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Hidden attractors in dynamical systems: systems with no equilibria, multistability and coexisting attractors

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Abstract: From a computational point of view it is natural to suggest the classification of attractors, based on the simplicity of finding basin of attraction in the phase space: *an attractor is called a hidden attractor if its basin of attraction does not intersect with small neighborhoods of equilibria, otherwise it is called a self-excited attractor*. Self-excited attractors can be localized numerically by the *standard computational procedure*, in which after a transient process a trajectory, started from a point of unstable manifold in a neighborhood of unstable equilibrium, is attracted to the state of oscillation and traces it. Thus one can determine unstable equilibria and check the existence of self-excited attractors. In contrast, for the numerical localization of hidden attractors it is necessary to develop special analytical-numerical procedures in which an initial point can be chosen from the basin of attraction analytically. For example, hidden attractors are attractors in the systems with no-equilibria or with the only stable equilibrium (a special case of multistability and coexistence of attractors); hidden attractors arise in the study of well-known fundamental problems such as 16th Hilbert problem, Aizerman and Kalman conjectures, and in applied research of Chua circuits, phase-locked loop based circuits, aircraft control systems, and others.

Keywords: hidden oscillation, hidden attractor, systems with no equilibria, coexisting attractors, coexistence of attractors, multistable systems, multistability, 16th Hilbert problem, nested limit cycles, Aizerman conjecture, Kalman conjecture, absolute stability, nonlinear control system, describing function method, harmonic balance, phase-locked loop (PLL), drilling system, aircraft control systems, Chua circuits

1. INTRODUCTION

An oscillation in dynamical system can be easily localized numerically if initial conditions from its open neighborhood lead to long-time behavior that approaches the oscillation. Such an oscillation (or a set of oscillations) is called an attractor and its attracting set is called the basin of attraction [1]. Thus, from a computational point of view it is natural to suggest the following classification of attractors, based on the simplicity of finding basin of attraction in the phase space:

Definition. *An attractor is called a hidden attractor if its basin of attraction does not intersect with small neighborhoods of equilibria, otherwise it is called a self-excited attractor.*

For a *self-excited attractor* its basin of attraction is connected with an unstable equilibrium and, therefore, self-excited attractors can be localized numerically by the *standard computational procedure*, in which after a transient process a trajectory, started from a point of unstable manifold in a neighborhood of unstable equilibrium, is attracted to the state of oscillation and traces it. Thus self-excited attractors can be easily visualized.

In contrast, for a hidden attractor its basin of attraction is not connected with unstable equilibria. For example, hidden attractors are attractors in the systems with no-equilibria or with the only stable equilibrium (a special case of multistable systems and coexistence of attractors). Multistability is often undesired situation in many applications, however coexisting self-excited attractors can be found by the standard computational procedure. In contrast, there is no regular way to predict the existence or coexistence of hidden attractors in a system. Note that one cannot guarantee the localization of an attractor by the integration of trajectories with random initial conditions (especially for multidimensional systems) since its basin of attraction can be very small.

For numerical localization of hidden attractors it is necessary to develop special analytical-numerical procedures in which an initial point can be chosen from the basin of attraction analytically since there are no similar transient processes leading to such attractors from the neighborhoods of equilibria.

[1] Rigorous definitions of attractors can be found, e.g., in (Broer et al., 1991; Boichenko et al., 2005; Milnor, 2006; Leonov, 2008)

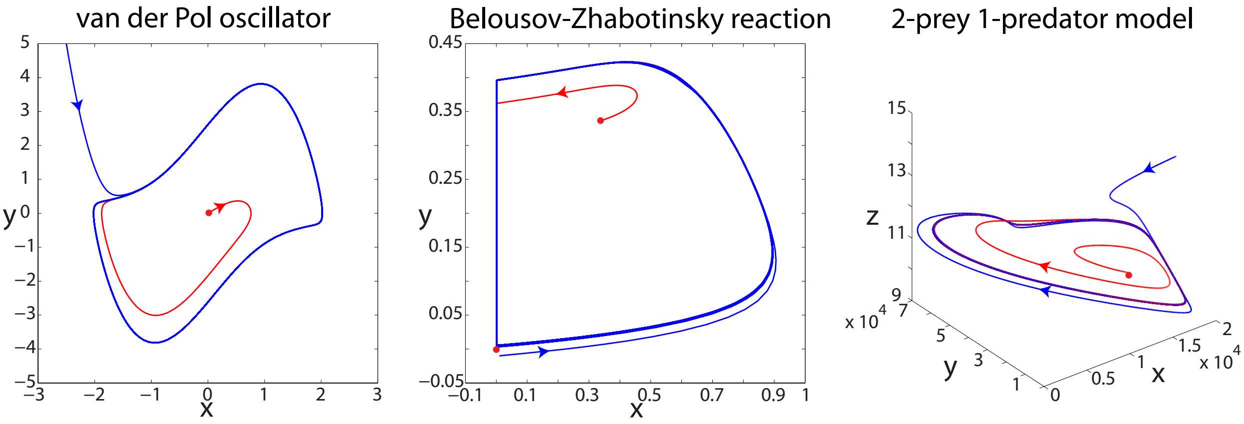

Fig. 1. Standard computation of classical self-excited periodic oscillations.

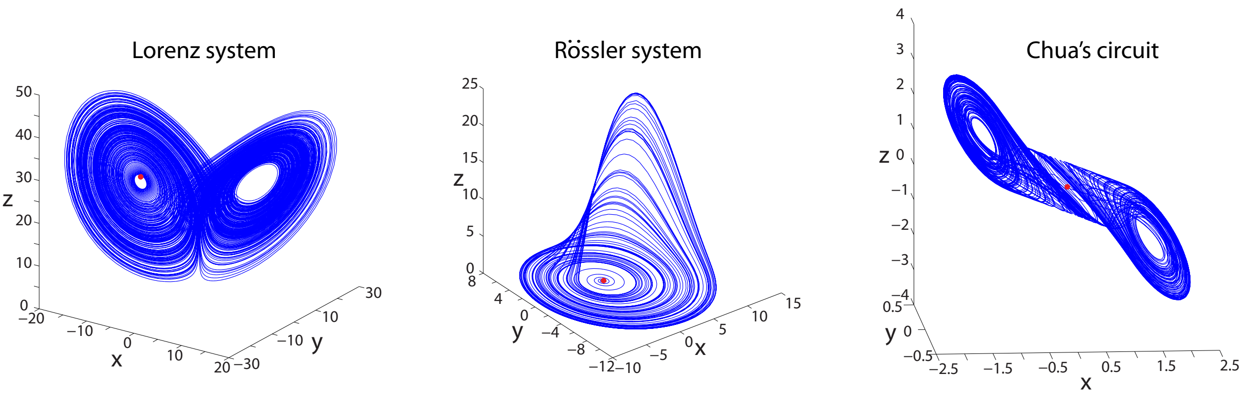

Fig. 2. Standard computation of classical self-excited chaotic attractors.

2. SELF-EXCITED ATTRACTOR LOCALIZATION

In the first half of the last century during the initial period of the development of the theory of nonlinear oscillations (Timoshenko, 1928; Krylov, 1936; Andronov et al., 1966; Stoker, 1950) much attention was given to analysis and synthesis of oscillating systems, for which the problem of the existence of oscillations can be studied with relative ease.

These investigations were encouraged by the applied research of periodic oscillations in mechanics, electronics, chemistry, biology and so on (see, e.g., (Strogatz, 1994)) The structure of many applied systems (see, e.g., Rayleigh (Rayleigh, 1877), Duffing (Duffing, 1918), van der Pol (van der Pol, 1926), Tricomi (Tricomi, 1933), Beluosov-Zhabotinsky (Belousov, 1959) systems) was such that the existence of oscillations was almost obvious since the oscillations were excited from unstable equilibria (*self-excited oscillations*). This allowed one, following Poincare's advice "*to construct the curves defined by differential equations*" (Poincaré, 1881), to visualize periodic oscillations by the standard computational procedure.

Then in the middle of 20th century it was found numerically the existence of chaotic oscillations (Ueda et al., 1973; Lorenz, 1963), which were also excited from unstable equilibria and could be computed by the standard computational procedure. Later many other famous self-excited attractors (Rossler, 1976; Chua et al., 1986; Sprott, 1994; Chen and Ueta, 1999; Lu and Chen, 2002) were discovered. Nowadays there is enormous number of publications devoted to the computation and analysis of various self-exited chaotic oscillations (see, e.g., recent publications (Awrejcewicz et al., 2012; Tuwankotta et al., 2013; Zelinka et al., 2013; Zhang et al., 2014) and many others).

In Fig. 1 is shown the computation of classical self-exited oscillations: van der Pol oscillator (van der Pol, 1926), one of the modifications of Belousov-Zhabotinsky reaction (Belousov, 1959), two preys and one predator model (Fujii, 1977).

In Fig. 2 is shown the computation of classical self-excited chaotic attractors: Lorenz system (Lorenz, 1963), Rössler system (Rossler, 1976), "double-scroll" attractor in Chua's circuit (Bilotta and Pantano, 2008).

3. HIDDEN ATTRACTOR LOCALIZATION

While the classical attractors are self-excited attractors and, therefore, they can be obtained numerically by the standard computational procedure, for the localization of hidden attractors it is necessary to develop special procedures since there are no similar transient processes leading to such attractors. At first the problem of investigating hidden oscillations arose in the second part of Hilbert's 16th problem (1900) on the number and possible dispositions of limit cycles in two-dimensional polynomial systems (see, e.g., (Reyn, 1994; Anosov et al., 1997; Chavarriga and Grau, 2003; Li, 2003; Dumortier et al., 2006; Lynch, 2010) and authors' works (Leonov and Kuznetsov, 2007a;

Kuznetsov and Leonov, 2008; Leonov et al., 2008; Leonov and Kuznetsov, 2010; Leonov et al., 2011a; Kuznetsov et al., 2013b; Leonov and Kuznetsov, 2013b)). The problem is still far from being resolved even for a simple class of quadratic systems. Here one of the main difficulties in computation is nested limit cycles — hidden oscillations.

Later, the problem of analyzing hidden oscillations arose in engineering problems. In 1956s in M. Kapranov's work (Kapranov, 1956) on the stability of PLL model, the qualitative behavior of this model was studied and the estimation of stability domain was obtained. In these investigations Kapranov assumed that in PLL systems there were self-excited oscillations only. However, in 1961, N. Gubar' revealed a gap in Kapranov's work and the possibility of the existence of hidden oscillations in two-dimensional PLL models was showed analytically (Gubar', 1961; Leonov and Kuznetsov, 2013b).

In 1957 R.E. Kalman stated the following (Kalman, 1957): "*If $f(e)$ in Fig. 1* [see Fig. 3] *is replaced by constants K corresponding to all possible values of $f'(e)$, and it is found that the closed-loop system is stable for all such K, then it intuitively clear that the system must be monostable; i.e., all transient solutions will converge to a unique, stable critical point.*" Kalman's conjecture is the strengthening

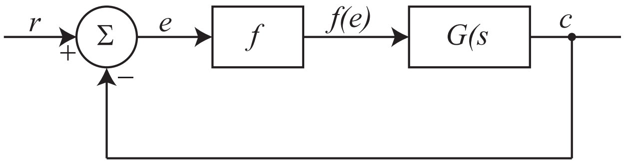

Fig. 3. Nonlinear control system. $G(s)$ is a linear transfer function, $f(e)$ is a single-valued, continuous, and differentiable function (Kalman, 1957)

of Aizerman's conjecture (Aizerman, 1949), in which in place of condition on the derivative of nonlinearity it is required that the nonlinearity itself belongs to linear sector. Here the application of widely-known describing function method (DFM)[2] leads to the conclusion on the absence of oscillations and the global stability of the only stationary point, what explains why these conjectures were put forward.

In the last century the investigations of Aizerman's and Kalman's conjectures on absolute stability have led to the finding of hidden oscillations in automatic control systems with a unique stable stationary point and with a nonlinearity, which belongs to the sector of linear stability (see, e.g., (Leonov et al., 2010a; Bragin et al., 2010; Leonov and Kuznetsov, 2011a; Kuznetsov et al., 2011b; Leonov and Kuznetsov, 2011b, 2013b) and surveys (Leonov et al., 2010b; Bragin et al., 2011; Leonov and Kuznetsov, 2013b)).

Discussions and recent developments on existence of periodic solution and absolute stability theory, related to Aizerman and Kalman conjectures, are presented, e.g., in (Kaszkurewicz and Bhaya, 2000; Lozano, 2000; Vidyasagar, 2002; Ackermann and Blue, 2002; Rasvan, 2004; Leigh, 2004; Gil, 2005; Margaliota and Yfoulis, 2006; Brogliato et al., 2006; Michel et al., 2008; Liao and Yu, 2008; Wang et al., 2009; Llibre et al., 2011; Haddad and Chellaboina, 2011; Grabowski, 2011; Alli-Oke et al., 2012; Nikravesh, 2013). The generalization of Kalman's conjecture to multidimensional nonlinearity is known as Markus-Yamabe conjecture (Markus and Yamabe, 1960) (also proved to be false (Bernat and Llibre, 1996; Cima et al., 1997; Leonov and Kuznetsov, 2013b)).

At the end of the last century the difficulties of numerical analysis of hidden oscillations arose in simulations of aircraft's control systems (anti-windup) (Lauvdal et al., 1997; Leonov et al., 2012a; Andrievsky et al., 2012, 2013a,b) and drilling systems (de Bruin et al., 2009; Kiseleva et al., 2012; Leonov and Kuznetsov, 2013b; Kiseleva et al., 2014; Leonov et al., 2014), which caused crashes.

Further investigations on hidden oscillations were greatly encouraged by the present authors' discovery (Kuznetsov et al., 2010; Leonov et al., 2010c; Kuznetsov et al., 2011c,b; Leonov et al., 2011c,b; Bragin et al., 2011; Leonov and Kuznetsov, 2012; Kuznetsov et al., 2013a; Leonov and Kuznetsov, 2013a), in 2009-2010 (for the first time), of *chaotic hidden attractor* in Chua's circuits — a simple electronic circuit with nonlinear feedback, which can be described by the following equations

$$\begin{aligned} \dot{x} &= \alpha(y - x - \psi(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -(\beta y + \gamma z) \\ \psi(x) &= m_1 x + (m_0 - m_1)\text{sat}(x) \end{aligned} \tag{1}$$

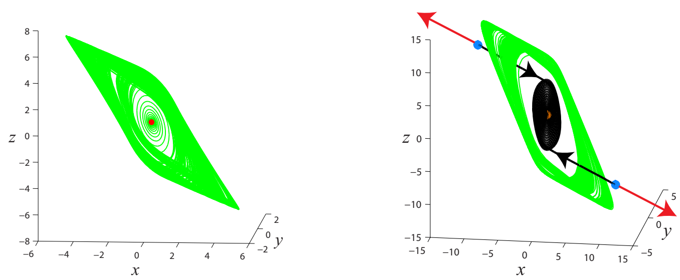

Fig. 4. a. Self-excited and b. Hidden Chua attractor with similar shapes

Until recently only self-excited attractors have been found in Chua circuits (see, e.g., Fig. 4 a.). Note that L. Chua himself, in analyzing various cases of attractors existence in Chua's circuit (Chua, 1992), does not admit the existence of hidden attractor (being discovered later) in his circuits. Now it is shown that Chua's circuit and its various modifications (Kuznetsov et al., 2010, 2011a; Leonov et al., 2011c, 2012b; Kuznetsov et al., 2013a) can exhibit hidden chaotic attractors (see, Fig. 4 b., Fig. 5[3]) with positive largest Lyapunov exponent (Leonov and Kuznetsov, 2007b)[4].

[2] In engineering practice for the analysis of the existence of periodic solutions it is widely used classical harmonic linearization and describing function method (DFM). However this classical method is not strictly mathematically justified and can lead to untrue results

[3] Applying more accurate analytical-numerical methods (Lozi and Ushiki, 1993), one might here distinguish a few close coexisting attractors.

[4] While there are known the effects of the largest Lyapunov exponent (LE) sign reversal for nonregular time-varying linearizations, the computation of Lyapunov exponents for linearization of nonlinear autonomous system along non stationary trajectories is widely used

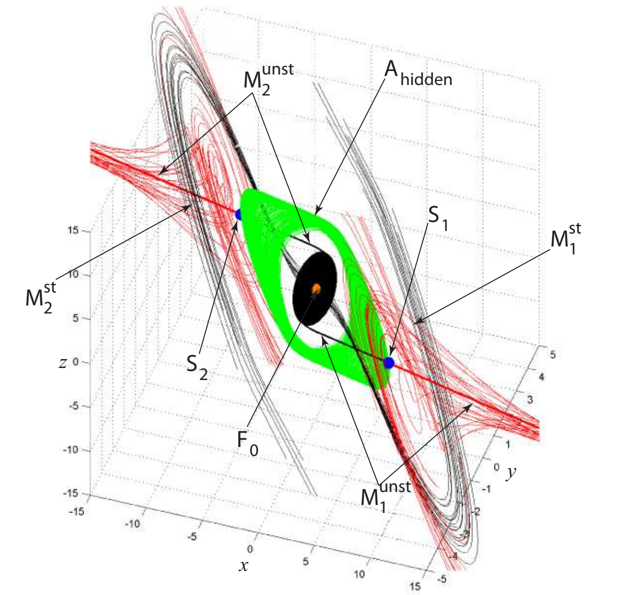

Fig. 5. Hidden chaotic attractor ($\mathrm{A_{hidden}}$ — green domain) in classical Chua circuit: locally stable zero equilibrium F_0 attracts trajectories (black) from stable manifolds $M^{st}_{1,2}$ of two saddle points $S_{1,2}$; trajectories (red) from unstable manifolds $M^{unst}_{1,2}$ tend to infinity; $\alpha = 8.4562$, $\beta = 12.0732$, $\gamma = 0.0052$, $m_0 = -0.1768$, $m_1 = -1.1468$.

Nowadays Chua circuits and other electronic generators of chaotic oscillations are widely used in various chaotic secure communication systems. The operation of such systems is based on the synchronization chaotic signals of two chaotic identical generators (transmitter and receiver) for different initial data (Kapitaniak, 1992; Ogorzalek, 1997; Yang, 2004; Eisencraft et al., 2013). The control signal, depending on the difference of signals of transmitter and receiver, changes the state of receiver. The existence of hidden oscillations and the improper choice of the form of control signal may lead to inoperability of such systems. For example, consider the synchronization of two Chua system (1), linearly coupled through the second equation by $K(y - \tilde{y})$, with initial data $x(0), y(0), z(0)$ in a small neighborhood of the zero equilibrium point for the first system and $\tilde{x}(0), \tilde{y}(0), \tilde{z}(0)$ on the attractor for the second system. Fig. 6 and Fig. 7 show that the synchronization is acquired for the classical double-scroll attractor ($\alpha = 9.3516$, $\beta = 14.7903$, $\gamma = 0.0161$, $m_0 = -1.1384$ $m_1 = -0.7225$) and may not be acquired for the hidden attractor.

Recent examples of hidden attractors can be found in (Jafari and Sprott, 2013; Molaie et al., 2013; Li and Sprott, 2014b,a; Lao et al., 2014; Li and Sprott, 2014a; Chaudhuri and Prasad, 2014; Wei et al., 2014; Li et al., 2014; Wei et al., 2014; Sprott, 2014), see also (Graham and Tl, 1986; Banerjee, 1997; Pisarchik et al., 2006; Feudel, 2008; Yang. et al., 2010; Wei, 2011; Wang and Chen, 2012b,a; Wei, 2012; Wei and Pehlivan, 2012; Liu et al., 2013; Abooee et al., 2013; Sprott et al., 2013; Galias and Tucker, 2013; Pisarchik, 2014).

for investigation of chaos (Leonov and Kuznetsov, 2007b; Kuznetsov and Leonov, 2005, 2001; Kuznetsov et al., 2014), where the positiveness of the largest Lyapunov exponent is often considered as indication of chaotic behavior in the considered nonlinear system.

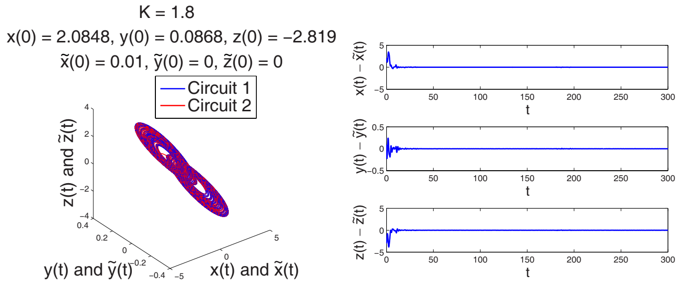

Fig. 6. Synchronization on self-excited attractor.

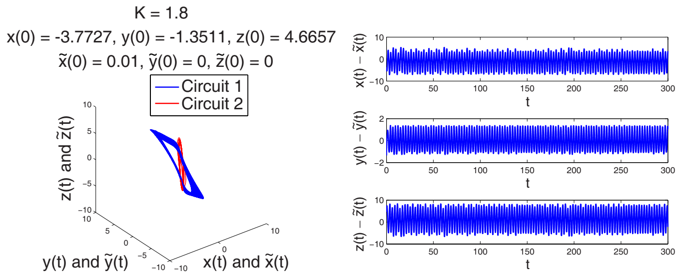

Fig. 7. No synchronization on hidden attractor.

4. ANALYTICAL-NUMERICAL PROCEDURE FOR HIDDEN OSCILLATIONS LOCALIZATION

An effective method for the numerical localization of hidden attractors in multidimensional dynamical systems is the method based on a *homotopy* and *numerical continuation*: it is necessary to constructed a sequence of similar systems such that for the first (starting) system the initial data for numerical computation of oscillating solution (starting oscillation) can be obtained analytically, e.g, it is often possible to consider the starting system with self-excited starting oscillation. Then the transformation of this starting oscillation is tracked numerically in passing from one system to another.

Further, it is demonstrated an example of effective analytical-numerical approach for hidden oscillations localization in the systems with scalar nonlinearity, which is based on the continuation principle, the method of small parameter and, a modification of the describing function method (DFM) [5].

Consider a system with one scalar nonlinearity

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{2}$$

Here $\mathbf{P}$ is a constant $(n \times n)$-matrix, $\mathbf{q}, \mathbf{r}$ are constant n-dimensional vectors, * is a transposition operation, $\psi(\sigma)$ is a continuous piecewise-differentiable scalar function, and $\psi(0) = 0$.

[5] DFM is used here for determining an initial periodic solution for continuation method. However, in many cases the initial periodic solution is a self-excited oscillation and may be obtained by the standard computational procedure.

Define a coefficient of harmonic linearization k (suppose that such k exists) in such a way that the matrix $\mathbf{P}_0 = \mathbf{P} + k\mathbf{q}\mathbf{r}^*$ has a pair of purely imaginary eigenvalues $\pm i\omega_0$ ($\omega_0 > 0$) and the rest eigenvalues have negative real parts . Rewrite system (2) as

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varphi(\mathbf{r}^*\mathbf{x}), \tag{3}$$

where $\varphi(\sigma) = \psi(\sigma) - k\sigma$.

Introduce a finite sequence of the functions

$$\varphi^0(\sigma), \varphi^1(\sigma), \ldots, \varphi^m(\sigma)$$

such that the graphs of neighboring functions $\varphi^j(\sigma)$ and $\varphi^{j+1}(\sigma)$ slightly differ from one another, the initial function $\varphi^0(\sigma)$ is small, and the final function $\varphi^m(\sigma) = \varphi(\sigma)$ (e.g., below it is considered $\varphi^j(\sigma) = \varepsilon^j\varphi(\sigma)$, $\varepsilon^j = j/m$). Since the initial function $\varphi^0(\sigma)$ is small over its domain of definition, the method of describing functions give mathematically correct conditions of the existence of a periodic solution for system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varphi^0(\mathbf{r}^*\mathbf{x}) \tag{4}$$

see, e.g. (Leonov, 2010; Leonov et al., 2010c,a).

Its application allows one to define a stable nontrivial periodic solution $\mathbf{x}^0(t)$ (a starting oscillating attractor is denoted further by $\mathcal{A}_0$).

Two alternatives are possible. The first case: all the points of $\mathcal{A}_0$ are in an attraction domain of the attractor $\mathcal{A}_1$, which is an oscillating attractor of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varphi^j(\mathbf{r}^*\mathbf{x}) \tag{5}$$

with $j = 1$. The second case: in the passing from system (4) to system (5) with $j = 1$ it is observed a loss of stability (bifurcation) and $\mathcal{A}_0$ vanishes. In the first case the solution $\mathbf{x}^1(t)$ can be found numerically by starting a trajectory of system (5) with $j = 1$ from the initial point $\mathbf{x}^0(0)$. If numerical integration over a sufficiently large time interval $[0, T]$ shows that the solution $\mathbf{x}^1(t)$ remains bounded and is not attracted by an equilibrium, then this solution reaches an attractor $\mathcal{A}_1$. In this case it is possible to proceed to system (5) with $j = 2$ and to perform a similar procedure of computation of $\mathcal{A}_2$ by starting a trajectory of system (5) with $j = 2$ from the initial point $\mathbf{x}^1(T)$ and computing a trajectory $\mathbf{x}^2(t)$.

Following this procedure, sequentially increasing j, and computing $\mathbf{x}^j(t)$ (a trajectory of system (5) with the initial data $\mathbf{x}^{j-1}(T)$), one can either find a solution around $\mathcal{A}_m$ (an attractor of system (5) with $j = m$, i.e. original system (3)), or observe at a certain step j a bifurcation, where the attractor vanishes.

Let $\varphi^0(\sigma) = \varepsilon\varphi(\sigma)$ with ε being a small parameter. To define the initial data $\mathbf{x}^0(0)$ of the initial periodic solution, system (4) with the nonlinearity $\varphi^0(\sigma)$ is transformed by a linear nonsingular transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$ to the form [6]

$$\begin{aligned} \dot{y}_1 &= -\omega_0 y_2 + b_1\varphi^0(y_1 + \mathbf{c}_3^*\mathbf{y}_3), \\ \dot{y}_2 &= \omega_0 y_1 + b_2\varphi^0(y_1 + \mathbf{c}_3^*\mathbf{y}_3), \\ \dot{\mathbf{y}}_3 &= \mathbf{A}_3\mathbf{y}_3 + \mathbf{b}_3\varphi^0(y_1 + \mathbf{c}_3^*\mathbf{y}_3). \end{aligned} \tag{6}$$

Here y_1, y_2 are scalars, $\mathbf{y}_3$ is $(n-2)$-dimensional vector; $\mathbf{b}_3$ and $\mathbf{c_3}$ are $(n-2)$-dimensional vectors, b_1 and b_2 are real numbers; $\mathbf{A}_3$ is an $((n-2) \times (n-2))$-matrix with all eigenvalues with negative real parts. Without loss of generality, it can be assumed that for the matrix $\mathbf{A}_3$ there exists a positive number $d > 0$ such that $\mathbf{y}_3^*(\mathbf{A}_3 + \mathbf{A}_3^*)\mathbf{y}_3 \leq -2d|\mathbf{y}_3|^2, \forall\, \mathbf{y}_3 \in \mathbb{R}^{n-2}$.

[6] Such a transformation exists for nondegenerate transfer functions.

Introduce the describing function

$$\Phi(a) = \int_0^{2\pi/\omega_0} \varphi\big(\cos(\omega_0 t)a\big)\cos(\omega_0 t)dt, \tag{7}$$

and assume the existence of its derivative.

Theorem 4.1. (Leonov, 2010; Bragin et al., 2011) If there is a positive a_0 such that

$$\Phi(a_0) = 0, \quad b_1 \frac{d\Phi(a)}{da}\bigg|_{a=a_0} < 0 \tag{8}$$

then there is a periodic solution

$$\mathbf{x}^0(0) = \mathbf{S}(y_1(0), y_2(0), \mathbf{y}_3(0))^*$$

with the initial data $y_1(0) = a_0 + O(\varepsilon)$, $y_2(0) = 0$, $\mathbf{y}_3(0) = \mathbf{O_{n-2}}(\varepsilon)$.

This theorem describes the procedure of the search for stable periodic solutions by the standard describing function method.

Note that condition (8) cannot be satisfied in the case when the conditions of Aizerman's and Kalman's conjectures are fulfilled (i.e. a nonlinearity belongs to the sector of linear stability). But it is possible to modify and justify the describing function method for the special nonlinearities satisfying the conditions of Aizerman's and Kalman's conjectures Leonov and Kuznetsov (2011a); Bragin et al. (2011).

5. HIDDEN OSCILLATIONS IN AIRCRAFT CONTROL SYSTEMS

The presence of control input saturation can dramatically degrade the closed-loop system performance. Since the feedback loop is broken when the actuator saturates, the unstable modes of the regulator may then drift to undesirable values. A terrifying illustration of this detrimental effect is given by the pilot-induced oscillations that entailed the YF-22 (Boeing) crash in 1992 and Gripen (SAAB) crash in 1993.

Consider the following equations (Andrievsky et al., 2013) of the aircraft short period angular motion along the longitudinal axis, linearized about trim operating points α_{trim}, $\delta_{e,\text{trim}}$, cf. (Reichert, 1992; Ferreres and Biannic, 2007; Biannic and Tarbouriech, 2009):

$$\begin{cases} \dot{\alpha} = Z_\alpha\alpha + q + Z_\delta\delta_e, \\ \dot{q} = M_\alpha\alpha + M_q q + M_\delta\delta_e, . \end{cases} \tag{9}$$

Here α, δ_e are the deviations of the angle-of-attack and the elevator deflection angle from a trim flying condition, the variable q represents the body axis pitch rate, M_α, M_q, M_δ, Z_α, Z_δ are the linearized model parameters for the given flight conditions. Let the elevator deflection angle be symmetrically bounded about trim value $\delta_{e,\text{trim}}$: $|\delta_e| \leq \bar{\delta}_e$. Suppose that the linear dynamics of the elevator servosystem between PID-controller output signal $u(t)$ and unsaturated elevator deflection $\tilde{\delta}_e(t)$ along with the

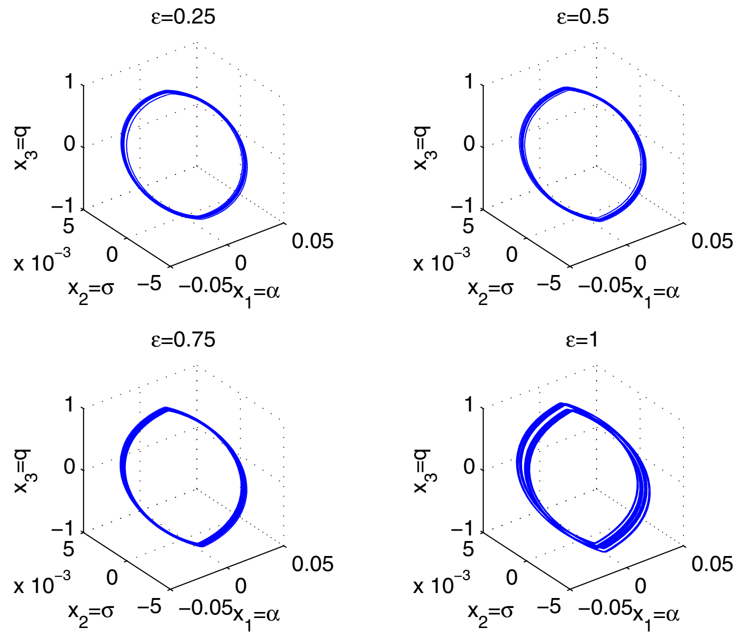

Fig. 8. Multistep localization of hidden oscillation: $\varepsilon^j = j/4$. At the last step (i.e. $j = m$ in (5)) there is stable zero equilibrium point coexist with stable oscillation (hidden oscillation).

compensating device model is described by the following transfer function

$$W(s) = \frac{\tilde{\delta}_e}{u} = \frac{k(T_2^2 s^2 + 2\xi_2 T_2 s + 1)}{T_1^2 s^2 + 2\xi_1 T_1 s + 1}, \qquad (10)$$

where k denotes the servo static gain, T_1, T_1 are the time constants and ξ_1, ξ_2 stand for the damping ratios. Finally, the saturated actuator output signal (the elevator deflection) is as follows

$$\delta_e(t) = \bar{\delta}_e \,\mathrm{sat}\left(\frac{\tilde{\delta}_e(t)}{\bar{\delta}_e}\right), \qquad (11)$$

where sat$(\cdot)$ denotes the saturation function.

In the sequel the following parameter values are taken (Barbu et al., 1999): $Z_\alpha = -1.0\ \mathrm{s}^{-1}$, $Z_\delta \approx 0$, $M_\alpha = 15\ \mathrm{s}^{-2}$, $M_q = 3.0\ \mathrm{s}^{-1}$, $M_\delta = -18\ s^{-2}$, $\bar{\delta}_e = 20$ deg (≈ 0.35 rad), $k = 10$, $T_1 = 0.083$ s, $T_2 = 0.057$ s, $\xi_1 = 0.1$, $\xi_2 = 0.4$.

It may be easily checked that for the given parameters the aircraft is weathercock unstable; the eigenvalues $s_{1,2}$ of system (9) are taken as $s_1 = -6$, $s_2 = 2$. Let the control goal be the tracking for the commanded angle-of-attack $\alpha^*(t)$. The following classical PID controller may be applied for eliminating the static tracking error and the achievement of the desired transient specification for the "nominal" (non-saturated) system:

$$u(t) = k_I \sigma_I(t) + k_P e(t) + k_D q(t),$$
$$\sigma_I(t) = \int_0^t e(\tau)\, d\tau, \quad \sigma(0) = 0, \qquad (12)$$

where $e(t) = \alpha(t) - \alpha^*(t)$ is a tracking error, k_P, k_q, k_I are proportional, derivative and integral controller gains (respectively).

The gains k_I, k_P, k_D are computed for the nominal (unsaturated) mode so as to place the nominal closed loop poles inside a truncated sector with a minimal degree of stability $\eta = 5.6$ (to ensure fulfillment of the settling time specification) and a minimal damping ratio $\xi = 0.3$ as follows: $k_P = 5.5$, $k_q = 0.55$ s, $k_I = 19\ \mathrm{s}^{-1}$. The H_∞ gain of the closed-loop system is found as $H_\infty = 1.13$ and the frequency stability margin is 75 deg.

Consider the closed-loop aircraft control system behavior imposed by saturation of the elevator servo. The focus of our attention is the possibility of hidden oscillations in the closed-loop system. Consider behavior analysis of the particular control system (9) – (12).

At the first step let us find matrices $\mathbf{P}$, $\mathbf{q}$, $\mathbf{r}$ in (2) for the considered case. After simple calculations one can find from (9), (10), (12) the following matrices:

$$\mathbf{P} = \begin{pmatrix} Z_\alpha & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ M_\alpha & 0 & M_q & 0 & 0 \\ -k_P & k_I & -k_q & -2\xi_1 T_1^{-1} & -T_1^{-2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} -Z_\delta \\ 0 \\ -M_\delta \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{r}^* = \Big(-kk_P T_2^2 T_1^{-2}, kk_I T_2^2 T_1^{-2}, -kk_q T_2^2 T_1^{-2},$$
$$k(2\xi_2 T_2 - 2\xi_1 T_2^2 T_1^{-1}) T_1^{-2}, k(1 - T_2^2 T_1^{-2}) T_1^{-2}\Big).$$

The nonlinearity $\psi(\cdot)$ in (2) has a form (11).

Numerically, for the given above parameter values, matrices $\mathbf{P}$, $\mathbf{q}$, $\mathbf{r}$ take the from:

$$\mathbf{P} = \begin{pmatrix} -1.0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 15.0 & 0 & -3.0 & 0 & 0 \\ -5.56 & 19.1 & -0.556 & -2.41 & -145 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ 18.0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{r}^* = (-26.1\ \ 89.6\ \ -2.61\ \ 54.5, 763)\,.$$

Application of the described above multistep localization procedure allows one to find hidden oscillation (see Fig. 8) in the considered system.

6. CONCLUSION

Since one cannot guarantee revealing complex oscillations regime by linear analysis and standard simulation, rigorous nonlinear analysis and special numerical methods should be used for investigation of nonlinear control systems.

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