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On the Manipulability of Velocity-constrained Serial Robotic Manipulators

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Abstract:
This paper presents a performance metric for the manipulability of constrained serial manipulators. The manipulability measure and manipulability ellipsoid are found from the manipulability Jacobian, which is easily found for serial manipulators as the standard Jacobian matrix, or for parallel manipulators as the Jacobian that gives the mapping from the active joint velocities to the end-effector velocities. For serial manipulators with constraints on the kinematic chain, however, this measure is not straightforwardly found because the manipulability Jacobian cannot be found in the same way as for the cases above. In this paper we propose to use the Constrained Jacobian Matrix, i.e., an analytical mapping between the end-effector and joint velocities that also takes the chain constraints into account. This analytical representation of the Constrained Jacobian Matrix allows us to find a performance metric describing the manipulability of a serial manipulator with chain constraints in the same way as for non-constrained kinematic chains and parallel manipulators. The approach allows us to compare the manipulability of the end effector of non-constrained serial manipulators and constrained serial manipulators and we study the important case of robotics-assisted minimally invasive surgery where the arm is constrained at the entry point.

Keywords: Robot Manipulator, Kinematic Constraints, Surgical Robotics.

1. INTRODUCTION

Robot control laws are often designed or evaluated based on performance metrics. One such metric is what is referred to as the manipulability of the robotic arm. The manipulability of a robotic manipulator tells us in what directions the end effector can move and how much effort it requires to move in each direction. The manipulability is often measured by the condition number or the maximum singular value of the Jacobian matrix, or more illustratively by the manipulability ellipsoid. For redundant robots the objective is generally to take advantage of the null-space of the robot to find a trajectory for which the manipulability remains high, or at least above some limit, while for non-redundant robots the objective is to avoid singular configurations by modifying the trajectory.

The concept of manipulability of serial manipulators was introduced in [**?**]. For the six degrees of freedom of the end-effector space the manipulability was found and a 6-DoF ellipsoid was constructed to define the manipulability. When a motion in one direction of the ellipsoid cannot be realized, the length of this axis becomes zero and the manipulability ellipsoid has no volume. This is often referred to as a singular configuration.

The manipulability can also be found for parallel manipulators [**??**]. In this case the length in one or more directions of the manipulability ellipsoid can become either zero or infinite. If one direction is zero it means that a velocity cannot be realized in this direction, this is often referred to as an unmanipulable singularity. On the other hand, if the length approaches infinity the manipulator cannot prevent the end effector to move in a certain direction. In this case the passive joints will move even though all the active joints are locked, which is referred to as an unstable singularity [**??**].

For serial manipulators the manipulability Jacobian is defined as the mapping from joint space velocities to the velocity of the end effector, i.e., the standard analytical or geometric Jacobian matrix. For parallel manipulators the manipulability Jacobian is normally found as the mapping from the active joints to the end-effector velocities. This paper endeavors to define a similar metric for serial manipulators with velocity constraints on one or more points in the kinematic chain. This is different from parallel robots with closed-loop constraints because the constraints are imposed on a point on the chain so that the joints after the constraint can move freely. We thus get two different sets of joints where one is constrained and the other one is free.

This is an important class of robots that needs to be treated differently from the two cases described above. One important example of such mechanisms is robots used for minimally invasive surgery where the point at which the robot is inserted into the body does not allow for lateral motion to avoid damaging the patient's skin. Other examples are serial robots in the presence of obstacles or virtual velocity constraints, or snake-like robots with several joints that need to enter into small openings, for example through a car window or into a building.

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The manipulability of serial and parallel manipulators have been studied extensively in literature. There does not, however, seem to be a corresponding analysis of the manipulability of serial manipulators subject to chain constraints. For these robots the manipulability cannot be determined by the standard Jacobian matrix as this relation does not give the mapping from joint to end-effector space when kinematic constraints are present in a kinematic chain. Constrained serial manipulators differ from serial unconstrained manipulators in that the standard geometric or analytical Jacobians do not give the mapping from joint to end-effector velocity space because of the constraint, and also differ from parallel manipulators because the end-effector motion of the serial robot is "free" while the chain motion is constrained.

In this paper we propose to use a modified Jacobian to determine the manipulability. More specifically we propose the Constrained Jacobian Matrix (CJM) introduced in [?] as a candidate for analyzing the manipulability of a serial robotic manipulator subject to one or more velocity constraints on the kinematic chain. The CJM finds the mapping between the end-effector velocities and a new set of velocity variables which again has a one-to-one relation to the joint velocities. It is thus a 2-step mapping from the joint velocities to the end-effector velocities which satisfies the velocity constraints. We suggest that this mapping can be used to find the manipulability measure of a serial robotic manipulator with velocity constraints on the chain.

The paper is organized as follows: Section 2 gives an overview of the problem discussed and sets the framework. Section 3 briefly introduces the most relevant concepts of differential kinematics and introduce the Constrained Jacobian Matrix. The new manipulability index for constrained serial manipulators is introduced in Section 4 and a simple example related to robotics-assisted minimally invasive surgery is shown in Section 5.

2. SYSTEM OVERVIEW AND PROBLEM FORMULATION

The system discussed in this paper consists of a serial robotic manipulator with one or more velocity constraints on the kinematic chain. The manipulator arm is normally redundant to maintain the full mobility of the end effector under chain constraints. The system setup for these constraints together with the most important configuration spaces used in this paper are shown in Fig. 1. We denote the frame of the joint located before the constraint at $\mathcal{F}_c$ in the chain by $\mathcal{F}_a$ and the joint that is located after the constraint is denoted $\mathcal{F}_b$. The desired end-effector motion is given by the frame $\mathcal{F}_e$. In this paper we are concerned with the manipulability of the frame $\mathcal{F}_e$. We will denote the velocity variables representing the velocities of $\mathcal{F}_e$ with respect to the inertial frame $\mathcal{F}_0$ in the following way

$$V_{0e}^B = \begin{bmatrix} v_x^e \ v_y^e \ v_z^e \ \omega_x^e \ \omega_y^e \ \omega_z^e \end{bmatrix}^\mathsf{T} \tag{1}$$

and similarly for the other frames. The superscript B state that the variables are written in body coordinates [?].

We consider robotic manipulators with n joints so that the unconstrained motion is an n-DoF motion. Furthermore we assume that the constrained motion has m degrees of freedom so the constraint has dimension $n - m$. The end-effector space has dimension r. Denote the joints so that the joint closest to the base is joint 1 and in increasing order until we reach the last joint and the end effector. We will divide the joints into two groups where the first group q_a consists of all joints that come before the constraint $\mathcal{F}_c$ in the kinematic chain and the second group q_b consists of all joints after the constraints.

The problem considered in this paper is to find the manipulability of these kinds of kinematic structures, i.e., serial robotic manipulators with "free" end-effector motion—which means there are no constraints imposed on the end effector itself—but with one or more kinematic constraints on the kinematic chain. Intuitively, this kind of constraints will potentially reduce the mobility of the system drastically and thus leads to reduced performance. In this paper we derive a performance metric for thorough analysis of these systems and study in detail the effects that these constraints have on the manipulability of constrained serial manipulator.

3. DIFFERENTIAL KINEMATICS

Differential kinematics is the study of how joint velocities relate to the link velocities, and in particular the end-effector velocities. Manipulability can be thought of as a measure of how well-defined this mapping is for a given configuration of the robot arm.

3.1 Manipulator Jacobian

In this section we will find the relation between the desired end-effector velocities and the corresponding joint velocities for serial manipulators. The standard body Jacobian matrix gives the mapping from the joint velocities $\dot{q}$ to the end-effector velocities V_{0e}^B in body coordinates which is the mapping [?]

$$V_{0e}^B = J_e^B(q)\dot{q}. \tag{2}$$

The body Jacobian matrix $J_e^B(q)$ is found by representing the twist of each joint i in the end-effector frame, i.e.,

$$\begin{aligned} J_e^B(q) &= \begin{bmatrix} X_1^\dagger \ X_2^\dagger \ \cdots \ X_n^\dagger \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Ad}_{g_{1e}}^{-1} X_1^1 \ \mathrm{Ad}_{g_{2e}}^{-1} X_2^2 \ \cdots \ X_n^n \end{bmatrix} \in \mathbb{R}^{n\times 6} \end{aligned} \tag{3}$$

where X_i^i is the constant twist in frame $\mathcal{F}_i$ and $\mathrm{Ad}_{g_{ie}}^{-1}$ is the Adjoint matrix that transforms X_i^i from frame $\mathcal{F}_i$ to $X_i^\dagger$ represented in the end-effector frame $\mathcal{F}_e$. The body Jacobian matrix can also be found for other links than the end effector, in which case it is denoted J_i^B which gives the velocities of link i. Particularly, the Jacobian matrix that gives the velocity of the link $\mathcal{F}_a$ located before the constraint is denoted J_a^B.

3.2 Constrained Jacobian Matrix

In this section we will find the body Jacobian matrices, as above, but subject to the constraints, i.e., we find the mapping from the joint velocity variables to the end effector subject to a constraint on the velocity at the constraint frame $\mathcal{F}_c$. For a large class of constraints the Constrained Jacobian Matrix can be written in the form [?]

$$\bar{J}_e^B = \left[\sum \alpha_i X_i^\dagger \ \sum \alpha_j X_j^\dagger \ \cdots \ \sum \alpha_k X_k^\dagger \right] \in \mathbb{R}^{m\times 6} \tag{4}$$

for some $(n-m)$-dimensional constraint. The bar in $\bar{J}_e^B$ distinguishes the Constrained Jacobian Matrix from the standard Jacobian J_e^B. $X_i^\dagger$ are the manipulator twists (represented in $\mathcal{F}_a$) while α_i are configuration-dependent functions of the manipulator and constraint kinematics.

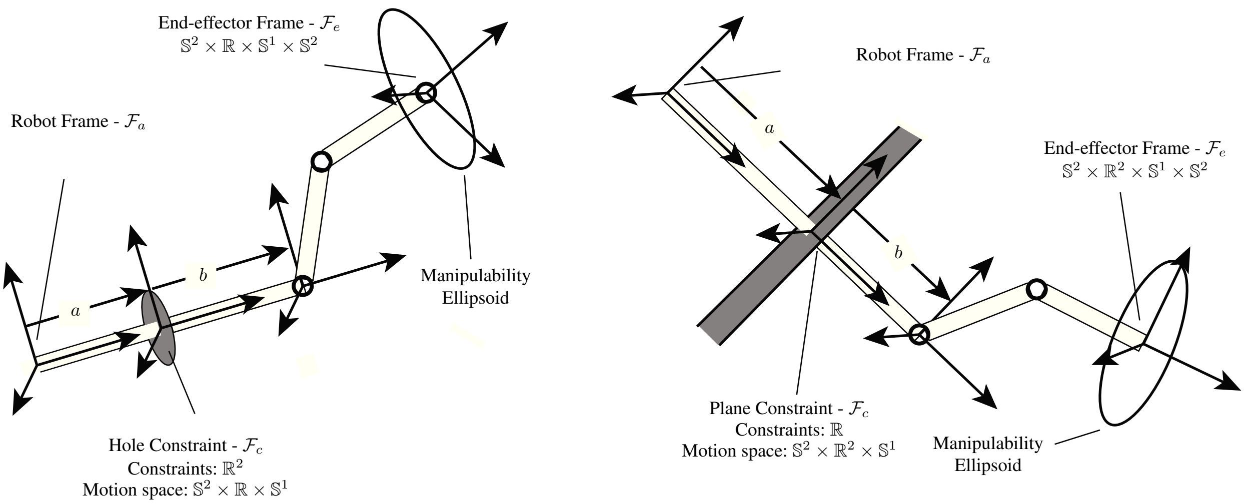

Fig. 1. We study the manipulability of the end-effector frame $\mathcal{F}_e$ in the presence of a constraint at $\mathcal{F}_c$. The figure shows two examples of the constraints discussed in this paper: on the left, a hole constraint which prevents any lateral motion of a specific point on the manipulator chain; and on the right, a plane constraint that restricts the linear motion of a point to a given direction in the plane. The motion spaces of the different frames are subgroups of $SE(3)$ defined by linear motion $\mathbb{R}$, circular motion $\mathbb{S}$, and the sphere $\mathbb{S}^2$.

Given the end-effector velocity we want to find the free and constrained velocity variables of the robot. We will find the Constrained Jacobian Matrix $\bar{J}_e^B$ which gives the relation $V_{0e}^B = \bar{J}_e^B v_m^a$ and the required velocities v_m^a are found from the desired end-effector velocities by the inverse of the Constrained Jacobian Matrix. These new velocity variables and the form of the Constrained Jacobian Matrix depends on the type of constraint. For plane and hole constraints, for example, the new velocity variables are found as

$$\begin{bmatrix} v_x^a \\ v_y^a \\ v_z^a \\ \omega_x^a \\ \omega_y^a \\ \omega_z^a \end{bmatrix} = \begin{bmatrix} v_x^a \\ v_1 \\ v_z^a \\ \frac{1}{a} v_1 \\ \omega_y^a \\ \omega_z^a \end{bmatrix} \qquad \begin{bmatrix} v_x^a \\ v_y^a \\ v_z^a \\ \omega_x^a \\ \omega_y^a \\ \omega_z^a \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_z^a \\ \frac{1}{a} v_2 \\ -\frac{1}{a} v_1 \\ \omega_z^a \end{bmatrix}, \tag{5}$$

respectively. We can now write the reduced velocity variables as

$$\bar{V}_{0a}^B = \begin{bmatrix} v_1 & v_x^a & v_z^a & \omega_y^a & \omega_z^a \end{bmatrix}^{\mathsf{T}}, \bar{V}_{0a}^B = \begin{bmatrix} v_1 & v_2 & v_z^a & \omega_z^a \end{bmatrix} \tag{6}$$

respectively. v_i for $i = 1, 2$ are the constrained variables while the remaining variables are denoted the free variables. We now present in brief the main idea of how to find the new velocity variables and derive the Constrained Jacobian Matrix. The following is taken from [**?**]:

The main idea is to find the velocity V_{0a}^B in terms of a set of new velocity variables parametrized in such a way that these variables can be chosen freely and at the same time guarantee that the constraints at $\mathcal{F}_c$ are satisfied. This means that certain directions in the velocity space are reduced from a higher to a lower-dimensional space represented by new velocity variables v_i.

As our main objective is to follow a desired end-effector motion V_{0e}^B we need to find the mapping from V_{0e}^B to the free variables, i.e., V_{0a}^B with the reduction in dimensionality represented by v_i. This is obtained in the following way:

(1) Define a desired end-effector velocity V_{0e}^B.
(2) Given a constraint at $\mathcal{F}_c$, define the velocities at this point which satisfy the constraints, i.e., the velocities at the previous joint $\mathcal{F}_a$ are given by
    - the free variables $\{v_x^a, v_y^a, v_z^a, \omega_x^a, \omega_y^a, \omega_z^a\}$, and
    - the constrained variables $\{v_1, v_2, v_3, \dots\}$.

    The free variables are the ones that can be chosen freely and do not affect the constraint. The constrained variables require a specific form and structure for the constraints to be satisfied. We will therefore replace some of the free variables with the constrained variables which gives us the required structure. These variables thus represent a freedom, but in a space with reduced dimensionality that satisfies the constraint. The constrained variables are thus written in terms of the free variables v as
$$V_{0a}^B = V_{0a}^B(v). \tag{7}$$
(3) Eliminate the redundant variables that arise as a result of the reduced dimensionality and denote the minimal representation of the velocity variables by $\bar{V}_{0a}^B$. The variables will now take the form shown in (6).
(4) Find a mapping from the end-effector velocities V_{0e}^B to the new reduced velocity variables $\bar{V}_{0a}^B$, which will take the form
$$v_m^a = \begin{bmatrix} \bar{V}_{0a}^B \\ \dot{q}_b \end{bmatrix} = \begin{bmatrix} \text{constrained variables} \\ \text{free variables} \\ \text{joint velocities} \end{bmatrix}. \tag{8}$$
    The mapping is given by the Constrained Jacobian Matrix $\bar{J}_e^B$ that gives the important relation $V_{0e}^B = \bar{J}_e^B v_m^a$, i.e., the transformation from the new reduced velocity variables v_m^a to the desired end-effector velocities V_{0e}^B. The *joint velocities* represent the joints that are determined by the end-effector velocity V_{0e}^B only and do not depend on the constraints. These are typically the joints q_b that are situated between the constraint and the end effector. The *free variables* are the velocities of $\mathcal{F}_a$ that do not depend on the constraint, but differently from the *joint velocities*, they depend on the joints q_a between the base

and the constraint. Finally, the *constrained variables* are constraint dependent and give the velocity at $\mathcal{F}_a$ the required structure so that the constraints are satisfied.

(5) From the new variables, find the robot velocity at $\mathcal{F}_a$.

We refer to [?] for details on this topic.

We note that there are two main steps: First we need to find a suitable representation of the velocity variables and define the Constrained Jacobian Matrix which gives us $\bar{V}_{0a}^B$ and $\dot{q}_b$. Then we find V_{0a}^B from $\bar{V}_{0a}^B$ and finally the joint velocities $\dot{q}_a$ from V_{0a}^B by the standard Jacobian matrix.

The kinematics of a serial robotic manipulator with chain constraints are now found by the two kinematic relations

$$V_{0e}^B = \bar{J}_e^B(q) v_m^a \tag{9}$$

$$V_{0a}^B = J_a^B(q)\dot{q} \tag{10}$$

where $\bar{J}_e^B(q)$ is the Constraint Jacobian Matrix and $J_a^B(q)$ is the standard body geometric Jacobian that gives the mapping from the (first m) joints to joint $\mathcal{F}_a$. We will see examples of these matrices in Section 5.

4. MANIPULABILITY

In this section we briefly review the manipulability measure of robots as presented in literature. Given the manipulability Jacobian $J_M \in \mathbb{R}^{r\times n}$, the manipulability matrix is given by

$$W = J_M J_M^{\mathsf{T}} \tag{11}$$

and the manipulability measure as

$$w = \sqrt{\det \left|J_M J_M^{\mathsf{T}}\right|}. \tag{12}$$

Let the dimension of the task space r be the same as the mobility of the manipulator n, i.e., a non-redundant robots with $r = n$. In this case the manipulability measure is given simply by

$$w = \left|\det J_M\right|. \tag{13}$$

The manipulability ellipsoid is found by the eigenvalues and eigenvectors of the manipulability matrix M. Let the singular value decomposition of J_M be written as [?]

$$J_M = U\Sigma V^{\mathsf{T}} \tag{14}$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 & \vdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \sigma_r & \vdots & 0 \end{bmatrix} \tag{15}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$. The manipulability measure can now be written as

$$w = \sigma_1\sigma_2\sigma_3\cdots\sigma_r. \tag{16}$$

Furthermore the manipulability ellipsoid is the ellipsoid with principal axes $\sigma_1 u_1, \sigma_2 u_2, \ldots, \sigma_r u_r$ where u_i are the columns of U.

In this section we will find the manipulability measure for constrained robotic arms. The manipulability measure is found by (12), so it only remains to find the manipulability Jacobian, which is found in the following way:

- Serial manipulators: The standard Jacobian matrix $J_e^B(q)$.
- Parallel manipulators: The Jacobian map from the active joint velocities to the end-effector velocities.
- Constrained serial manipulators: The Jacobians $\bar{J}_e^B(q)$ and $J_a^B(q)$ found in Eqs. (9) and (10).

The two first cases are studied in detail in literature. In the next section we will study in more detail the manipulability of constrained serial manipulator arms.

4.1 The Manipulability Measure for Constrained Kinematic Chains

The manipulability of constrained serial manipulators needs to be found in two steps. We first look at the manipulability of the end effector assuming the first joints q_a can generate the required motion. This mapping depends only on the geometry of the constraint and the links q_b located after the constraint. This is given by (9).

The Constraint Jacobian Matrix $\bar{J}_e^B(q)$ tells us whether it is possible to generate an end-effector velocity $\bar{V}_{0e}^B$ using the new velocity variables v_m^a, i.e., whether the constrained velocities V_{0a}^B and the joints after the constraint can generate the desired end-effector velocities. The condition number and singularity analysis of this matrix thus gives the effects of the constraints on the manipulability. This depends only on the kinematics of the constraint itself and the kinematics of the joints after the constraint.

The manipulability ellipsoid is given by choosing $J_M = \bar{J}_e^B(q)$, and thus gives the manipulability in the directions of the end-effector frame V_{0e}^B. The interpretation of this is as follows:

It gives us the mobility of V_{0e}^B given V_{0a}^B and $\dot{q}_b$. We see that because we use V_{0a}^B the mobility depends on the constraint. We thus interpret this as the constraint-dependent manipulability of the system, and denote this the Constrained Manipulability Measure (CMM).

In the above we have assumed that the robot can always generate the desired motion V_{0a}^B. This is of course not always the case. We thus need to find the same performance measure for V_{0a}^B. This is similar to the standard manipulability measure for non-constrained serial manipulators.

The manipulability of the standard geometric Jacobian $J_a^B(q)$ is a measure of how efficiently the manipulator can generate the motion V_{0a}^B required to generate the desired end-effector motion V_{0e}^B. This is thus equivalent to the standard manipulability analysis of a serial manipulator, but with respect to joint $\mathcal{F}_a$ as opposed the the end effector $\mathcal{F}_e$. We will denote this the Manipulator Manipulability Measure (MMM).

The CMM and MMM together give a measure of the manipulability of a constrained serial manipulator. If the manipulability ellipsoid of both these are non-vanishing we know that an arbitrary motion can be realized in the end-effector space, even though there are constraints on the kinematic chain. Furthermore, the manipulability ellipsoids gives us valuable information about the cause of the singularity and how it can be resolved.

5. CASE STUDY—MANIPULABILITY OF A SERIAL MANIPULATOR WITH HOLE CONSTRAINTS

YZ-Manipulator Arm Consider an 8-DoF manipulator with 6 joint before and 2 joints after a hole constraint. Assume further

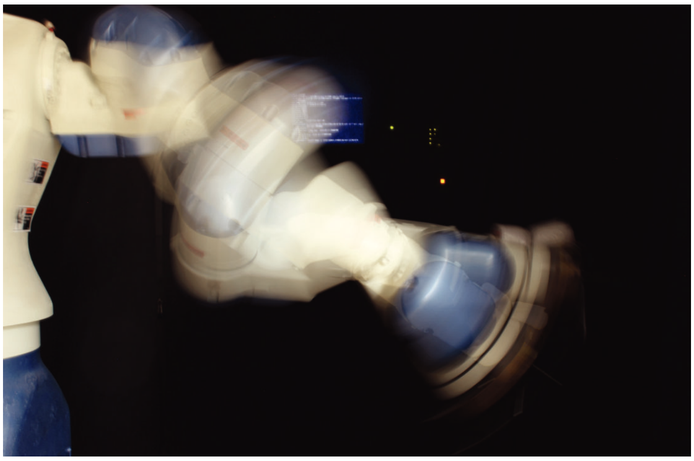

Fig. 2. An example of a robot with two links after a hole constraint

that the last two joints rotate around the y- and the z-axes, respectively. An overlayed photo of such a robot with imposed hole constraint on a point on the arm is showed in Fig. 2 and illustrated in Fig. 3. For the hole constraints the Constrained Jacobian Matrix can be written by the relation

$$\begin{bmatrix} v_x^e \\ v_y^e \\ v_z^e \\ \omega_x^e \\ \omega_y^e \\ \omega_z^e \end{bmatrix} = \begin{bmatrix} -\alpha_1 & cq_8 sq_7 & \beta_1 & -l_8 sq_7 sq_8 & -l_8 cq_8 & 0 \\ \frac{1}{a} b sq_7 & cq_7 & 0 & 0 & 0 & 0 \\ -\alpha_2 & sq_7 sq_8 & \beta_2 & l_8 sq_7 cq_8 & -l_8 sq_8 & 0 \\ -\frac{1}{a} sq_8 & 0 & -\frac{1}{a} cq_7 cq_8 & sq_7 cq_8 & -sq_8 & 0 \\ 0 & 0 & \frac{1}{a} sq_7 & cq_7 & 0 & 1 \\ \frac{1}{a} cq_8 & 0 & -\frac{1}{a} cq_7 sq_8 & sq_7 sq_8 & scq_8 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_z^a \\ \omega_z^a \\ \dot{q}_7 \\ \dot{q}_8 \end{bmatrix} \tag{17}$$

where

$$\alpha_1 = \frac{1}{a} cq_8 (l_8 + bcq_7),$$
$$\alpha_2 = \frac{1}{a} sq_8 (l_8 + bcq_7),$$
$$\beta_1 = \frac{1}{a} sq_8 (b + l_8 cq_7),$$
$$\beta_2 = \frac{1}{a} cq_8 (b + l_8 cq_7),$$

b is the distance from the constraint to the last joint and q_8 is the length of the last link.

The condition of this matrix tells us whether the desired velocities at the end effector can be realized or not. As we can see, the matrix only depends on the link geometry (length) of the last link, the joint angles q_b and the distance between the frames $\mathcal{F}_a$, $\mathcal{F}_c$, and $\mathcal{F}_b$ given by the variables a and b. If we plot the manipulability measure with respect to q_7 and q_8 it will look like the plot in Fig. 4 where it is plotted for two different values of a.

We note that the manipulability measure only depends on the first joint q_7 right after the constraint. The manipulability measure vanishes for $q_7 = 0, \pm\pi$. Note that this singularity is different from the well known singularity of the 2-DoF planar robot which also has a singularity when it is stretched out (second joint equals zero). For the planar case this singularity is a limit singularity which arises when the manipulator is at the limit of its workspace, so that no motion in the direction of the stretched

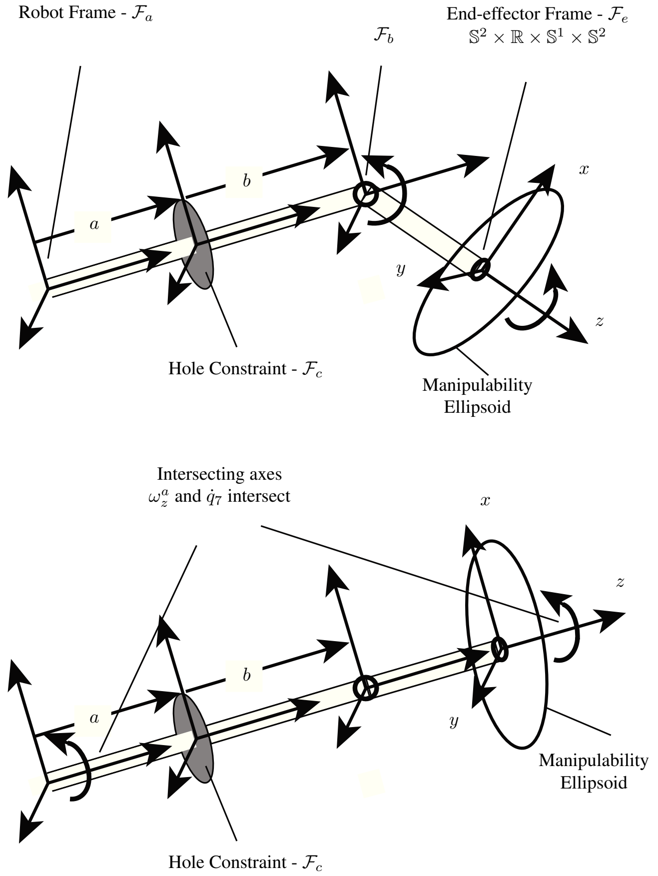

Fig. 3. The robot has links q_b that rotates about the y- and z-axes, and has singularities at $q_7 = 0, \pm\pi$ (bottom figure).

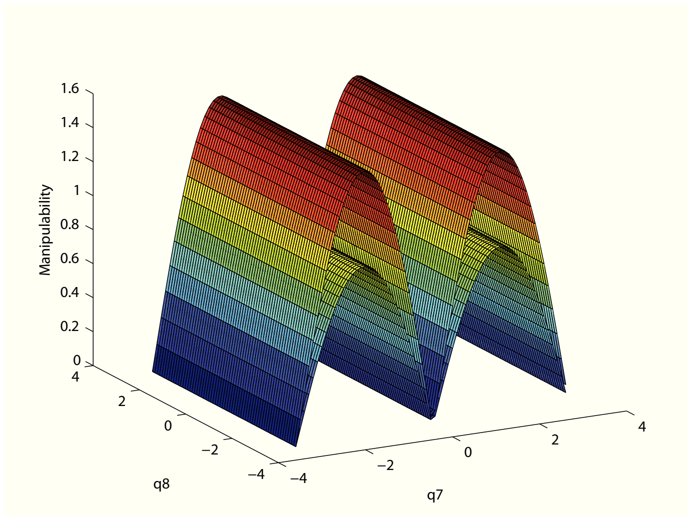

Fig. 4. The manipulability measure of the Constrained Jacobian Matrix with respect to the joint variables $q_b = q_7$, q_8 for two different distances between $\mathcal{F}_a$ and $\mathcal{F}_c$ given as $a = 0.4$ (the surface with the highest manipulability) and $a = 0.5$ (the surface with the lowest manipulability). The link length is set to $l_8 = 1$.

out arm can be realized. In our case, however, the singularity is due to the alignment of two velocity variables, namely ω_z^a and $\dot{q}_7$. It is important to note that this is different from singularities that arise in unconstrained manipulators where two of the axes

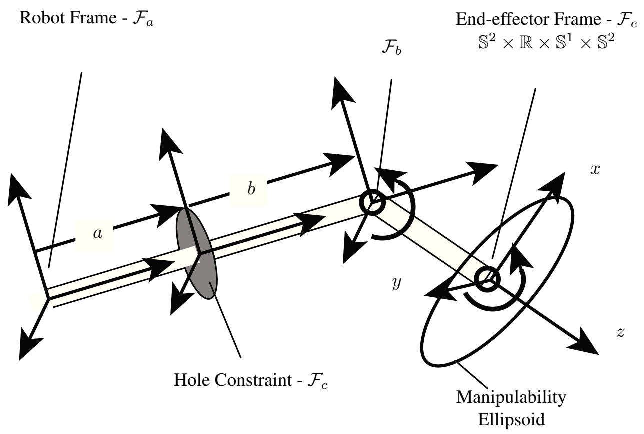

Fig. 5. For this robot both links q_b rotate about the y-axis and the manipulability measure vanishes for all values of q_7 and q_8.

are aligned. In our case the direction one of the free velocity variables ω_z^a is aligned with the axis of $\dot{q}_7$, which is different from aligning two axes of the manipulator in that it depends on the geometry of the constraint, and not the geometry of the manipulator joints q_a.

YY-Manipulator Arm In Fig. 5 we see another example of manipulator geometry where both joints q_b rotate around the same axis, namely the y-axis. In this case the Constrained Jacobian Matrix takes the form

$$\begin{bmatrix} v_x^e \\ v_y^e \\ v_z^e \\ \omega_x^e \\ \omega_y^e \\ \omega_z^e \end{bmatrix} = \begin{bmatrix} -\alpha_1 & 0 & 0 & l_7 s q_7 & 0 & 0 \\ 0 & -\beta_1 & s q_{78} & 0 & -l_7 c q_8 & 0 \\ 0 & \beta_2 & c q_{78} & 0 & l_7 s q_8 & 0 \\ 0 & \frac{1}{a} & 0 & 0 & 1 & 1 \\ -\frac{1}{a} c q_{78} & 0 & 0 & s q_{78} & 0 & 0 \\ \frac{1}{a} s q_{78} & 0 & 0 & c q_{78} & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_z^a \\ \omega_z^a \\ \dot{q}_7 \\ \dot{q}_8 \end{bmatrix} \quad (18)$$

where we have defined

$$\alpha_1 = \frac{1}{a}(b + l_7 c q_7),$$
$$\beta_1 = \frac{1}{a}(b c q_{78} + l_7 c q_8),$$
$$\beta_2 = \frac{1}{a}(b s q_{78} + l_7 s q_8).$$

The manipulability measure is always vanishing, so we have an ill-conditioned mapping for all joint positions q_b. The reason for this can be seen from a geometric point of view: Consider the xz-plane as seen from the frame $\mathcal{F}_a$. Then we see that we use the four variables v_1, v_z^a, $\dot{q}_7$, and $\dot{q}_8$ to generate the 3-DoF motion of this plane. It thus only remains two degrees of freedom to generate the remaining 3-DoF, which is the reason for the vanishing manipulability measure for all q_7 and q_8.

6. CONCLUSION

In this paper we have proposed a new manipulability measure for constrained serial manipulators. We have shown that the Constrained Jacobian Matrix previously proposed by the authors can be used to find the manipulability of serial manipulators with velocity constraints on the chain. We show through two simple examples how the approach can be used to analyze the manipulability of two different robotic arms with hole constraints on the chain and show how two different joint setups can give very different results when it comes to manipulability. This can be used to identify geometric structures for which the mobility is lost/maintained, and thus important in manipulator design and control in the presence of obstacles. We particularly focus on robotic setups such as the ones found in robotics-assisted minimally invasive surgery.

As future work we want to generalize the approach to include obstacles with different geometry and also study how the dynamic manipulability of the manipulator is affected by the constraint.

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