control problems can be found, for instance, in Agrachev et al. [2004], Bloch [2003], Jurdjevic [1997]. See, also, Jurdjevic [2011], Sachkov [2009], Puta [1996].

2. INVARIANT CONTROL SYSTEMS AND EQUIVALENCE

2.1 Invariant control affine systems

Invariant control affine systems were first considered in Brockett [1972] and Jurdjevic et al. [1972]. An ℓ-input *left-invariant control affine system* Σ takes the form

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

Here G is a connected (matrix) Lie group with Lie algebra $\mathfrak{g}$, and $A, B_1, \ldots, B_\ell$ are elements of $\mathfrak{g}$. The (affine) parametrization map

$$\Xi\,(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \quad u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$$

is assumed to be injective. The *trace* $\Gamma = \operatorname{im} \Xi\,(\mathbf{1}, \cdot) \subseteq \mathfrak{g}$ of the system Σ is the affine subspace

$$A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle.$$

The system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Σ is said to have *full rank* if its trace Γ generates the whole Lie algebra $\mathfrak{g}$.

When the state space G is fixed, we specify the system Σ by simply writing

$$\Sigma\,:\, A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

Remark 1. For systems evolving on three-dimensional Lie groups, it is a simple matter to characterize the full-rank condition. A single-input inhomogeneous system has full rank if and only if A, B_1, and $[A, B_1]$ are linearly independent. A two-input homogeneous system has full rank if and only if B_1, B_2, and $[B_1, B_2]$ are linearly independent. Any two-input inhomogeneous system or three-input (homogeneous) system has full rank.

2.2 Detached feedback equivalence

Two (invariant control affine) systems Σ and Σ' are (locally) *feedback equivalent* if there exists a (local) diffeomorphism $g' = \phi(g)$ between their state spaces and an invertible transformation $u' = \varphi(g, u)$ such that the diffeomorphism $\Phi(g, u) = (\phi(g), \varphi(g, u))$ brings Σ into Σ'. Feedback equivalence (of smooth control systems) has been extensively studied in the last few decades (see Respondek et al. [2006] and the references therein).

We specialize feedback equivalence, by requiring that the transformation $u' = \varphi(g, u)$ is constant over the state space. Such (feedback) transformations Φ are exactly those that are compatible with the Lie group structure (cf. Biggs et al. [2012a]). Σ and Σ' are called (locally) *detached feedback equivalent* if there exist neighbourhoods N and N' of (the unit elements) $\mathbf{1}$ and $\mathbf{1}'$, respectively, and diffeomorphisms $\phi : N \to N'$, $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that $\phi(\mathbf{1}) = \mathbf{1}'$ and $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. (Here $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$.)

Theorem 2. Two full-rank systems Σ and Σ' are detached feedback equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.

Proof. Suppose Σ and Σ' are detached feedback equivalent. Then $T_\mathbf{1}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and so $T_\mathbf{1}\phi \cdot \Gamma = \Gamma'$. As $T_\mathbf{1}\phi$ is a linear isomorphism, it remains only to show that it preserves the Lie bracket. Let $u, v \in \mathbb{R}^\ell$, and let $\Xi_u = \Xi(\cdot, u)$ and $\Xi_v = \Xi(\cdot, v)$ denote the corresponding (local) vector fields. Then $\phi_*[\Xi_u, \Xi_v] = [\phi_*\Xi_u, \phi_*\Xi_v]$ and so $T_\mathbf{1}\phi \cdot [\Xi_u(\mathbf{1}), \Xi_v(\mathbf{1})] = [\Xi'_{\varphi(u)}(\mathbf{1}'), \Xi'_{\varphi(v)}(\mathbf{1}')] = [T_\mathbf{1}\phi \cdot$

$\Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})]$. As Σ has full rank, the elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^\ell$ generate the Lie algebra $\mathfrak{g}$. Hence $T_{\mathbf{1}}\phi$ is a Lie algebra isomorphism. Conversely, suppose we have a Lie algebra isomorphism ψ such that $\psi \cdot \Gamma = \Gamma'$. Then there exist neighbourhoods N and N' of $\mathbf{1}$ and $\mathbf{1}'$, respectively, such that $\phi : N \to N'$ is a (local) group isomorphism with $T_{\mathbf{1}}\phi = \psi$ (see, e.g., Knapp [2002]). Also, the equation $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ defines an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$. Consequently $T_g\phi \cdot \Xi(g,u) = T_{\mathbf{1}}L_{\phi(g)} \cdot \psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\phi(g), \varphi(u))$. Hence Σ and Σ' are detached feedback equivalent.

Henceforth, we shall refer to detached feedback equivalence simply as equivalence.

3. CLASSIFICATION OF SYSTEMS

We classify all full-rank systems evolving on three-dimensional Lie groups. More precisely, for each three-dimensional Lie algebra $\mathfrak{g}$, we classify all systems evolving on a corresponding (connected) Lie group G.

Remark 3. If Σ' is a system on G' and $\mathfrak{g}' \cong \mathfrak{g}$, then Σ' is equivalent to some system Σ on G. Therefore, it is sufficient to consider only one Lie algebra from each isomorphism class. Moreover, only one connected Lie group need be considered for each Lie algebra.

Remark 4. By theorem 2, the classification problem essentially reduces to the classification of the affine subspaces of each Lie algebra $\mathfrak{g}$.

The classification of real three-dimensional Lie algebras is well known (see, e.g., MacCallum [1999], Krasiński et al. [2003], Mubarakzyanov [1963]). There are eleven types of three-dimensional Lie algebras (in fact, nine algebras and two parametrized infinite families of algebras). In terms of an appropriate ordered basis (E_1, E_2, E_3), the commutator operation is given by

$$[E_2, E_3] = n_1 E_1 - a E_2$$
$$[E_3, E_1] = a E_1 + n_2 E_2$$
$$[E_1, E_2] = n_3 E_3.$$

The (Bianchi-Behr) structure parameters are given by

Type	Bianchi	a	n_1	n_2	n_3	
$3\mathfrak{g}_1$	I	0	0	0	0	$\mathbb{R}^3$
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	III	1	1	-1	0	$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$
$\mathfrak{g}_{3.1}$	II	0	1	0	0	$\mathfrak{h}_3$
$\mathfrak{g}_{3.2}$	IV	1	1	0	0	
$\mathfrak{g}_{3.3}$	V	1	0	0	0	
$\mathfrak{g}_{3.4}^0$	VI_0	0	1	-1	0	$\mathfrak{se}(1,1)$
$\mathfrak{g}_{3.4}^a$	VI_a	$a>0$, $a\neq1$	1	-1	0	
$\mathfrak{g}_{3.5}^0$	VII_0	0	1	1	0	$\mathfrak{se}(2)$
$\mathfrak{g}_{3.5}^a$	VII_a	$a>0$	1	1	0	
$\mathfrak{g}_{3.6}$	$VIII$	0	1	1	-1	$\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{g}_{3.7}$	IX	0	1	1	1	$\mathfrak{so}(3)$

An enumeration of all (connected) three-dimensional Lie groups can be found in Onishchik et al. [1994]. However, we shall not make explicit reference to the connected Lie groups involved in the statements of the classification.

3.1 The solvable case

The classification procedure is as follows. First the group of automorphisms is determined; a standard computation yields the result (see, e.g, Harvey [1979], Ha et al. [2009], Popovych et al. [2003]). Equivalence class representatives are then constructed by considering the action of an automorphism on the trace of a typical system. Lastly, one verifies that none of the representatives are equivalent.

Theorem 5. (Biggs et al. [2013b,c]). Let Σ be a full-rank system evolving on a solvable group G.

(0) If $\mathfrak{g} \cong 3\mathfrak{g}_1$, then Σ is equivalent to exactly one of the following systems
$$\Sigma^{(2,1)} : E_1 + u_1 E_2 + u_2 E_3$$
$$\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

(1) If $\mathfrak{g} \cong \mathfrak{g}_{3.1}$, then Σ is equivalent to exactly one of the following systems
$$\Sigma^{(1,1)} : E_2 + u E_3$$
$$\Sigma^{(2,0)} : u_1 E_2 + u_2 E_3$$
$$\Sigma^{(2,1)}_1 : E_1 + u_1 E_2 + u_2 E_3$$
$$\Sigma^{(2,1)}_2 : E_3 + u_1 E_1 + u_2 E_2$$
$$\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

(2) If $\mathfrak{g} \cong \mathfrak{g}_{3.2}$, then Σ is equivalent to exactly one of the following systems
$$\Sigma^{(1,1)}_1 : E_2 + u E_3$$
$$\Sigma^{(1,1)}_{2,\beta} : \beta E_3 + u E_2$$
$$\Sigma^{(2,0)} : u_2 E_2 + u_2 E_3$$
$$\Sigma^{(2,1)}_1 : E_1 + u_1 E_2 + u_2 E_3$$
$$\Sigma^{(2,1)}_2 : E_2 + u_1 E_3 + u_2 E_1$$
$$\Sigma^{(2,1)}_{3,\beta} : \beta E_3 + u_1 E_1 + u_2 E_2$$
$$\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

(3) If $\mathfrak{g} \cong \mathfrak{g}_{3.3}$, then Σ is equivalent to exactly one of the following systems
$$\Sigma^{(2,1)}_1 : E_1 + u_1 E_2 + u_2 E_3$$
$$\Sigma^{(2,1)}_{2,\beta} : \beta E_3 + u_1 E_1 + u_2 E_2$$
$$\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

(4) If $\mathfrak{g} \cong \mathfrak{g}^0_{3.4}$, then Σ is equivalent to exactly one of the following systems
$$\Sigma^{(1,1)}_1 : E_2 + u E_3$$
$$\Sigma^{(1,1)}_{2,\alpha} : \alpha E_3 + u E_2$$
$$\Sigma^{(2,0)} : u_1 E_2 + u_2 E_3$$
$$\Sigma^{(2,1)}_1 : E_1 + u_1 E_2 + u_2 E_3$$
$$\Sigma^{(2,1)}_2 : E_1 + u_1 (E_1 + E_2) + u_2 E_3$$
$$\Sigma^{(2,1)}_{3,\alpha} : \alpha E_3 + u_1 E_1 + u_2 E_2$$
$$\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

(4a) If $\mathfrak{g} \cong \mathfrak{g}_{3.4}^a$ (resp. $\mathfrak{g} \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 = \mathfrak{g}_{3.4}^1$), then Σ is equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma_1^{(1,1)} &: E_2 + uE_3 \\
\Sigma_{2,\beta}^{(1,1)} &: \beta E_3 + uE_2 \\
\Sigma^{(2,0)} &: u_1 E_2 + u_2 E_3 \\
\Sigma_1^{(2,1)} &: E_1 + u_1 E_2 + u_2 E_3 \\
\Sigma_2^{(2,1)} &: E_1 + u_1(E_1 + E_2) + u_2 E_3 \\
\Sigma_3^{(2,1)} &: E_1 + u_1(E_1 - E_2) + u_2 E_3 \\
\Sigma_{4,\beta}^{(2,1)} &: \beta E_3 + u_1 E_1 + u_2 E_2 \\
\Sigma^{(3,0)} &: u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

(5) If $\mathfrak{g} \cong \mathfrak{g}_{3.5}^0$, then Σ is equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma_1^{(1,1)} &: E_2 + uE_3 \\
\Sigma_{2,\alpha}^{(1,1)} &: \alpha E_3 + uE_2 \\
\Sigma^{(2,0)} &: u_1 E_2 + u_2 E_3 \\
\Sigma_1^{(2,1)} &: E_1 + u_1 E_2 + u_2 E_3 \\
\Sigma_{2,\alpha}^{(2,1)} &: \alpha E_3 + u_1 E_1 + u_2 E_2 \\
\Sigma^{(3,0)} &: u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

(5a) If $\mathfrak{g} \cong \mathfrak{g}_{3.5}^a$, then Σ is equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma_1^{(1,1)} &: E_2 + uE_3 \\
\Sigma_{2,\beta}^{(1,1)} &: \beta E_3 + uE_2 \\
\Sigma^{(2,0)} &: u_1 E_2 + u_2 E_3 \\
\Sigma_1^{(2,1)} &: E_1 + u_1 E_2 + u_2 E_3 \\
\Sigma_{2,\beta}^{(2,1)} &: \beta E_3 + u_1 E_1 + u_2 E_2 \\
\Sigma^{(3,0)} &: u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

Here $\alpha > 0$ and $\beta \neq 0$ parametrize families of distinct (non-equivalent) class representatives.

Proof. We consider, as typical cases, only items (1) and (5), i.e., the Heisenberg Lie algebra and the Euclidean Lie algebra.

(1) The group of automorphisms $\mathsf{Aut}(\mathfrak{g}_{3.1})$, with respect to the appropriate basis (E_1, E_2, E_3), is

$$\left\{ \begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix} : u, v, w, x, y, z \in \mathbb{R},\ yw - vz \neq 0 \right\}.$$

Let Σ be a single-input inhomogeneous system with trace $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i \right\rangle \subset \mathfrak{g}_{3.1}$. Then

$$\psi = \begin{bmatrix} a_2 b_3 - a_3 b_2 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ 0 & a_3 & b_3 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma^{(1,1)} = \Gamma$.

Let Σ be a two-input homogeneous system with trace $\Gamma = \langle B_1, B_2 \rangle$. Then $\widehat{\Sigma} : B_1 + \langle B_2 \rangle$ is a (full-rank) single-input inhomogeneous system. Therefore, there exists an automorphism ψ such that $\psi \cdot (B_1 + \langle B_2 \rangle) = E_2 + \langle E_3 \rangle$. Thus $\psi \cdot \langle B_1, B_2 \rangle = \langle E_2, E_3 \rangle$ and so Σ is equivalent to $\Sigma^{(2,0)}$.

Let Σ be a two-input inhomogeneous system with trace $\Gamma = A + \Gamma^0$. Suppose $E_1 \notin \Gamma^0$. Let $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i, \sum_{i=1}^3 c_i E_i \right\rangle$. Then

$$\psi = \begin{bmatrix} v_1 & v_2 & v_3 \\ 0 & 1 & 0 \\ 0 & 0 & v_1 \end{bmatrix} \qquad [v_1\ v_2\ v_3] \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = [1\ 0\ 0]$$

defines an automorphism ψ such that $\psi \cdot \Gamma = \Gamma_1^{(2,1)}$. (As $E_1 \notin \Gamma^0$, it follows that $v_1 \neq 0$.) On the other hand, suppose $E_1 \in \Gamma^0$. Then $\Gamma = a_2 E_2 + a_3 E_3 + \langle E_1, b_2 E_2 + b_3 E_3 \rangle$. Hence

$$\psi = \begin{bmatrix} b_2 a_3 - a_2 b_3 & 0 & 0 \\ 0 & b_2 & a_2 \\ 0 & b_3 & a_3 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_2^{(2,1)} = \Gamma$. Thus Σ is equivalent to $\Sigma_2^{(2,1)}$.

If Σ is a three-input system, then clearly $\Gamma = \Gamma^{(3,0)}$ and so Σ is equivalent to $\Sigma^{(3,0)}$.

Clearly, no homogeneous system can be equivalent to an inhomogeneous one. Also, if the number of inputs for two systems differ, then they cannot be equivalent. As E_1 is an eigenvector of every automorphism, it follows that $\Sigma_1^{(2,1)}$ and $\Sigma_2^{(2,1)}$ are not equivalent.

(5) The group of automorphisms $\mathsf{Aut}(\mathfrak{g}_{3.5}^0)$ is

$$\left\{ \begin{bmatrix} x & y & u \\ -\sigma y & \sigma x & v \\ 0 & 0 & \sigma \end{bmatrix} : x, y, u, v \in \mathbb{R},\ x^2 + y^2 \neq 0,\ \sigma = \pm 1 \right\}.$$

Let Σ be a single-input inhomogeneous system with trace $\Gamma = A + \Gamma^0 \subset \mathfrak{g}_{3.5}^0$. Suppose $E_3^*(\Gamma^0) \neq \{0\}$. (Here E_3^* is the corresponding element of the dual basis.) Then $\Gamma = a_1 E_1 + a_2 E_2 + \langle b_1 E_1 + b_2 E_2 + E_3 \rangle$. Thus

$$\psi = \begin{bmatrix} a_1 & a_1 & b_1 \\ -a_1 & a_2 & b_2 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_1^{(1,1)} = \Gamma$. So Σ is equivalent to $\Sigma_1^{(1,1)}$. On the other hand, suppose $E_3^*(\Gamma^0) = \{0\}$. Then $\Gamma = a_1 E_1 + a_2 E_2 + a_3 E_3 + \langle b_1 E_1 + b_2 E_2 \rangle$ with $a_3 \neq 0$. Hence

$$\psi = \begin{bmatrix} b_2 \operatorname{sgn}(a_3) & b_1 & \frac{a_1}{a_3 \operatorname{sgn}(a_3)} \\ -b_1 \operatorname{sgn}(a_3) & b_2 & \frac{a_2}{a_3 \operatorname{sgn}(a_3)} \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_{2,\alpha}^{(1,1)} = \Gamma$, where $\alpha = a_3 \operatorname{sgn}(a_3)$.

Let Σ be a two-input homogeneous system with trace $\Gamma = \langle B_1, B_2 \rangle$. Then $\widehat{\Sigma} : B_1 + \langle B_2 \rangle$ is a (full-rank) single-input inhomogeneous system. Therefore, there exists an automorphism ψ such that $\psi \cdot (B_1 + \langle B_2 \rangle)$ equals either $E_2 + \langle E_3 \rangle$ or $\alpha E_3 + \langle E_2 \rangle$. Hence, in either case, we get $\psi \cdot \langle B_1, B_2 \rangle = \langle E_2, E_3 \rangle$. Thus Σ is equivalent to $\Sigma^{(2,0)}$.

Let Σ be a two-input inhomogeneous system with trace $\Gamma = A + \Gamma^0$. Suppose $E_3^*(\Gamma^0) \neq 0$. Then $\Gamma = a_1 E_1 + a_2 E_2 + \langle b_1 E_1 + b_2 E_2, c_1 E_1 + c_2 E_2 + E_3 \rangle$. Thus

$$\psi = \begin{bmatrix} v_2 b_2 & v_2 b_1 & c_1 \\ -v_2 b_1 & v_2 b_2 & c_2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} b_2 & -b_1 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$

defines an automorphism ψ such that $\psi \cdot \Gamma_1^{(2,1)} = \Gamma$ (We have that $v_2 \neq 0$ as Σ is inhomogeneous.) Indeed, $\psi \cdot \langle E_2, E_3\rangle = \Gamma^0$ and

$$\begin{aligned}\psi \cdot E_1 &= v_2 b_2 E_1 - v_2 b_1 E_2 \\ &= (a_1 - v_1 b_1)E_1 + (a_2 - v_1 b_2)E_2 \\ &= a_1 E_1 + a_2 E_2 - v_1(b_1 E_1 + b_2 E_2) \in \Gamma.\end{aligned}$$

On the other hand suppose $E_3^*(\Gamma)^0 = \{0\}$. Then $\Gamma = a_3 E_3 + \langle E_1, E_2\rangle$. Thus $\psi = \operatorname{diag}(1, \operatorname{sgn}(a_3), \operatorname{sgn}(a_3))$ is an automorphism such that $\psi \cdot \Gamma = \Gamma_{2,\alpha}^{(2,1)}$ for some $\alpha > 0$.

If Σ is a three-input system, then it is equivalent to $\Sigma^{(3,0)}$.

Again, most pairs of systems cannot be equivalent due to different homogeneities or different number of inputs. As the subspace $\langle E_1, E_2\rangle$ is invariant (under the action of automorphisms), $\Sigma_1^{(1,1)}$ is not equivalent to any system $\Sigma_{2,\alpha}^{(1,1)}$. For $A \in \mathfrak{g}_{3.5}^0$ and $\psi \in \mathsf{Aut}(\mathfrak{g}_{3.5}^0)$, we have that $E_3^*(\psi \cdot \alpha E_3) = \pm\alpha$. Thus $\Sigma_{2,\alpha}^{(1,1)}$ and $\Sigma_{2,\alpha'}^{(1,1)}$ are equivalent only if $\alpha = \alpha'$. For the two-input inhomogeneous systems, similar arguments hold.

3.2 The semisimple case

The procedure for classification is similar to that of the solvable groups. However, here we employ an invariant bilinear product ω (the Lorentzian product and the dot product, respectively); the inhomogeneous systems are (partially) characterized by the level set $\{A \in \mathfrak{g} : \omega(A, A) = \alpha\}$ that their trace is tangent to.

Theorem 6. (Biggs et al. [2013a]). Let Σ be a full-rank system on a semisimple group G with Lie algebra $\mathfrak{g}$.

(6) If $\mathfrak{g} \cong \mathfrak{g}_{3.6}$, then Σ is equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma_1^{(1,1)} &: E_3 + u(E_2 + E_3) \\
\Sigma_{2,\alpha}^{(1,1)} &: \alpha E_2 + u E_3 \\
\Sigma_{3,\alpha}^{(1,1)} &: \alpha E_1 + u E_2 \\
\Sigma_{4,\alpha}^{(1,1)} &: \alpha E_3 + u E_2 \\
\Sigma_1^{(2,0)} &: u_1 E_1 + u_2 E_2 \\
\Sigma_2^{(2,0)} &: u_1 E_2 + u_2 E_3 \\
\Sigma_1^{(2,1)} &: E_3 + u_1 E_1 + u_2(E_2 + E_3) \\
\Sigma_{2,\alpha}^{(2,1)} &: \alpha E_1 + u_1 E_2 + u_2 E_3 \\
\Sigma_{3,\alpha}^{(2,1)} &: \alpha E_3 + u_1 E_1 + u_2 E_2 \\
\Sigma^{(3,0)} &: u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

(7) If $\mathfrak{g} \cong \mathfrak{g}_{3.7}$, then Σ is equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma_\alpha^{(1,1)} &: \alpha E_2 + u E_3 \\
\Sigma^{(2,0)} &: u_1 E_2 + u_2 E_3 \\
\Sigma_\alpha^{(2,1)} &: \alpha E_1 + u_1 E_2 + u_2 E_3 \\
\Sigma^{(3,0)} &: u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

Here $\alpha > 0$ parametrizes families of distinct (non-equivalent) class representatives.

Proof. We consider only item (7), i.e., the orthogonal Lie algebra $\mathfrak{g}_{3.7}$. (The proof for item (6), although more involved, is similar.) The group of automorphisms of $\mathfrak{g}_{3.7}$ is $\mathsf{SO}(3) = \{g \in \mathbb{R}^{3\times 3} : gg^\top = \mathbf{1}, \det g = 1\}$. The dot product $\bullet$ on $\mathfrak{g}_{3.7}$ is given by $A \bullet B = a_1 b_1 + a_2 b_2 + a_3 b_3$. (Here $A = \sum_{i=1}^3 a_i E_i$ and $B = \sum_{i=1}^3 b_i E_i$.) The level sets $\mathcal{S}_\alpha = \{A \in \mathfrak{so}(3) : A \bullet A = \alpha\}$ are spheres of radius $\sqrt{\alpha}$ (and are preserved by automorphisms). The group of automorphisms acts transitively on each sphere $\mathcal{S}_\alpha$. The *critical point* $\mathfrak{C}^\bullet(\Gamma)$ (at which an inhomogeneous affine subspace is tangent to a sphere $\mathcal{S}_\alpha$) is given by

$$\mathfrak{C}^\bullet(\Gamma) = A - \frac{A \bullet B}{B \bullet B} B$$

$$\mathfrak{C}^\bullet(\Gamma) = A - [B_1 \ B_2] \begin{bmatrix} B_1 \bullet B_1 & B_1 \bullet B_2 \\ B_1 \bullet B_2 & B_2 \bullet B_2 \end{bmatrix}^{-1} \begin{bmatrix} A \bullet B_1 \\ A \bullet B_2 \end{bmatrix}.$$

Critical points behave well under the action of automorphisms, i.e., $\psi \cdot \mathfrak{C}^\bullet(\Gamma) = \mathfrak{C}^\bullet(\psi \cdot \Gamma)$ for any automorphism ψ. (The critical point of Γ is well defined as it is independent of parametrization.)

Let Σ be a single-input inhomogeneous system with trace Γ. There exists an automorphism ψ such that $\psi \cdot \Gamma = \alpha \sin\theta\, E_1 + \alpha\cos\theta\, E_2 + \langle E_3\rangle$, where $\alpha = \sqrt{\mathfrak{C}^\bullet(\Gamma) \bullet \mathfrak{C}^\bullet(\Gamma)}$. Hence

$$\psi' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi' \cdot \psi \cdot \Gamma = \Gamma_\alpha^{(1,1)}$.

Let Σ be a two-input homogeneous system with trace $\Gamma = \langle B_1, B_2\rangle$. Then $\widehat{\Sigma} : B_1 + \langle B_2\rangle$ is a (full-rank) single-input inhomogeneous system. Therefore, there exists an automorphism ψ such that $\psi \cdot (B_1 + \langle B_2\rangle) = \alpha E_2 + \langle E_3\rangle$. Hence, $\psi \cdot \langle B_1, B_2\rangle = \langle E_2, E_3\rangle$. Thus Σ is equivalent to $\Sigma^{(2,0)}$.

Let Σ be a two-input inhomogeneous system with trace Γ. We have $\mathfrak{C}^\bullet(\Gamma) \bullet \mathfrak{C}^\bullet(\Gamma) = \alpha^2$ for some $\alpha > 0$. As $\mathfrak{C}^\bullet(\Gamma_{1,\alpha}) \bullet \mathfrak{C}^\bullet(\Gamma_{1,\alpha}) = \alpha^2$, there exists an automorphism ψ such that $\psi \cdot \mathfrak{C}^\bullet(\Gamma) = \mathfrak{C}^\bullet(\Gamma_{1,\alpha})$. Hence $\psi \cdot \Gamma$ and $\Gamma_{1,\alpha}$ are both equal to the tangent plane of $\mathcal{S}_{\alpha^2}$ at $\psi \cdot \mathfrak{C}^\bullet(\Gamma)$, and are therefore identical.

If Σ is a three-input system, then it is equivalent to $\Sigma^{(3,0)}$.

Lastly we note that none of the representatives obtained are equivalent. (Again, we first distinguish representatives in terms of homogeneity and number of inputs.) As $\alpha^2 = \mathfrak{C}^\bullet(\Gamma_\alpha^{(1,1)}) \bullet \mathfrak{C}^\bullet(\Gamma_\alpha^{(1,1)})$ (resp. $\alpha^2 = \mathfrak{C}^\bullet(\Gamma_\alpha^{(2,1)}) \bullet \mathfrak{C}^\bullet(\Gamma_\alpha^{(2,1)})$) is an invariant quantity, the systems $\Sigma_\alpha^{(1,1)}$ and $\Sigma_{\alpha'}^{(1,1)}$ (resp. $\Sigma_\alpha^{(2,1)}$ and $\Sigma_{\alpha'}^{(2,1)}$) are equivalent only if $\alpha = \alpha'$.

3.3 Characterization of equivalence classes

The equivalence classes (in the classification) may be characterized in a simple algebraic manner. As typical examples, we characterize the equivalence classes obtained for the Heisenberg Lie algebra $\mathfrak{g}_{3.1}$, the Euclidean Lie algebra $\mathfrak{g}_{3.5}^0$, and the orthogonal Lie algebra $\mathfrak{g}_{3.7}$. The appropriate characterization (or "classifying condition") of the class corresponding to each representative is given below.

	Type	Characterization	Class
$\mathfrak{g}_{3.1}$	$(1,1)$		$\Sigma^{(1,1)}$
	$(2,0)$		$\Sigma^{(2,0)}$
	$(2,1)$	$E_1 \notin \Gamma^0$	$\Sigma_1^{(2,1)}$
		$E_1 \in \Gamma^0$	$\Sigma_2^{(2,1)}$
	$(3,0)$		$\Sigma^{(3,0)}$
$\mathfrak{g}_{3.5}^0$	$(1,1)$	$E_3^*(\Gamma^0) \neq \{0\}$	$\Sigma_1^{(1,1)}$
		$E_3^*(\Gamma^0) = \{0\},\ E_3^*(A) = \pm\alpha$	$\Sigma_{2,\alpha}^{(1,1)}$
	$(2,0)$		$\Sigma^{(2,0)}$
	$(2,1)$	$E_3^*(\Gamma^0) \neq \{0\}$	$\Sigma_1^{(2,1)}$
		$E_3^*(\Gamma^0) = \{0\},\ E_3^*(A) = \pm\alpha$	$\Sigma_{2,\alpha}^{(2,1)}$
	$(3,0)$		$\Sigma^{(3,0)}$
$\mathfrak{g}_{3.7}$	$(1,1)$	$\mathfrak{C}^\bullet(\Gamma) \bullet \mathfrak{C}^\bullet(\Gamma) = \alpha^2$	$\Sigma_\alpha^{(1,1)}$
	$(2,0)$		$\Sigma^{(2,0)}$
	$(2,1)$	$\mathfrak{C}^\bullet(\Gamma) \bullet \mathfrak{C}^\bullet(\Gamma) = \alpha^2$	$\Sigma_\alpha^{(2,1)}$
	$(3,0)$		$\Sigma^{(3,0)}$

4. CONCLUSION

We have classified, under detached feedback equivalence, all (full-rank) left-invariant control affine systems evolving on three-dimensional Lie groups. The presentation of the results follows closely the Bianchi-Behr enumeration (of real three-dimensional Lie algebras). However, due to space limitations, we have opted to provide details only for three groups, namely the Heisenberg group, the Euclidean group, and the orthogonal group. (These groups rank among the most "popular" three-dimensional Lie groups, in terms of control-oriented applications.) In present-day literature, one can find a sizable body of works dedicated to the study of invariant control systems (on lower dimensional Lie groups) and their applications. In particular, there are a number of notable contributions to geometric control on the three above-mentioned groups (see, e.g., Jurdjevic [1999], Monroy-Perez et al. [1999], Moiseev et al. [2010], Sachkov [2010], Jurdjevic [1995], Remsing [2010], Remsing [2011]).

Apart from detached feedback equivalence, there is another natural equivalence relation: state space equivalence (cf. Jakubczyk [1990], Krener [1973]). This equivalence relation is stronger and as such less promising. For instance, on the Euclidean group, we have the following classification of (a subclass of) systems (Adams et al. [2012]). Any two-input inhomogeneous system is equivalent to exactly one of the following systems

$$\Sigma_{1,\alpha\beta\boldsymbol{\gamma}}^{(2,1)} : \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2)$$
$$\Sigma_{2,\alpha\beta\boldsymbol{\gamma}}^{(2,1)} : \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2$$
$$\Sigma_{3,\alpha\beta\boldsymbol{\gamma}}^{(2,1)} : \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3).$$

Here $\alpha > 0$, $\beta \neq 0$, and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ parametrize families of class representatives.

Detached feedback equivalence has a natural extension to invariant optimal control problems (Biggs et al. [2012b]). More precisely, to an invariant optimal control problem (with quadratic cost) we associate a cost-extended system. (By specification of the boundary data, we recover the problem.) Equivalence of cost-extended systems is (partially) based on the equivalence of the underlying control systems. We have the following example of a classification under "cost equivalence." Any controllable two-input inhomogeneous cost-extended system on the Heisenberg group is cost equivalent to exactly one of the cost-extended systems

$$\begin{cases} \Sigma_1^{(2,1)} : E_1 + u_1 E_2 + u_2 E_3 \\ L(u) = (u_1 - \alpha)^2 + u_2^2. \end{cases}$$

Here $\alpha \geq 0$ parametrizes a family of class representatives.

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