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Scalability and Performance Improvement of Distributed Sequential Command Governor Strategies via Graph Colorability Theory

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Abstract: Graph colorability theory is shown here to be instrumental for the improvement of scalability and performance of recently introduced distributed non-cooperative sequential Command Governor (CG) strategies. Such properties were limited in earlier distributed CG schemes because the structure of constraints was not taken into account in their implementation. Here, by exploiting the idea that agents that are not jointly involved in any coupling constraint can simultaneously update their control actions, this is done by grouping all agents in the network into particular subsets (Turns) and allowing all agents belonging to a single turn to update simultaneously their commands while agents in other turns are instructed to keep applying their current commands. Graph colorability concepts and results are used to determine the minimum number of independent turns and distribute exclusively agents into them. A Turn-Based distributed CG strategy is therefore proposed and its main properties analyzed. A final example is also presented to illustrate the effectiveness of the proposed strategy by comparisons.

1. INTRODUCTION

The problem of interest here is the design of distributed supervision strategies based on multi-agent Command Governor (CG) ideas for networked interconnected systems in situations where the use of a centralized coordination unit is impracticable because requiring unrealistic or unavailable communication and/or computation infrastructures.

The distributed context under consideration is depicted in Figure 1, where the supervisory task is distributed amongst many agents, which are assumed to be able to communicate each other through a communication network. There, each agent is in charge of supervising and coordinating one specific subsystem.

In particular, let r_i, g_i, x_i, y_i and c_i represent respectively: the nominal reference, the applied reference, the state, performance and the coordination related output vectors of the $i-$th subsystem. In such a context, the supervision task can be expressed as the requirement of satisfying some tracking performance, viz. $y_i \approx r_i$, whereas the coordination task consists of enforcing some pointwise-in-time constraints $c_i \in \mathcal{C}_i$ and/or $f(c_1, c_2, ..., c_N) \in \mathcal{C}$ on each subsystem and/or on the overall network evolutions. To this end, each $i-$th supervisor is in charge of modifying its nominal local reference r_i into a feasible one g_i, whenever the joint application of all nominal references would produce constraint violations and hence loss of coordination. Earlier works on distributed CG strategies relied on a non-

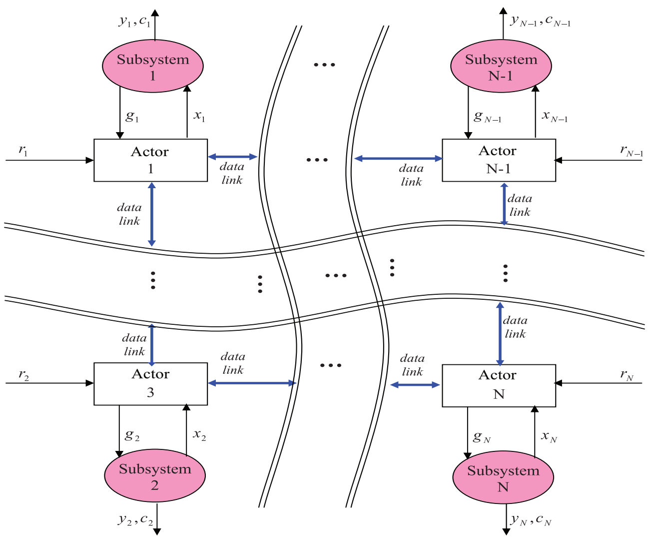

Fig. 1. Multi-agent architectures

cooperative game theoretical approach and several strategies have been proposed, both *sequential* (Casavola *et al.*. (2011-A)), (Tedesco *et al.* (2012-A)) where, according to a prefixed order, only one agent at each sampling time is

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allowed to update its control action while all others keep applying their previous applied commands, and *parallel* (Tedesco *et al.* (2012-B)), (Casavola *et al.* (2014-A)), (Tedesco *et al.* (2012-C)) where, on the contrary, all agents update their control actions simultaneously at each sampling time.

In this paper, a Turn-Based distributed CG approach is proposed that generalizes the Sequential distributed CG scheme introduced in (Tedesco *et al.* (2012-A)) for the supervision and coordination of dynamic subsystems characterized by decoupled dynamics but sharing coupling constraints involving state and input evolutions.

In order to improve the scalability and optimality performance of the earlier sequential CG schemes, the idea that agents that are not jointly involved in any coupling constraint can simultaneously update their control actions without violating constraints is here exploited. The main intuition of this paper is to use graph colorability theory to systematically group agents into particular subsets (turns). At each sampling time instant, on the basis of a round-robin policy for the turns, only agents belonging to a specific turn are allowed to update simultaneously their commands, while all agents in all other turns keep applying constantly their current commands until their turn becomes active.

In this paper we will discuss and highlight the existing link between the grouping policy and the graph minimal vertex coloring problem. In the current application, the graph characterizes the existing coupling interconnections arising among the agents induced by the constraints. Such a problem has been widely studied in the last century (see Jensen and Toft (1994) for a survey on the subject), efficient algorithms and heuristics exist and several interesting results have been proved. Among many, of particular interest here is the fact that, for many classes of graph topologies and in particular for sparse graphs, the minimal number of colors of a graph is bounded (and often by a small integer) regardless of the number of nodes that the graph consists of.

The use of such graph theoretical concepts allows one to remarkably improve the scalability and the performance of the proposed supervision scheme with respect to the earlier version described in (Tedesco *et al.* (2012-A)). Specifically, in the final example it is shown that the resources (CPU time and data exchanges) required to accomplish the coordination task do not increase with the increasing number of agents and the performance almost coincide with the one of the centralized solution.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider a set of N subsystems $\mathcal{A} = \{1, \ldots, N\}$. Each subsystem is assumed to be a LTI closed-loop dynamical system regulated by a local controller which ensures asymptotic stability and good closed-loop properties when the constraints are not active (small-signal regimes). Let the i-th closed-loop subsystem be described by the following discrete-time model

$$\left\{\begin{array}{lcl} x_i(t+1) & = & \Phi_i x_i(t) + G_i g_i(t) \\ y_i(t) & = & H_i^y x_i(t) \\ c_i(t) & = & H_i^c x_i(t) + L_i g_i(t) \end{array}\right. \tag{1}$$

where: $t \in \mathbb{Z}_+$, $x_i \in \mathbb{R}^{n_i}$ is the state vector (which includes the controller states under dynamic regulation), $g_i \in \mathbb{R}^{m_i}$ the manipulable reference vector which, if no constraints (and no CG) were present, would coincide with the desired reference $r_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{m_i}$ is the output vector which is required to track r_i. As the system is pre-stabilized, the matrix Φ_i has all eigenvalues strictly inside the unitary ball. Finally, $c_i \in \mathbb{R}^{n_i^c}$ represents the local constrained vector.

Let the aggregate constrained vector $c = [c_1^T, \ldots, c_N^T]^T \in \mathbb{R}^{n^c}$, with $n^c = \sum_{i=1}^N n_i^c$. It is assumed that at each time instant $c(t)$ must be contained in a convex and compact polytopic set defined as follows:

$$Ac(t) \le b \tag{2}$$

with $A \in \mathbb{R}^{z \times n^c}$ and $b \in \mathbb{R}^z$, z is the number of constraints and where $\le$ is meant component-wise.

The distributed CG design problem is the one of locally determine, at each time step t and for each agent $i \in \mathcal{A}$, a suitable reference signal $g_i(t)$ which is the best approximation of $r_i(t)$ such that its application do not produces constraints violation, i.e. $Ac(t) \le b$, $\forall t \in \mathbb{Z}_+$.

3. TURN-BASED DISTRIBUTED CG

3.1 Centralized CG

The main idea of classical centralized solutions to the CG design problem (see Bemporad *et al.* (1997)) for a system in the form

$$\left\{\begin{array}{lcl} x(t+1) & = & \Phi x(t) + G g(t) \\ y(t) & = & H^y x(t) \\ c(t) & = & H^c x(t) + L g(t) \end{array}\right. \tag{3}$$

characterized by matrices Φ, G, H^y, H^c, L having proper dimensions and subject to constraints (2) is to choose at each time instant a set point g approximating $r(t)$ such that:

1) the associated steady-state constrained output
$$c_g = H^c(I - \Phi)^{-1} G g \tag{4}$$
satisfies the constraints with a margin $\delta > 0$, i.e. $Ac_g \le b - \delta[1, ..., 1]^T$.
2) if the command g is kept constant from t onward, constraints are never violated, i.e. $Ac(x(t), k, g) \le b, \forall k \ge 0$, where
$$c(k, x, g) := H^c\left(\Phi^k x + \sum_{\tau=0}^{k-1} \Phi^{k-\tau-1} G g\right) + Lg \tag{5}$$

As proven in Bemporad *et al.* (1997), the latter two conditions translates into confining g in a finitely-determinable convex set which can be described by the following constraints

$$Ac_g \le b_\delta \tag{6}$$

$$Ac(k, x(t), g) \le b \;\; k = 0, 1, ..., k_0 \tag{7}$$

where k_0 is an integer that can be computed on the basis of the system dynamics and b_δ is a vector depending on $\delta > 0$. For details please refer to Bemporad *et al.* (1997). It has been proven that this scheme always ensure constraints satisfaction. Moreover, under constant references r, the applied command $g(t)$ converges in finite time to a constant reference $\hat{r}$ which is the best approximation of r compatible with the steady-state constraint (6).

3.2 Constraints Associated to the i-th Agent

The first step towards the development of a distributed CG is to note that, under local dynamics (1), the aggregated system can be written as (3) where $x(t) = [x_1^T(t), ..., x_n^T(t)]^T, g(t) = [g_1^T(t), ..., g_n^T(t)]^T$, $y(t) = [y_1^T(t), ..., y_n^T(t)]^T$, and $\Phi = \text{diag}(\Phi_1, ..., \Phi_N)$ $G = \text{diag}(G_1, ..., G_N)$, $H^y = \text{diag}(H_1^y, ..., H_N^y)$, $H^c = \text{diag}(H_1^c, ..., H_N^c)$ and $L = \text{diag}(L_1, ..., L_N)$. In this case c_g in (4) and $c(k, x, g)$ in (5) are given by:

$$c_g = \left[c_{1,g_1}^T, ...c_{N,g_N}^T\right]^T,\ c(k, x, g) = \left[c_1^T(k, x_1, g_1), ..., c_N^T(k, x_N, g_N)\right]^T$$

where

$$c_{i,g_i} = H_i^c(I - \Phi_i)^{-1}G_i g_i$$

and

$$c_i(k, x_i, g_i) := H_i^c\left(\Phi_i^k x_i + \sum_{\tau=0}^{k-1}\Phi_i^{k-\tau-1}G_i g_i\right) + L_i g_i.$$

Please note that each c_{i,g_i} and $c_i(k, x_i, g_i)$ only depend on the state and the command of the ith subsystem. Moreover, conditions (6)-(7) become:

$$A\left[c_{1,g_1}^T, ...c_{N,g_N}^T\right]^T \le b_\delta \tag{8}$$

$$A\left[c_1^T(k, x_1, g_1), ..., c_N^T(k, x_N, g_N)\right]^T \le b,\ k = 0, 1, ..., k_0 \tag{9}$$

Using the classical CG approach, if at each time instant t the new command g complies with the above conditions, the constraints are never violated. Here, on the contrary, the aim is at finding a distributed local strategy able to take decisions subject to (8) and (9) on the basis of local information only.

The main observation is that for this class of systems, the i-th agent may concur only to the violation of constraints where its constrained variables $c_i(t)$ are present. To put in evidence these constraints, let $a_j^T = [a_{j,1}^T, ..., a_{j,N}^T]$ be the j-th row of A, with sub-rows $a_{j,i}^T \in \mathbb{R}^{n_i^c}, \forall i \in \mathcal{A}$, and $b_j \in \mathbb{R}$ the j-th element of b. Then, for each $i \in \mathcal{A}$ it is possible to define the block-matrices $A_i = [A_{i,1}|......|A_{i,N}]$, with $A_{i,j} \in \mathbb{R}^{z_i \times n_i^c}$ and vector $\tilde{b}_i \in \mathbb{R}^{z_i}$ collecting respectively all and only the, say z_i, rows a_j of A and b_j of b in which the i-th agent is involved, i.e. such that the sub-rows $a_{j,i}^T \ne 0_{n_i^c}$. As a consequence, the constraints associated to the i-th agent can simply be described as

$$A_i c(t) \le \tilde{b}_i \tag{10}$$

As a matter of fact, in many cases the matrix A is quite sparse and only a subset of agents are involved in the constraints associated to the i-th agent. In this respect, the following definition is in order:

Definition 3.1. (**Neighborhood of the i-th Agent**) The neighborhood $\mathcal{N}_i$ of the i-th agent consists of all agents whose subsystem evolutions are jointly constrained with the i-th subsystem evolution as defined in (10), i.e.

$$\mathcal{N}_i = \{i\} \cup \{j \in \mathcal{A} : A_{i,j} \ne 0\}. \tag{11}$$

In this formulation, from the perspective of the i-th agent, constraints (2) become

$$\tilde{A}_i\tilde{c}_i(t) \le \tilde{b}_i \tag{12}$$

where $\tilde{c}_i(t) = S_{c,\mathcal{N}_i}c(t)$ denote the c-vectors associated to the neighbors of the i-th agent and $\tilde{A}_i = A_i S_{c,\mathcal{N}_i}^T$ is the associated matrix. Notice that $S_{c,\mathcal{N}_i}$ is a selection matrix which extracts all rows of c related with agents in $\mathcal{N}_i$.

Next, assume that at time t agent i receives from its neighbors their state measurements and their previously applied commands, i.e. $\tilde{x}_i(t) = S_{x,\mathcal{N}_i}x(t)$ and $\tilde{g}_i(t-1) = S_{g,\mathcal{N}_i}g(t-1)$, where $S_{x,\mathcal{N}_i}$ and $S_{g,\mathcal{N}_i}$ are the selection matrix extracting the elements of the x and g vectors related to agents in $\mathcal{N}_i$.

If at time t, all agents in $\mathcal{N}_i$ except the i-th were holding the commands applied at time $t-1$, the i-th agent could select a local command g_i satisfying constraints (10) by fulfilling the following inequalities

$$\tilde{A}_i\tilde{c}_{\tilde{g}_i} \le \tilde{b}_{i,\delta} \tag{13}$$

$$\tilde{A}_i\tilde{c}_i(k, \tilde{x}_i(t), \tilde{g}_i) \le \tilde{b}_i\ \ k = 0, 1, ..., k_0 \tag{14}$$

where $\tilde{g}_i$ is set equal to $\tilde{g}_i(t-1)$ for all entries except g_i, and

$$c_{\tilde{g}_i} = S_{c,\mathcal{N}_i}c_{\tilde{g}} = \tilde{H}_i^c(I - \tilde{\Phi}_i)^{-1}\tilde{G}_i\tilde{g}_i$$

$$\tilde{c}_i(k, \tilde{x}_i(t), \tilde{g}_i) := \tilde{H}_i^c\left(\tilde{\Phi}_i^k\tilde{x}_i + \sum_{\tau=0}^{k-1}\tilde{\Phi}_i^{k-\tau-1}\tilde{G}_i\tilde{g}_i\right) + \tilde{L}_i\tilde{g}_i.$$

where: $\tilde{\Phi}_i = S_{x,\mathcal{N}_i}\Phi S_{x,\mathcal{N}_i}^T$, $\tilde{G}_i = S_{x,\mathcal{N}_i}GS_{g,\mathcal{N}_i}^T$, $\tilde{H}_i^c = S_{c,\mathcal{N}_i}H^c S_{x,\mathcal{N}_i}^T$, $\tilde{L}_i = S_{c,\mathcal{N}_i}H^c S_{g,\mathcal{N}_i}^T$.

Interestingly enough, this idea can be easily extended. To this end, let the following set of agent be defined

Definition 3.2. (**Turn**) A turn $\mathcal{T} \subset \mathcal{A}$ is a *subset of non-neighboring nodes*, i.e. $\forall i, j \in \mathcal{T}$ such that $i \ne j$, $j \notin \mathcal{N}_i$ (none of them is a neighbor of the others).

The following proposition proves that the same discussion made for a single agent, can be extended to all agents contained in a turn $\mathcal{T}$

Proposition *1.* Let $\mathcal{T}_t \subset \mathcal{A}$ be a turn selected at time t. Then, if at time t all agents not in $\mathcal{T}_t$ keep applying their previously applied commands, i.e. $g_i(t) = g_i(t-1), \forall i \notin \mathcal{T}_t$ and all agents in $i \in \mathcal{T}_t$ update their commands g_i accordingly to (13)-(14), then the overall constraints (2) are never violated.

Proof: It is sufficient to look at the structures of constraints (8)-(9) and (13)-(14). □

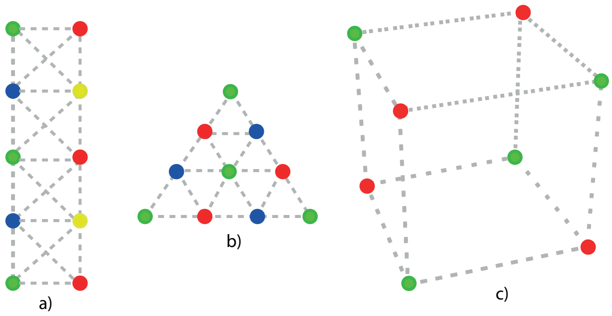

Fig. 2. Particular lattice structures: a) four colors are needed to cover the graph, b) three colors are needed, c) only two colors are required

3.3 The Overall Algorithm

At this point, given a sequence of q *turns* $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3...\mathcal{T}_q$, and assuming that each agent is assigned to a single turn, the problem of locally determining at each time t the best $g_i(t)$ approximating $r_i(t)$ such that global constraints are satisfied can be solved by allowing only the agents in the current turn $\mathcal{T}_t$ to update their commands in accordance with constraints (8)-(9). This idea can be formalized as follows:

Algorithm 1: ***The Turn-Based CG (TB-CG)***

REPEAT AT EACH TIME t
1 IF $i \in \mathcal{T}_{t\text{mod}_q}$
  1.1 RECEIVE $\tilde{x}_i(t), \tilde{g}_i(t-1)$ FROM NEIGHBORS;
  1.2 SOLVE
$$g_i(t) = \arg\min_{g_i} ||g_i - r_i(t)||^2_{\Psi_i} \tag{15}$$
  SUBJECT TO (13)-(14)
2 ELSE
  2.1 SET $g_i(t) = g_i(t-1)$
3 APPLY $g_i(t)$
4 SEND $g_i(t)$ AND $x_i(t)$ TO THE NEIGHBORING AGENTS

where $\|x\|^2_{\Psi_i}, \Psi_i = \Psi_i^T > 0, \forall i \in \mathcal{A}$, denotes the quadratic form $x^T \Psi_i x$. Please note that the latter algorithm allows one to satisfy constraints using only local data. In fact, each agent in the turn $\mathcal{T}_t$ only needs to know the commands, the states and the dynamic models of its neighbors.

4. TURNS AND GRAPH COLORABILITY THEORY

A crucial point of the proposed algorithm is the way the turn sets $\mathcal{T}_t$ are selected. A first general requisite to guarantee some convergence property is that, periodically, all agents of the network are selected, i.e. $\exists t' : \forall t > 0, \cup_{i=0}^{t'} \mathcal{T}_{t+i} = \mathcal{A}$.

In order to make the overall system behaving "fast" in response to fast changing reference signals, a further guideline is to choose a sequence of $\mathcal{T}_t$ that maximizes the frequency with which agents are allowed to update their local commands. Although not optimal w.r.t. some specific performance criteria, from a practical viewpoint a reasonable choice is to resort to a periodic scheduling $\mathcal{T}_1, ..., \mathcal{T}_q$ of length q, with q as small as possible and such that $\cup_{i=1,..,q} S_i = \mathcal{A}$. In this way, one can ensure that each agent can update its command with a frequency that is at least $1/q$.

It is worth pointing out that the problem of determining the minimum q for which a collection of shifts $\mathcal{T}_1, ..., \mathcal{T}_q$ exists such that $\cup_{i=1,..,r} \mathcal{T}_i = \mathcal{A}$ is equivalent to the minimal vertex coloring problem (Jensen and Toft (1994)) for the graph $\Gamma(\mathcal{A}, \mathcal{E})$, whose set of nodes coincides with $\mathcal{A}$ and where $\mathcal{E} \subset \mathcal{A} \times \mathcal{A}$ is the set of edges connecting neighbor agents, i.e. the edge (i, j) belongs to $\mathcal{E}$ if and only if $j \in \mathcal{N}_i$, more formally such a problem can formulated as follows

Definition 4.1. Find the minimal assignment of colors to each vertex of a $\Gamma(\mathcal{A}, \mathcal{E})$ such that, forall $(i, j) \in \mathcal{A} \times \mathcal{A}$, $(i, j) \in \mathcal{E} \Rightarrow \{i, j$ have a different color$\}$.

The connection with our turnation design problem is quite evident. After the latter problem is solved and a color is assigned to each agent, a minimal collection of turns $\mathcal{T}_1, ..., \mathcal{T}_q$ is found such that $\cup_{i=1,..,q} \mathcal{T}_i = \mathcal{A}$ by regrouping all agents with the same color in a single turn $\mathcal{T}_i$.

In view of the supervision scheme here proposed, it is important to remark that, for many classes of graph topologies and in particular for sparse graphs, the minimal number of colors of a graph is bounded (and often by a small integer) regardless of the number of nodes the graph consists of. Is this the case, for instance, for lattices where, as seen in Figure 2, easy coloring rules can be be determined. A general result that it is important to recall here is the Brooks' Theorem (Brooks (1941)) that states that for any graph with maximum degree (number of edges for each nodes) Δ, the chromatic number (minimal number of colors) is at most $q = \Delta + 1$. Please note that this ensures that, in the case of sparse topologies, the number of turns and then the frequency with which each agent updates its command remain typically bounded by a small integer, independently on the number of agents.

5. PROPERTIES

Thanks to the following Lemma it is possible to prove that the scheme here proposed is equivalent to the sequential scheme (S-CG) proposed in Tedesco *et al.* (2012-A), of which it shares all the properties.

Lemma *1.* Given a turn $\mathcal{T}_t$, the solutions of all local optimization problems

$$\begin{array}{l} \min_{g_i} ||g_i - r_i(t)||^2_{\Psi_i} \\ \textit{subject to } (13) - (14), \forall i \in \mathcal{T}_t \end{array} \tag{16}$$

are equivalent to the solution of the following single optimization problem

$$\begin{array}{ll} \min_{g_i, i \in T_t} \sum_{i \in \mathcal{T}_t} ||g_i - r_i(t)||^2_{\Psi_i} & \\ \textit{subject to} & \\ \tilde{A}_i \tilde{c}_{\tilde{g}_i} \leq \tilde{b}_{i,\delta}, \ \forall i \in \mathcal{T}_t & \\ \tilde{A}_i \tilde{c}_i(k, \tilde{x}_i(t), \tilde{g}_i) \leq \tilde{b}_i, \quad k = 0, 1, ..., k_0, \forall i \in \mathcal{T}_t & \end{array} \tag{17}$$

Proof: Each optimization problem (16) governs a single command g_i. Thanks to the definition of turn, if a constraint is influenced by $g_i, i \in \mathcal{T}_t$ then it is not influenced by any other $g_j, j \in \mathcal{T}_t$. As a consequence, a vector collecting local feasible solutions for (16) for all $i \in \mathcal{T}_t$ is also a feasible solution for problem (17). Moreover, because each agent deals with a decoupled objective function in (16), clearly

$$\sum_{i \in \mathcal{T}_t} \min_{g_i} ||g_i - r_i(t)||^2_{\Psi_i} = \min_{g_i, i \in T_t} \sum_{i \in \mathcal{T}_t} ||g_i - r_i(t)||^2_{\Psi_i}$$

which means that problem (17) is an aggregation of $|\mathcal{T}_t|$ decoupled optimization problems. □

The main consequence of the above Lemma is that we can look at the proposed scheme as to a sequential scheme, like those presented in (Tedesco *et al.* (2012-A))and (Casavola *et al.*. (2011-A)), where each turn $\mathcal{T}_t$ may be there interpreted as a *super*-agent that governs all $g_i, i \in \mathcal{T}_t$ by solving the optimization problem (17). This allows one to recover several interesting properties of the scheme.

The main properties of the proposed Turn-Based CG scheme are summarized in the following Theorem

Theorem 1. Consider the asymptotically stable systems (1) along with the distributed **TB-CG** (Algorithm 1) selection rule performed by agents in $\mathcal{A}$ distributed into q turns (*super*-agents) $\mathcal{T}_i$. Let assume that at time $t = 0$ an admissible solution $g(0)$ exists for problem (2). Then:

1) for each agent $i \in \mathcal{A}$, at each decision time t, the minimizer in (15) uniquely exists and can be obtained

by locally solving a convex constrained optimization problem;

2) the overall system acted by the agents implementing the TB-CG policy never violates the constraints (2) $t \in \mathbb{Z}_+$;
3) whenever $r(t) \equiv [\ r_1^T, \ldots, r_N^T\ ]^T, \forall t$, with r_i a constant set-point, the sequence of solutions $g(t) = [g_1^T(t), \ldots, g_N^T(t)]^T$ asymptotically converges to a Pareto-Optimal (PO) stationary (constant) solution of the following problem

$$\begin{array}{l}\min\limits_{g}[\| g_1 - r_1 \|^2_{\Psi_1}, \ldots, \| g_i - r_i \|^2_{\Psi_2}, \ldots, \| g_N - r_N \|^2_{\Psi_N}] \\ \text{subject to } g = [g_1^T, .., g_i^T, ..., g_N^T]^T \in \mathcal{W}_\delta\end{array} \tag{18}$$

where $\mathcal{W}_\delta := \{g \in \mathbb{R}^m : Ac_g \le b_\delta\}$ represents a viable steady-state admissible region (see definition of viability in Casavola *et al.* (2014-B); Casavola *et al.*. (2011-B)). The PO solution is given by r whenever $r \in \mathcal{W}_\delta$, or by any other solution $\hat{r} \in \mathcal{W}_\delta$ otherwise. □

Proof: Item 1) is true by construction. Item 2) directly follows from Proposition 1. For what concerns Item 3), because of Lemma 1, the TB-CG scheme is equivalent to the S-CG one carried out by q *super*-agents $\mathcal{T}_i$ which, under viability properties converge to a PO solution for the problem (Casavola *et al.* (2014-B))

$$\begin{array}{l}\min\limits_{g}\left[\sum\limits_{i \in \mathcal{T}_1} ||g_i - r_i(t)||^2_{\Psi_i}, \ldots, \sum\limits_{i \in \mathcal{T}_q} ||g_i - r_i(t)||^2_{\Psi_i}\right] \\ \text{subject to } g = [g_1^T, .., g_i^T, ..., g_N^T]^T \in \mathcal{W}_\delta\end{array} \tag{19}$$

The proof is concluded by noticing that a PO solution for problem (19) is also PO for (18). □

6. ILLUSTRATIVE EXAMPLE

In order to show the effectiveness of the proposed method a set of 21 decoupled particle masses representing vehicles has been considered and depicted in Figure 3. The following equations describe the (i, j)-th mass dynamics

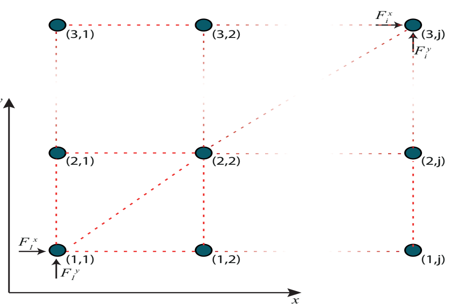

Fig. 3. Planar particle masses. The red dashed edges indicates presence of coupling constraints between two masses according to equations (21)

$$m\ddot{x}_{i,j} = F^x_{i,j}, \quad m\ddot{y}_{i,j} = F^y_{i,j} \tag{20}$$

where $(x_{i,j}, y_{i,j})$, $i \in \{1, 2, ..7\}$, $j \in \{1, 2, 3\}$ are the coordinates of the (i, j)-th mass position w.r.t a fixed Cartesian reference frame and $(F^x_{i,j}, F^y_{i,j})$, $i \in \mathcal{A}$, the components along the same reference frame of the forces acting as inputs for the subsystems. The value $m = 1\ [Kg]$ will be assumed in the simulations. For CG design purposes the models have been discretized with a sampling time of $T_c = 0.1\ [sec]$ and an optimal LQ state-feedback local controller is used as a precompensator for each mass.

Masses are subject to the following local and coupling constraints

$$\begin{cases} \left|F^p_{i,j}(t)\right| \le 2\ [N] \quad p = x, y \\ 0.125[m] \le |x_{i,j+1}(t) - x_{i,j}(t)| \le 0.375[m] \\ 0.125[m] \le |x_{i,j}(t) - x_{i,j-1}(t)| \le 0.375[m] \\ 0.125[m] \le |y_{i+1,j}(t) - y_{i,j}(t)| \le 0.375[m] \\ 0.125[m] \le |y_{i,j}(t) - y_{i-1,j}(t)| \le 0.375[m] \\ 0.125[m] \le |x_{i+1,j+1}(t) - x_{i,j}(t)| \le 0.375[m] \\ 0.125[m] \le |y_{i+1,j+1}(t) - y_{i,j}(t)| \le 0.375[m] \\ 0.125[m] \le |x_{i,j}(t) - x_{i-1,j-1}(t)| \le 0.375[m] \\ 0.125[m] \le |y_{i,j}(t) - y_{i-1,j-1}(t)| \le 0.375[m] \\ \forall t \in \mathbb{Z}_+ \end{cases} \tag{21}$$

The first set of inequalities represents input-saturation constraints on the forces $F^x_{i,j}$ and $F^y_{i,j}$ acting as inputs to the vehicles. They have to be taken into account in order to avoid the generation of control sequences out of the actuator ranges. The second set of constraints represents collision avoidance prescriptions among the vehicles.

Such a constraints structure can be modelled by means of the graph depicted in Figure 3, where each mass represents a vertex while the red dashed edges denote existence of constraints between each two masses. In this case the minimal vertex coloring problem is solved by using only three colors (blu, red, green). As a consequence, the whole CG supervision action is spread among three groups of agents which adopt the Turn-Based policy introduced in Section 3. In particular, in the undertaken simulations, each vehicle has been instructed to track a "circular" reference

$$\begin{array}{rcl} r^x_{i,j}(t) &=& \rho \cos\left((-1)^{i+j} 2\pi t/75\right) + \bar{r}^x_{i,j}, \quad t \in \mathbb{Z}_+ \\ r^y_{i,j}(t) &=& \rho \sin\left((-1)^{i+j} 2\pi t/75\right) + \bar{r}^y_{i,j}, \quad t \in \mathbb{Z}_+ \end{array} \tag{22}$$

with $\rho = 0.125[m]$ is the radius and scalars $\bar{r}^x_{i,j} = -i0.0884[m]$, $\bar{r}^y_{i,j} = -j0.0884[m]$. Simulation results involve a comparison among three CG supervision strategies: the standard CG (Centralized CG) (Bemporad *et al.* (1997)), the distributed sequential agent-based CG (S-CG) (Tedesco *et al.* (2012-A)) and the proposed turn-based distributed sequential CG hereafter referred to as Color-CG where agents perform Algorithm 1.

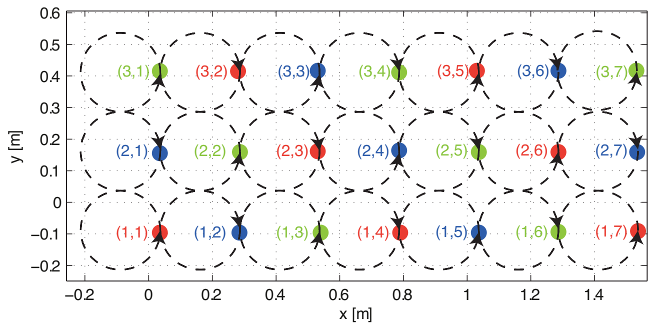

Fig. 4. Planar masses (colored spots) with desired references (black dashed lines)

A video showing the positions of the masses during the simulation (in that video $T = 300[steps]$) for each applied supervising strategy can be downloaded at the url "http://youtu.be/msu1Lu8STus". Within the given simulation horizon T, the centralized CG and the Color-CG strategies are able to cover the entire circle. On the contrary, masses supervised by the S-CG method are not able to track in a proper way the desired reference. These behavior could be also analyzed by observing how the quantity $J(t) := \sum_{i=1}^{7}\sum_{j=1}^{3}(r_{i,j}(t) - g_{i,j}(t))^2$ varies during the simulations. In this respect, Figure 5 shows that Centralized CG and Color-CG have a very similar behavior although the Color-CG makes use of a reduced amount of resources (see Figures 7-8) and each agent only knows local information. As expected the S-CG exhibits the worst performance.

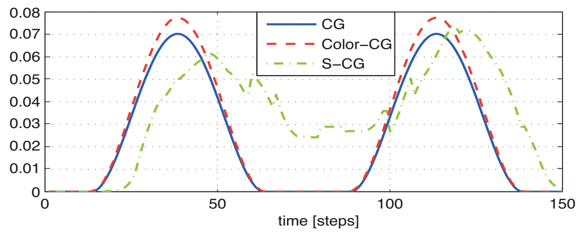

Fig. 5. Trend related to $J(t)$ during the simulation

Further simulations have been carried out by considering scenarios characterized by an increasing number of masses. In particular, systems ranging from 4 masses up to 256 masses (agents) have been taken into account. Comparisons in this case have been performed in terms of scalability of the proposed approaches. Simulation results have been reported in Figures 6-8. Also in this case, proposed Color-CG and centralized CG exhibit the best performance. Moreover, it is evident in Figures 7-8 that one of the main advantages of such turn-based distributed schemes is in the low amount of data exchanged for its implementation and in the related negligible computational burdens that do not increase with the number of agents.

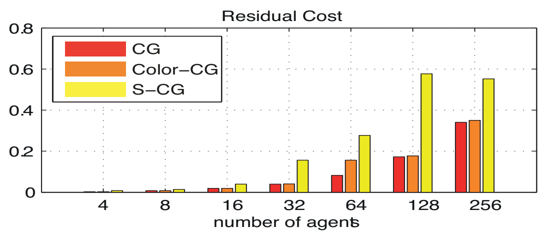

Fig. 6. Residual Cost $\frac{1}{T}\sum_{t=0}^{T} J(t)$

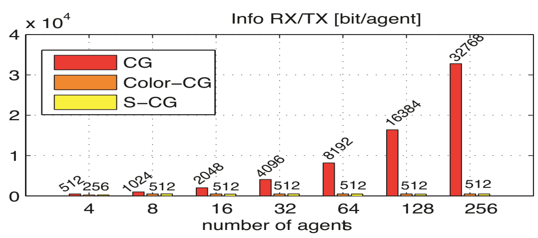

Fig. 7. Information received/transmitted by assuming that 32 bit are needed to encode each scalar

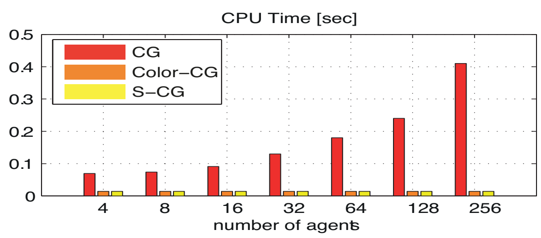

Fig. 8. Mean CPU time (seconds per step) per agent

7. CONCLUSIONS

In this work a novel Turn-Based sequential distributed CG scheme has been proposed as a generalization of the earlier sequential distributed CG scheme of (Tedesco *et al.* (2012-A)).

The round-robin policy that in the earlier scheme allows only a single agent at the time to update its command, here is applied turn-wise, where a turn is a group of agents that can simultaneously update their commands without consequences on the fulfillment of the constraints.

Graph minimal vertex coloring problems has been shown to be instrumental for the determination of turns and the implementation of the Turn-based CG strategy, whose main properties concerning optimality, stability and feasibility have been discussed.

In the final example, the performance of the proposed scheme has been compared with those pertaining to both the centralized and the agent-based distributed schemes presented in (Tedesco *et al.* (2012-A)). It was found that the proposed scheme achieves better scalability and performance.

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