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Sampled measurement adaptive observer for a class of state-affine nonlinear

Théo Folin*,[$], Tarek Ahmed-Ali*, Fouad Giri*, Françoise Lamnabhi-Lagarrigue**

** Université de Caen Basse-Normandie, GREYC-CNRS, 14032 Caen, France*
(e-mail: Theo.Folin@unicaen.fr, Tarek.Ahmed-Ali@ensicaen.fr, fouad.giri@unicaen.fr).
[$] BodyCap, 14200 Hérouville Saint Claire, France
*** LSS-CNRS, SUPELEC, EECI, 91192 Gif-sur-Yvette, France (e-mail: Francoise.Lamnabhi-Lagarrigue@lss.supelec.fr)}*

Abstract: The problem of state observation, based on sampled output measurements, is addressed for a class of nonlinear state-affine systems. The difficulty lies in the fact that the state equation involve the undisturbed output with unknown parameters. A hybrid adaptive observer is designed and formally analysed. Sufficient conditions are established for the observer to be exponentially convergent in the absence of output disturbance. The sufficient conditions include a usual persistent excitation condition as well as an explicit upper bound on the admissible sampling time. In presence of nonzero disturbance, the L_∞-gain between the disturbance input and the (state and parameter) estimation errors is explicitly established in function of the design parameters.

1. INTRODUCTION

Designing adaptive observers for continuous-time nonlinear systems has also been given a great deal of interest, especially over the last two decades, and some techniques for obtaining exponentially convergent observers are now available, e.g. (Bastin and Gevers, 1988; Marino and Tomei, 1996; Besancon, 2000; Besançon et al., 2006; Zhang, 2002). The point is that, the nonlinearity of the observer and its adaptive nature makes its exact discretization (necessary for implementation purpose) a highly complex issue. Furthermore, there is no guarantee that approximate discrete-time versions will (and generally they will not) preserve the performances of the original continuous-time observers. Therefore, a great deal of attention has been paid to the problem of directly designing sampled-data based observers for continuous-time nonlinear systems. One approach consists in discretizing the system model and using the discrete-time model in the observer design. The point is that, for most nonlinear systems, the exact discretization, when this is possible, leads to quite complex useless discrete-time models. A practical alternative is to use approximate discrete-time models obtained using Euler-like discretization techniques. The observers based on these tractable discrete-time models provide only state estimates (of the continuous-time system) at sampling instants and are generally shown to guarantee semi-global practical stability of the observation error, e.g. (Assoudi et al., 2002; Laila and Astolfi, 2006). A different approach, developed in e.g. (Deza et al., 1992; Hammouri et al., 2002; Nadri et al., 2004, Nadri and Hammouri, 2003), consists in designing discrete-continuous time observers, based on the continuous-time system model. The design principle entails a division of the observation process into two main tasks: open-loop state prediction between two successive sampling times and feedback state correction at sampling times. Another approach has been proposed by (Raff et al., 2008) and consists in letting the innovation term be constant between two successive sampling times. More recently, a new design approach has been developed in (Karafyllis and Kravaris, 2009) that consists in using a hybrid observer in association with an inter-sample output predictor. In this approach, only the output predictor is reinitialized at each sampling time, while the state estimate is continuously updated which makes it a continuous function of time. The approach has proved to be applicable to several classes of systems including linear detectable systems and triangular globally Lipschitz systems. Compared to the continuous-discrete observer design technique, the hybrid approach features the exponential convergence of the observation error as well as the simplicity of implementation as only one equation of the observer is reinitialized at sampling times (namely, that of the output predictor).

So far, quite a few studies have focused on the design of sampled-data adaptive observers, for nonlinear systems subject to parametric uncertainty. A first attempt has been made in (Ahmed-Ali et al., 2009) where the work of (Nadri and Hammouri, 2003) has been extended to a class of state affine system with unknown parameters. A continuous-discrete adaptive observer has thus been obtained and shown to ensure asymptotically the performances of the underlying nonadaptive observer. Inspired by the work of (Karafyllis and Kravaris, 2009), a sampled-data hybrid adaptive observer has been designed in (Hann and Ahmed-Ali, 2012). This adaptive observer, which involves an inter-sample predictor between sampling instants, has turned to be simpler than the one proposed in (Ahmed-Ali et al., 2009), though both observers applies to the same class of systems.

In the present paper, we present a new adaptive observer, using sampled measurements, for a class of state affine systems. The novelty of the present study is twofold: (i) the class of systems under study is output injection type; (ii) the system includes two unknown parameter vectors and one of them is multiplied by a not fully accessible system output i.e. an output that only accessible to measurement at sampling times, up to measurement noise. It turns out that the state-affine property is missing most of the time (it is only recovered up to noise at the sampling times). This is a major difficulty compared to the works of (Ahmed-Ali et al., 2009) and (Hann and Ahmed-Ali, 2012) where all unknown parameters came in multiplied by perfectly available quantities. In this respect, one may note that, in the context of sampled-data observers, state-affine systems with output injection have been considered in (Nadri et al., 2004). But, the observer proposed there is not adaptive (no parametric uncertainty was considered).

The problem of sampled-data adaptive state observation is presently dealt with using a hybrid adaptive observer including two parameter adaptive laws that allows a separate estimation of the two unknown parameter vectors. Following the observer design principle in (Karafyllis and Kravaris, 2009), each adaptive law involves a correction of the parameter estimate trajectory using the error between the estimated output (i.e. $c\hat{x}$) and a predicted output. The output predictor is an instrumental component of the observer. It is reinitialized at each sampling instant to get benefit of the output measurements. In the rest of the time, it imitates the (continuous-time) system in open loop (no correcting error term is added). The adaptation gains in the two parameter adaptive laws are also separately updated using two exponentially stable ODEs. The first ODE resembles to the one used in (Hann and Ahmed-Ali, 2012) as it is driven with the (perfectly known) model term multiplying the first parameter vector. The second ODE differs from the preceding one in that the driving input is not a given component of the model but a constructed signal. Specifically, the latter is a saturated version of the predicted output z. The saturation function is resorted to ensure the boundedness of the corresponding adaptation gain despite the fact that the predicted output is not a priori bounded. The hybrid adaptive observer thus obtained is formally shown, under ad hoc sufficient conditions, to be exponentially convergent in the absence of output disturbance. The sufficient conditions include a usual persistent excitation condition as well as an explicit upper bound on the admissible sampling time. In presence of nonzero disturbance, the L_∞-gain between the disturbance input and the (state and parameter) estimation errors is explicitly expressed in function of the design parameters.

The paper is organized as follows: the class of systems dealt with is described along with the proposed observer in Section 2; the main theorem describing the observer performances is presented in Section 3 and its proof is placed in the Appendix; a conclusion and a reference list end the paper.

2. CLASS OF SYSTEMS AND ADAPTIVE OBSERVER

The system under study is described by the following observable canonical form:

$$\dot{x}(t) = A_0 x(t) + a x_1(t) + \psi(u(t))\theta \quad (1a)$$

$$y(t_k) = cx(t_k) + \xi(t_k) \quad (1b)$$

with,

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbf{R}^{n\times n}; \quad a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbf{R}^n \quad (2a)$$

$$\theta_1 \in \mathbf{R}^m; \quad \psi(u(t)) \in \mathbf{R}^{n\times m} \quad c = [1\ 0 \dots 0] \in \mathbf{R}^{1\times n} \quad (2b)$$

where u and y denote the system input and output, respectively; ξ is an external disturbance that is just supposed to be bounded. The state vector $x \in \mathbf{R}^n$ is not accessible to measurements and the parameter vectors a and θ_1 are unknown.

It is supposed that the dynamic part (1a) is L_∞-stable and its order n is known. The matrix function $\psi(.)$ is arbitrary but is continuous. The input u is any bounded and piecewise continuous signal. Then, it immediately follows that the state x and the output y are bounded. The output equation (1b) emphasizes the fact that the signal y is only accessible to measurements at the sampling instants t_k. Furthermore, it is readily seen from (2b) that, the state variable x_1 is related to the output by the relation $y = x_1 + \xi$. It turns out that, x_1 is only accessible to measurements up-to-noise and only at the sampling times. For this reason, it will be referred to the 'not-fully accessible' output. Now, as the state equation (1a) involves x_1, the system (1a-b) is called output injection type (e.g. Besançon et al., 2006).

The problem at hand consists in designing an adaptive observer that provides accurate online estimates of the state $x(t)$ and the parameter vectors a and θ_1. The state and the parameter estimation must only rely on the prior knowledge of the control input $u(t)$ and the sampled output measurements $y(t_k)$.

Sampled-data adaptive observers for state-affine systems have been proposed in (Hann and Ahmed-Ali, 2012, Ahmed-Ali *et al.*, 2009). However, the class of systems considered in those studies are not output injection type. Inversely, the class of systems considered in (Nadri *et al.*, 2004) is state-affine with output injection, but the sampled-data observer proposed there is not adaptive. A major feature of the present study is that the system under study (1a-b) is output injection type and the proposed observer is adaptive (i.e. the system is not subject to parametric uncertainty). Furthermore, in the state equation (1a), the undisturbed output x_1 comes in affected by the uncertain parameter a. This makes all previous observers (adaptive or not) useless in the present case. A major difficulty in the present study lies in the fact that the undisturbed output x_1 is not accessible to measurements between sampling times. Moreover, even at these times, no exact measurement of x_1 is possible because of the disturbance ξ in equation (1b).

The problem of estimating the state and parameters of the system (1a-b) is presently coped with using the following sampled-output adaptive observer:

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + \hat{a}(t)sat(z(t)) + \psi(u(t))\hat{\theta}_1(t)$$
$$- K\left(c\hat{x}(t) - z(t)\right) + \lambda(t)\dot{\hat{\theta}}(t) \quad (3a)$$

$$\dot{\hat{\theta}}(t) = -R(t)\lambda^T(t)c^T\left(c\hat{x}(t) - z(t)\right) \quad (3b)$$

$$\dot{z}(t) = cA_0\hat{x}(t) + c\hat{a}(t)sat(z(t)) + c\psi(u(t))\hat{\theta}_1(t)$$
$$\text{for } t \in [t_k, t_{k+1}) \quad (3c)$$

$$z(t_k) = y(t_k) \quad (3d)$$

$$\dot{\lambda}(t) = (A_0 - Kc)\lambda(t) + \left[sat(z(t))I_{n\times n}\ \psi(u(t))\right] \quad (3e)$$

$$\dot{R} = R - R\lambda^T c^T c\lambda R \quad (3f)$$

with:

$$\hat{\theta}(t) = \left[\hat{a}^T(t)\ \hat{\theta}_1^T(t)\right]^T,\ \hat{a}(t) \in \mathbf{R}^n, \hat{\theta}_1(t) \in \mathbf{R}^m,\ \lambda \in \mathbf{R}^{n\times(n+m)}$$

where $R(0) = I_{(n+m)\times(n+m)}$ (identity matrix) and all initial estimates ($\hat{a}(0), \hat{\theta}(0), \hat{x}(0)$ ...) may be arbitrarily chosen. In (13a-g), the observer gain $K \in \mathbf{R}^n$ is chosen so that the matrix $A_0 - Kc$ is Hurwitz . The time-varying adaptive gain λ, in the parameter adaptive laws (3b-c), is provided by equation (3e). The variable $z(t)$ represents, in view of (3d-e), a prediction of the output $y(t)$ over the interval (t_k, t_{k+1}), given the preceding output samples $y(t_k), y(t_{k-1})\ldots$

The saturation function in (3a) and (3c), is defined as follows,

$$sat(z) = \text{sgn}(z)\min(|z|, x_{1M}) \quad (4)$$

where $x_{1M} > 0$ is any upper bound of the undisturbed output i.e.

$$x_{1M} \geq \sup_{0\leq t<\infty} |x_1(t)| \quad (5)$$

The sampled-output adaptive observer (3a-f) features a separate estimation of the two unknown parameter vectors. The parameter vector θ is estimated using the parameter adaptive law defined by equations (3b) and (3f). The latter generates the adaptation gain $\lambda_1(t)$ from $\psi(u(t))$ which is a 'given' component of the model. Note that this adaptation gain law resembles to those involved in the adaptive observers of (Hann and Ahmed-Ali, 2012, Ahmed-Ali *et al.*, 2009).

The parameter vector a (which comes in the model (1a) multiplied by the not-completely measurable signal x_1) is estimated by the parameter adaptive law (3b). The latter generates the adaptation gain $\lambda_0(t)$ from the 'constructed' signal $sat(z(t))$. This is coherent with the fact that $z(t)$ is an estimate of x_1. The saturation of $z(t)$ is resorted because, at this stage, this signal is not yet proved to be bounded. Without using that saturation, it will not be possible to ensure the boundedness of the gain λ. Note that no parameter adaptive law similar to (3b) has been used in previous works (Hann and Ahmed-Ali, 2012, Ahmed-Ali *et al.*, 2009).

Now, to practically implement the saturation function $sat(z(t))$, a suitable value of x_{1M} will be supposed to be available. Based on these observations, it turns out that real scalars, say $\psi_M > 0$ and $\lambda_M > 0$, can be a priori determined using (3e), such that:

$$\sup_t |\psi(u(t))| \leq \psi_M\ ,\ \sup_t |\lambda(t)| \leq \lambda_M \quad (6)$$

3. ADAPTIVE OBSERVER ANALYSIS

The adaptive observer defined by equations (3a-f) will now be analyzed. As pointed out earlier, a major difficulty in the analysis is to cope with the term ax_1 (in equation (1a)) which involves an uncertain parameter multiplied by a signal that is fully and exactly accessible to measurements. Another difficulty lies in the hybrid (continuous-discrete) nature of the subsystem (3c-d) of the observer. That subsystem defines an output predictor between two successive sampling periods.

To formally analyze observer (3a-f), introduce the following errors:

$$\tilde{x} = \hat{x} - x\ , e = z - x_1, \quad (7a)$$

$$\tilde{\theta} = \hat{\theta} - \theta\ , \theta = \left[a^T\ \theta_1^T\right]^T,\ \tilde{\theta} = \left[\tilde{a}^T\ \tilde{\theta}_1^T\right]^T, \quad (7b)$$

Also, introduce the change of coordinates,

$$\eta = \tilde{x} - \lambda\tilde{\theta} \quad (8)$$

Then, the observer equations (3a-f) can be rewritten as follows, in terms of the errors (7-8):

$$\dot{\eta} = (A_0 - Kc)\eta + Ke + a(sat(z) - x_1) \quad (9a)$$

$$\dot{\tilde{\theta}} = -R\left(\lambda^T c^T c\lambda\ \tilde{\theta} - \lambda^T c^T c\eta - \lambda^T c^T e\right) \quad (9b)$$

$$\dot{\lambda}(t) = (A_0 - Kc)\lambda + \left[sat(z(t))I_{n\times n}\ \psi(u(t))\right] \quad (9c)$$

$$\dot{e} = cA_0\tilde{x}(t) + c\psi(u)\tilde{\theta} + c\tilde{a}sat(z) - ca(x_1 - sat(z))$$
$$t \in [t_k, t_{k+1}) \quad (9d)$$

$$e(t_k) = \xi(t_k) \quad (9e)$$

The analysis of the above error system takes benefit from the fact, shown in many places (see e.g. Besançon *et al.*, 2006; Zhang, 2002), that the time-varying matrix R (i.e. the solution of (3f)) does exist and is symmetric and positive definite, provided the following persistent excitation condition holds: $\exists\varepsilon > 0, \exists\delta > 0, \forall t > 0$:

$$\varepsilon_0 I_{(n+m)\times(n+m)} < \int_t^{t+\delta} \lambda^T(s)c^T c\lambda(s)ds < \varepsilon_1 I_{(n+m)\times(n+m)} \quad (10)$$

where $I_{(n+m)\times(n+m)}$ denotes the identity matrix of dimension $(n+m)\times(n+m)$. Under this condition, the matrix inverse R^{-1} in turn is bounded, symmetric and positive definite. More formally, there are two positive real numbers $(r, \bar{r})$, such that for all $s \geq 0$:

$$rI_{(n+m)\times(n+m)} \leq R^{-1} = (R^{-1})^T \leq \bar{r}I_{(n+m)\times(n+m)} \quad (11)$$

In the sequel, condition (10) is supposed to hold. Then, the following Lyapunov function is defined:

$$V = \tilde{\theta}^T R^{-1}\tilde{\theta} + \eta^T P\eta \tag{12}$$

where $P = P^T$ is any positive definite matrix satisfying the following inequality:

$$P(A_0 - Kc) + (A_0 - Kc)^T P \le -\mu I \tag{13}$$

and μ is any positive constant such that:

$$\mu > \frac{4\lambda_M^2}{\alpha} \tag{14}$$

where λ_M is as in (6) and α is any positive real scalar such that:

$$r \ge 2\alpha \tag{15}$$

where r is as in (11). The following additional notations will also prove to be useful:

$$\tau = \sup_k (t_k - t_{k-1}), \tag{16}$$

τ^* be the largest positive real number satisfying the two following inequalities:

$$\tau^* e^{\sigma\tau^*/2} < \frac{1}{|ca|} \quad \text{and} \quad \frac{\tau^* e^{\sigma\tau^*/2}}{1-|ca|\tau^* e^{\sigma\tau^*/2}} < \frac{1}{\gamma M_2} \tag{17a}$$

σ is any scalar such that:

$$0 < \sigma < \frac{\sigma_0}{2}, \quad M_1 = \sqrt{\frac{V(t_0)}{\sigma_2}} e^{\sigma_0 t_0/2} \tag{17b}$$

$$M_2 = \max\left(|cA_0|, |cA_0|\lambda_M, \psi_M, x_{1M}\right) \tag{17c}$$

$$\sigma_0 = \min\left\{\frac{1}{2}, \frac{1}{\lambda_{\max}(P)}\left(\frac{\mu}{2} - \frac{2\lambda_M^2}{\alpha}\right)\right\} \tag{17d}$$

$$\sigma_1 = \frac{4}{\mu}\|P\|^2(|K|^2 + |a|^2) + \frac{2\lambda_M}{\alpha}|c|^2 \tag{17e}$$

$$\sigma_2 = \inf\{\lambda_{\min}(P), r\} \tag{17f}$$

where $\lambda_{\max}(.)$ (resp. $\lambda_{\min}(.)$) denotes the maximum (resp. minimum) eigenvalue of a matrix. Note that τ^* in (17a) exists because the left sides of the two inequalities in (17) vanish as $\tau^* \to 0$.

Theorem 1. Let the sampled-output adaptive observer (3a-f) be applied to the system (1a-b). Then, one has the following properties:

1) For all $t \ge t_0$:

$$\sup_{t_0\le s\le t}\left(\left\|\begin{bmatrix}\tilde{\theta}(s)\\ \eta(s)\end{bmatrix}\right\| e^{\sigma s/2}\right) \le M_1 + \gamma \sup_{t_0\le s\le t}\left(e^{\sigma s/2}|e(s)|\right).$$

2) If $\tau < \tau^*$ then, for all $t \ge t_0$:

$$\left\|\begin{bmatrix}\tilde{\theta}(t)\\ \eta(t)\end{bmatrix}\right\| \le \frac{M_1(1-|ca|\tau e^{\sigma\tau/2})}{1-|ca|\tau e^{\sigma\tau/2} - \gamma M_2 \tau e^{\sigma\tau/2}} e^{-\sigma t/2}$$
$$+ \frac{\gamma}{1-|ca|\tau e^{\sigma\tau/2} - \gamma M_2\tau e^{\sigma\tau/2}} \sup_{t_0\le s\le t}|\xi(s)|.$$

In the particular case of undisturbed output (i.e. $\xi = 0$), $\left[\tilde{\theta}(t) \;\; \eta(t)\right]$ exponentially converges to the origin ■

The proof of this theorem is placed in the Appendix.

4. CONCLUDING REMARKS

This study has addressed the problem of adaptive state and parameter estimation for the state-affine systems (1a-b), using sampled output measurements in presence of noise. A key feature of the study is that even the term involving the true output x_1 in the state equation (1a) is affected by an uncertain parameter vector. Theorem 1 shows that, in the case of no output disturbance, the proposed adaptive observer, defined by (3a-f), guarantees the exponential convergence of all errors to the origin. This results goes beyond the existing results on sampled-data adaptive state observers.

REFERENCE

Ahmed-Ali T., R. Postoyan, F. Lamnabhi-Lagarrigue (2009). Continuous discrete adaptive observers for state affine systems. Automatica, vol. 45, pp. 2986-2990

Assoudi A, Yaagoubi E, Hammouri H. (2002). Non-linear observer based on the euler discretization. International Journal of Control, 75, pp.784–791.

Bastin G., Gevers M.R. (1988). Stable adaptive observers for nonlinear time-varying systems. IEEE Transactions on Automatic Control; vol. 33, pp. 650–658.

Besancon G. (2000). Remarks on nonlinear adaptive observer design. Systems and Control Letters; vol. 41 (4), pp. 271–280.

Besançon G., J. De León-Morales, O. Huerta-Guevara (2006). On adaptive observers for state affine systems. International Journal of Control, vol. 79 (6), pp. 581-591.

Deza F., E. Busvelle, J.P. Gauthier, D. Rakotopara (1992). High gain estimation for nonlinear systems. Systems & Control Letters, vol. 18 (4), pp. 295–299.

Hammouri H, P. Kabore, S. Othman, J. Biston (2002). Failure diagnosis and nonlinear observer. Application to a hydraulic process. Journal of the Franklin Institute, vol. 339 (4-5), pp. 455–478.

Hann C.A.B. and T. Ahmed-Ali (2012). Continuous adaptive observer for state affine sampled-data systems. Int. J. Robust. Nonlinear Control.

Karafyllis I, C. Kravaris (2009). From continuous-time design to sampled-data design of observers. IEEE Transactions on Automatic Control, vol. 54 (9), pp. 2169–2174.

Laila D.S., Astolfi A. (2006) Sampled-data observer design for a class of nonlinear systems with applications. 17th International Symposium on Mathematical theory of Networks and Systems, Kyoto, Japan.

Marino R, Tomei P. (1996). Nonlinear control design: geometric, adaptive and robust. Prentice Hall, UK.

Nadri M, Hammouri H. (2003). Design of a continuous-discrete observer for state affine systems. Applied Mathematics Letters, vol. 16 (6), pp. 967–974.

Nadri M., H. Hammouri, C. Astorga (2004). Observer design for continuous-discrete time state affine systems up to output injection. European Journal of Control 2004, vol. 10, pp. 252–263.

Raff T, Kögel M, Allgöwer F. Observer with sample-and-hold updating for lipschitz nonlinear systems with nonuniformly sampled measurements. American Control Conference, Seattle, Washington, USA, 2008.

Zhang Q. (2002). Adaptive observer for multiple-input-multiple-output (mimo) linear time-varying systems. IEEE Transactions on Automatic Control, vol. 47 (3), pp. 525–529.

APPENDIX. PROOF OF THEOREM 1

Part 1. Time-derivation of V gives, using (12) and (9a-e):

$$\begin{aligned}\dot{V} &= \tilde{\theta}^T \dot{R}^{-1}\tilde{\theta} + 2\tilde{\theta}^T R^{-1}\dot{\tilde{\theta}} + 2\eta^T P\dot{\eta} \\ &= \tilde{\theta}^T\left(-R^{-1} + \lambda^T c^T c\lambda\right)\tilde{\theta} - 2\tilde{\theta}^T\left(\lambda^T c^T c\lambda\tilde{\theta} + \lambda^T c^T c\eta - \lambda^T c^T e\right) \\ &\qquad + 2\eta^T P\left((A_0 - Kc)\eta + Ke + \left(sat(z) - x_1\right)a\right) \\ &\le -\tilde{\theta}^T R^{-1}\tilde{\theta} + \tilde{\theta}^T\lambda^T c^T c\lambda\tilde{\theta} + \alpha\left|\tilde{\theta}\right|^2 + \frac{1}{\alpha}\left|\lambda^T c^T c\eta - \lambda^T c^T e\right|^2 \\ &\qquad - \frac{\mu}{2}|\eta|^2 + \frac{4}{\mu}\|P\|^2\left(|K|^2 + |a|^2\right)e^2 \end{aligned} \tag{A1}$$

with α being any positive constant satisfying (15). Using (11) and (15), inequality (A1) develops further as follows:

$$\begin{aligned}\dot{V} &\le -\frac{1}{2}\tilde{\theta}^T R^{-1}\tilde{\theta} - \left(\frac{\mu}{2} - \frac{2}{\alpha}\left|\lambda^T c^T c\right|^2\right)|\eta|^2 \\ &\quad + \left(\frac{4}{\mu}\|P\|^2\left(|K|^2 + |a|^2\right) + \frac{2}{\alpha}\left\|\lambda^T c^T\right\|^2\right)e^2 \end{aligned} \tag{B2}$$

It is readily checked that $\left|\lambda^T c^T c\right| \le |\lambda| \le \lambda_M$, where λ_M is as in (6). By (14), μ is selected such that $\frac{\mu}{2} > \frac{2\lambda_M^2}{\alpha}$. Then one has, for all $s \ge 0$:

$$\frac{\mu}{2} - \frac{2}{\alpha}\left|\lambda^T(s)c^T c\right|^2 > \frac{\mu}{2} - \frac{2\lambda_M^2}{\alpha} > 0 .$$

Then, it follows from (B2) that:

$$\dot{V} \le -\sigma_0 V + \sigma_1 e^2 \tag{B3}$$

where σ_0 and σ_1 are defined by (17d-e). Integrating (B3), one gets for all $0 \le t_0 < t$:

$$V(t) \le e^{\sigma_0(t-t_0)}V(t_0) + \sigma_1\int_{t_0}^t e^{-\sigma_0(t-s)}e^2(s)ds$$

Given any scalar σ such that $0 < \sigma < \sigma_0/2$, it follows multiplying both sides of (B3) by $e^{\sigma t}$:

$$\begin{aligned} e^{\sigma t}V(t) &\le e^{-(\sigma_0-\sigma)t}e^{\sigma_0 t_0}V(t_0) + \sigma_1 e^{\sigma t}\int_{t_0}^t e^{-\sigma_0(t-s)}e^2(s)ds \\ &\le M_0 + \sigma_1 e^{\sigma_0 t}\int_{t_0}^t e^{-\sigma_0(t-s)}e^2(s)ds \end{aligned} \tag{B4}$$

with $M_0 = e^{\sigma_0 t_0}V(t_0)$, using the fact that $e^{-(\sigma_0-\sigma)t} < 1$. Inequality (B4) implies successively:

$$\begin{aligned} e^{\sigma t}V(t) &\le M_0 + \sigma_1 e^{\sigma t}\int_{t_0}^t e^{-\sigma_0(t-s)}e^{-\sigma s}e^{\sigma s}e^2(s)ds \\ &\le M_0 + \frac{2\sigma_1}{\sigma_0}\sup_{t_0\le s\le t} e^{\sigma s}e^2(s) \end{aligned}$$

Taking root-squares of both sides of the last inequality gives:

$$e^{\sigma t/2}\sqrt{V(t)} \le \sqrt{M_0} + \sqrt{\frac{2\sigma_1}{\sigma_0}}\sup_{t_0\le s\le t}\left(e^{\sigma s/2}|e(s)|\right) \tag{B5}$$

On the other hand, it readily follows from (12):

$$V = \tilde{\theta}^T R^{-1}(t)\tilde{\theta} + \eta^T P\eta \ge \sigma_2\left(\left|\tilde{\theta}\right|^2 + |\eta|^2\right)$$

where $\sigma_2 > 0$ is as in (17e). Taking the root square of both sides of the above inequality, one immediately obtains $\sqrt{V} \ge \sqrt{\sigma_2}\left\|\left[\tilde{\theta} \quad \eta\right]\right\|$ which, together with (A5), yields:

$$\left\|\left[\tilde{\theta} \quad \eta\right]\right\|e^{\sigma t/2} \le \sqrt{\frac{M_0}{\sigma_2}} + \sqrt{\frac{2\sigma_1}{\sigma_0\sigma_2}}\sup_{t_0\le s\le t}\left(e^{\sigma s/2}|e(s)|\right)$$

Clearly, the right side of this inequality is increasing. Then, it follows that:

$$\sup_{t_0\le s\le t}\left(\left\|\left[\tilde{\theta} \quad \eta\right]\right\|e^{\sigma s/2}\right) \le \sqrt{\frac{M_0}{\sigma_2}} + \sqrt{\frac{2\sigma_1}{\sigma_0\sigma_2}}\sup_{t_0\le s\le t}\left(e^{\sigma s/2}|e(s)|\right) \tag{A6}$$

Part 2. To establish Part 2 of Theorem 1, it follows integrating (9f-g) that, for all $t \in [t_k, t_{k+1})$:

$$\begin{aligned} e(t) &= \xi(t_k) \\ &+ \int_{t_k}^t cA_0\tilde{x}(s) + c\psi(u(s))\tilde{\theta}_1(s) + sat(z(s))c\tilde{a}(s)ds \\ &\qquad + \int_{t_k}^t \left(sat(z(s)) - x_1(s)\right)cads \end{aligned} \tag{A7}$$

By (8), $\eta + \lambda\tilde{\theta}$ can be substituted to $\tilde{x}$ on the right side of (B7). Then, taking absolute value of both sides of (A7) and multiplying the obtained inequality by $e^{\sigma t/2}$ one gets, for all $k \in \mathbf{N}$ and all $t \in [t_k, t_{k+1})$:

$$\begin{aligned} e^{\sigma t/2}|e(t)| &\le e^{\sigma t/2}|\xi(t_k)| \\ &+ e^{\sigma t/2}\int_{t_k}^t \left|cA_0\left(\eta(s) + \lambda(s)\tilde{\theta}(s)\right) + c\psi(u(s))\tilde{\theta}_1(s) + sat(z(s))c\tilde{a}(s)ds\right| \\ &\quad + e^{\sigma t/2}|ca|\int_{t_k}^t |x_1(s) - sat(z(s))|ds \\ &\le e^{\sigma t/2}|\xi(t_k)| + e^{\sigma t/2}M_2\int_{t_k}^t \left\|\left[\tilde{\theta}(s) \quad \eta(s)\right]\right\|ds \\ &\quad + e^{\sigma t/2}|ca|\psi_M^*\int_{t_k}^t |e(s)|ds \end{aligned} \tag{A8}$$

with $M_2 = \max\left(|cA_0|, |cA_0|\lambda_M, \psi_M, x_{1M}\right)$ where we have used (6) and the fact (already pointed out in Part 1 of this proof) that $|sat(z) - x_1| \le |e|$. Inequality (A8) develops further as follows:

$$e^{\sigma t/2}|e(t)| \le e^{\sigma t/2}|\xi(t_k)|$$

$$+e^{\sigma t/2}M_2\left(\sup_{t_k\le s\le t} e^{\sigma s/2}\left\|\left[\tilde{\theta}(s)\ \eta(s)\right]\right\|\right)\int_{t_k}^{t} e^{-\sigma s/2}ds$$

$$+e^{\sigma t/2}|ca|\left(\sup_{t_k\le s\le t} e^{\sigma s/2}|e(s)|\right)\int_{t_k}^{t} e^{-\sigma s/2}ds \qquad \text{(A9)}$$

It is readily checked that $0<\int_{t_k}^{t} e^{-\sigma s/2}ds \le \tau e^{-\sigma t_k/2}$. Then, one gets from (A9):

$$e^{\sigma t/2}|e(t)| \le e^{\sigma t/2}|\xi(t_k)| + M_2\tau e^{\sigma\tau/2}\left(\sup_{t_0\le s\le t} e^{\sigma s/2}\left\|\left[\tilde{\theta}(s)\quad \eta(s)\right]\right\|\right)$$

$$+|ca|\tau e^{\sigma\tau/2}\left(\sup_{t_0\le s\le t} e^{\sigma s/2}|e(s)|\right) \qquad \text{(A10)}$$

where we have used the inequalities $0<t-t_k<\tau$, with $\tau=\sup_k(t_k-t_{k-1})$, and the fact that t_k is an increasing sequence. It is readily seen that, the right side of inequality (A10) is an increasing function of t. Then, it follows that:

$$\sup_{t_0\le s\le t}\left(e^{\sigma s/2}|e(s)|\right) \le e^{\sigma t/2}\sup_{t_0\le s\le t}|\xi(s)|$$

$$+M_2\tau e^{\sigma\tau/2}\sup_{t_0\le s\le t}\left(e^{\sigma s/2}\left\|\left[\tilde{\theta}(s)\quad \eta(s)\right]\right\|\right)$$

$$+|ca|\tau e^{\sigma\tau/2}\sup_{t_0\le s\le t}\left(e^{\sigma s/2}|e(s)|\right) \qquad \text{(A11)}$$

Now, by letting τ be sufficiently small so that:

$$|ca|\tau e^{\sigma\tau/2}<1 \qquad \text{(A12)}$$

one gets from (A11):

$$\sup_{t_0\le s\le t}\left(e^{\sigma t/2}|e(t)|\right) \le \frac{e^{\sigma t/2}}{1-|ca|\tau e^{\sigma\tau/2}}\sup_{t_0\le s\le t}|\xi(s)|$$

$$+M_2\frac{\tau e^{\sigma\tau/2}}{1-|ca|\tau e^{\sigma\tau/2}}\sup_{t_0\le s\le t}\left(e^{\sigma s/2}\left\|\left[\tilde{\theta}(s)\ \ \eta(s)\right]\right\|\right) \qquad \text{(A13)}$$

Substituting the right side of (A13) to $\sup_{t_0\le s\le t}\left(e^{\sigma s/2}|e(s)|\right)$ in (A6) one gets:

$$\sup_{t_0\le s\le t}\left(\left\|\left[\tilde{\theta}(s)\ \eta(s)\right]\right\|e^{\sigma s/2}\right) \le M_1 + \frac{\gamma\, e^{\sigma t/2}}{1-|ca|\tau e^{\sigma\tau/2}}\sup_{t_0\le s\le t}|\xi(s)|$$

$$+\frac{\gamma M_2\tau e^{\sigma\tau/2}}{1-|ca|\tau e^{\sigma\tau/2}}\sup_{t_0\le s\le t}\left(e^{\sigma s/2}\left\|\left[\tilde{\theta}(s)\quad \eta(s)\right]\right\|\right) \qquad \text{(A13)}$$

Let τ be sufficiently small so that, in addition to (A21), the following inequality holds:

$$\gamma M_2\frac{\tau e^{\sigma\tau/2}}{1-|ca|\tau e^{\sigma\tau/2}}<1 \qquad \text{(A14)}$$

Then, (A13) yields:

$$\sup_{t_0\le s\le t}\left(\left\|\left[\tilde{\theta}(s)\quad \eta(s)\right]\right\|e^{\sigma s/2}\right) \le \frac{M_1(1-|ca|\tau e^{\sigma\tau/2})}{1-|ca|\tau e^{\sigma\tau/2}-\gamma M_2\tau e^{\sigma\tau/2}}$$

$$+\frac{\gamma e^{\sigma t/2}}{1-|ca|\tau e^{\sigma\tau/2}-\gamma M_2\tau e^{\sigma\tau/2}}\sup_{t_0\le s\le t}|\xi(s)|$$

This particularly gives:

$$\left\|\left[\tilde{\theta}(t)\quad \eta(t)\right]\right\| \le \frac{M_1(1-|ca|\tau e^{\sigma\tau/2})}{1-|ca|\tau e^{\sigma\tau/2}-\gamma M_2\tau e^{\sigma\tau/2}}e^{-\sigma t/2}$$

$$+\frac{\gamma}{1-|ca|\tau e^{\sigma\tau/2}-\gamma M_2\tau e^{\sigma\tau/2}}\sup_{t_0\le s\le t}|\xi(s)|$$

which establishes Part 2 and completes the proof of Proposition 2 ■