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Switching Piecewise Bilinear Control of Nonlinear Systems with Singularities ⋆

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Abstract: We propose the stabilization of nonlinear systems with singularities via switching piecewise bilinear (PB) control. The approximated model is fully parametric. Input-output (I/O) feedback linearization is applied to stabilize PB control systems. We construct piecewise Lyapunov-like function for each piecewise region and stabilize the state space using a hybrid control method. Examples confirm the feasibility of our proposals.

1. INTRODUCTION

Piecewise linear (PL) systems have been intensively studied in connection with nonlinear systems (see Imura and van der Schaft [2000], Johansson and Rantzer [1998], Sontag [1981]). The original idea was to parametrically approximate a nonlinear function with PL functions by Babayev [1997], Grandin [1986]. An important class of hybrid systems is PL systems with a set of rules for switching among systems (see Imura and van der Schaft [2000]), where state space is divided into polyhedral or polytopic regions, each region associated with a linear (or affine) system (see Babayev [1997], Johansson and Rantzer [1998]). Gain scheduling was also considered with the PL approach by Shamma and Athans [1990]. The PL system concept appears in T-S systems (see Takagi and Sugeno [1985], Tanaka and Wang [2001]) that approximate general nonlinear systems with a number of rules, but unlike the conventional PL approximation, these systems are not fully parametric.

This work concerns parametric piecewise approximation of nonlinear control systems based on the original idea of PL approximation. PL approximation has a general approximation capability for nonlinear functions with given precision, but the PL system obtained is too complex to use for control purposes. To overcome this difficulty, it has been suggested to use piecewise bilinear (PB) approximation by Sugeno [1999]. The PB model has the following features: 1) It is built on hyper cubes partitioned in state space. 2) It has a general approximation capability for nonlinear systems. 3) It is a piecewise nonlinear model and second simplest after the PL model. 4) It is continuous and fully parametric.

Taniguchi and Sugeno [2010a,b, 2012] have derived stabilizing conditions based on feedback linearization, where Taniguchi and Sugeno [2010a] and Taniguchi and Sugeno [2012] apply input-output linearization and Taniguchi and Sugeno [2010b] apply full-state linearization. In feedback linearization, we design a state feedback controller that transforms a nonlinear system into an equivalent linear system. Feedback linearization is a very powerful tool for synthesizing nonlinear control systems, but it is not always applicable because of strict linearization conditions, i.e., the linearizable region is often local.

For this reason, the last three decades have been spent studying approximate linearization via feedback. Approximate linearization was proposed in the literature based on four streams by Guarabassi and Savaresi [2001]: partial linearization (see Krener [1984], Reboulet and Champetier [1984], Rugh [1984]), linearization-oriented modeling (see Guardabassi et al. [1986], Hauser [1991]), nonlinearity measures (see Desoer and Wang [1980], Stack and III [1995]) and linear model matching (see Allgöwer et al. [1994], Isermann et al. [1997]).

In this paper, we propose stabilizing control of nonlinear systems with singularities via switching PB control and apply this method to ball and beam system as nonlinear systems with singularities. The ball and beam model was introduced by Hauser et al. [1989] and an approximate input-output linearization was applied to this system by Hauser et al. [1989], Sastry [1999].

The switching PB control is based on piecewise Lyapunov-like function of a hybrid control (see Pettersson and Lennartson [1997]). Most important advantage is the switching piecewise bilinear control systems are applicable to a wider class of nonlinear systems than conventional piecewise control. Because the piecewise bilinear model can be built on hyper cubes of any size and the stabilizing conditions based on piecewise Lyapunov-like function are less conservative than general Lyapunov theory.

Although the PB models and the PB controllers are simpler than the original nonlinear model and the conventional I/O linearizing controller, the control performance of the switching controller is better than the conventional I/O linearization. The simulation results confirm that the switching controller can easily adjust the control performance.

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This paper is organized as follows. Section 2 presents the canonical form of PB models. In Section 3, we introduce conventional I/O linearization of nonlinear system with singularities. Section 4 proposes the switching controller based on PB models for nonlinear systems with singularities. In Section 5, we show the simulation results, and Section 6 concludes this paper.

2. CANONICAL FORMS OF PIECEWISE BILINEAR MODELS

In this section, we introduce PB models suggested in Sugeno [1999]. We deal with the two-dimensional case without loss of generality. We consider the following nonlinear control system.

$$\begin{cases} \dot{x} = f_o(x) + g_o(x)u(x), \\ y = h_o(x). \end{cases} \tag{1}$$

The PB model (2) is constructed from a nonlinear system (1).

$$\begin{cases} \dot{x} = f(x) + g(x)u(x), \\ y = h(x), \end{cases} \tag{2}$$

where

$$\begin{cases} f(x) = \displaystyle\sum_{i_1=\sigma_1}^{\sigma_1+1}\sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_1^{i_1}(x_1)\omega_2^{i_2}(x_2) f_o(i_1,i_2), \\ g(x) = \displaystyle\sum_{i_1=\sigma_1}^{\sigma_1+1}\sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_1^{i_1}(x_1)\omega_2^{i_2}(x_2) g_o(i_1,i_2), \\ h(x) = \displaystyle\sum_{i_1=\sigma_1}^{\sigma_1+1}\sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_1^{i_1}(x_1)\omega_2^{i_2}(x_2) h_o(i_1,i_2), \\ x = \displaystyle\sum_{i_1=\sigma_1}^{\sigma_1+1}\sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_1^{i_1}(x_1)\omega_2^{i_2}(x_2) d(i_1,i_2), \end{cases} \tag{3}$$

and $f_o(i_1,i_2)$, $g_o(i_1,i_2)$, $h_o(i_1,i_2)$ and $d(i_1,i_2)$ are vertices of the nonlinear system (1). In the above, we assume $f(0,0)=0$ and $d(0,0)=0$ to guarantee $\dot{x}=0$ for $x=0$. $\omega_1^{\sigma_1}$ is normalized membership function of a triangular form as follows:

$$\omega_1^{\sigma_1}(x_1) = \begin{cases} \dfrac{x_1 - d_1(\sigma_1-1)}{d_1(\sigma_1) - d_1(\sigma_1-1)}, x_1 \in [d_1(\sigma_1-1), d_1(\sigma_1)] \\ \dfrac{d_1(\sigma_1+1) - x_1}{d_1(\sigma_1+1) - d_1(\sigma_1)}, x_1 \in [d_1(\sigma_1), d_1(\sigma_1+1)] \end{cases} \tag{4}$$

where $i=1,2$. $\omega_2^{\sigma_2}$ has also the same form. Define vector $d(\sigma_1,\sigma_2)$ and rectangle $R_{\sigma_1\sigma_2}$ in two-dimensional space as

$$\begin{aligned} &d(\sigma_1,\sigma_2) \equiv (d_1(\sigma_1), d_2(\sigma_2))^T, \\ &R_{\sigma_1\sigma_2} \equiv [d_1(\sigma_1), d_1(\sigma_1+1)] \times [d_2(\sigma_2), d_2(\sigma_2+1)]. \end{aligned} \tag{5}$$

σ_1 and σ_2 are integers: $-\infty < \sigma_1, \sigma_2 < \infty$ where $d_1(\sigma_1) < d_1(\sigma_1+1), d_2(\sigma_2) < d_2(\sigma_2+1)$ and $d(0,0) \equiv (d_1(0), d_2(0))^T$. Superscript ^{T} denotes a transpose operation.

A key point in the system is that state variable x is also expressed by a convex combination of $d(i_1,i_2)$ for $\omega_1^i(x_1)$ and $\omega_2^j(x_2)$, just as in the case of $\dot{x}$. As seen in equation (4), x is located inside $R_{\sigma_1\sigma_2}$ which is a rectangle: a hyper cube in general. That is, the expression of x is polytopic with four vertices $d(i_1,i_2)$. The model of $\dot{x} = f(x)$ is built on a rectangle including x in state space, it is also polytopic with four vertices $f(i_1,i_2)$. We call this form of the canonical model (2) parametric expression.

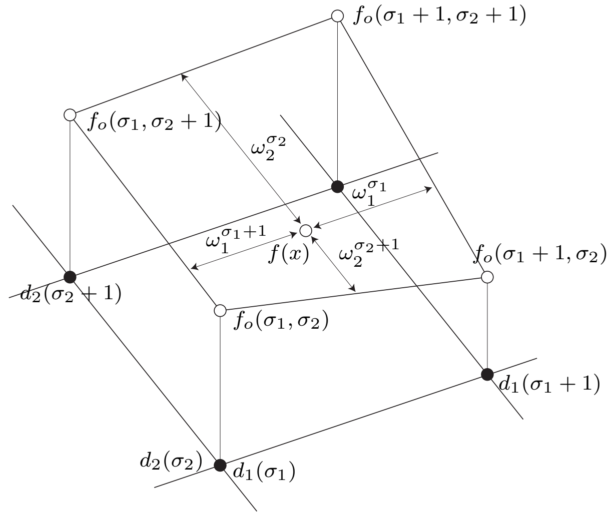

Fig. 1. Piecewise region ($f_o(x)$, $x \in R_{\sigma_1\sigma_2}$)

Representing $\dot{x}$ with x in Eqs. (2) and (4), we obtain the state space expression of the model found to be bilinear (biaffine) (see Sugeno [1999]), so the derived PB model has simple nonlinearity. In PL approximation, a PL model is built on simplexes partitioned in state space, triangles in the two-dimensional case. Note that any three points in three-dimensional space are spanned with an affine plane: $y = a + bx_1 + cx_2$. A PL model is continuous. It is, however, difficult to handle simplexes in the rectangular coordinate system.

Note that any four points in three-dimensional space are spanned with a biaffine plane: $y = a + bx_1 + cx_2 + dx_1x_2$. In contrast to a PL model, a PB model as such is built on rectangles with the four vertices $d(i_1,i_2)$, on hyper cubes in general dimensional space, partitioned in state space; it matches the rectangular coordinate system well, so PB models are applicable to control purposes.

The modeling procedure in region $R_{\sigma_1\sigma_2}$ is as follows:

(1) Assign vertices $d(i_1,i_2)$ for $x_1 = d_1(\sigma_1)$, $d_1(\sigma_1+1)$, $x_2 = d_2(\sigma_2)$, $d_2(\sigma_2+1)$ of state vector x, then partition state space into piecewise regions (Figure 1).
(2) Compute vertices $f_o(i_1,i_2)$, $g_o(i_1,i_2)$ and $h_o(i_1,i_2)$ in equation (3) by substituting values of $x_1 = d_1(\sigma_1)$, $d_1(\sigma_1+1)$ and $x_2 = d_2(\sigma_2)$, $d_2(\sigma_2+1)$ into original nonlinear functions $f_o(x)$, $g_o(x)$ and $h_o(x)$ in the system (1). Fig. 1 shows the expression of $f(x)$ and $x \in R_{\sigma_1\sigma_2}$.

The overall PB model is obtained automatically when all vertices are assigned. Note that $f(x)$, $g(x)$ and $h(x)$ in the PB model coincide with those in the original system at vertices of all regions.

3. INPUT-OUTPUT LINEARIZATION OF NONLINEAR SYSTEMS WITH SINGULARITIES

Consider the ball and beam system (see Sastry [1999]):

$$\begin{cases} \dot{x} = f_o + g_o u \\ y = h_o \end{cases} \tag{6}$$

where

$$x = (x_1,\ x_2,\ x_3,\ x_4)^T = \left(r,\ \dot{r},\ \theta,\ \dot{\theta}\right)^T,$$

$$f_o = \begin{pmatrix} x_2 \\ B(x_1 x_4^2 - g\sin x_3) \\ x_4 \\ 0 \end{pmatrix},\ g_o = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ h_o = x_1$$

Fig. 2 shows the ball and beam system. In this system,

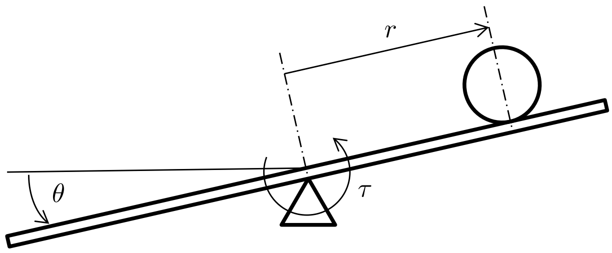

Fig. 2. Ball and beam system

r is the ball position, θ is the beam angle, $B(= 0.7143)$ is the constant and $g(= 9.81)$ is the acceleration of gravity.

We calculate the time derivatives of the output y until the input u appears.

$$\begin{cases} y = h_o = x_1, \\ \dot{y} = L_{f_o} h_o = x_2, \\ \ddot{y} = L_{f_o}^2 h_o = B(x_1 x_4^2 - g\sin x_3), \\ y^{(3)} = L_{f_o}^3 h_o + L_{g_o} L_{f_o}^2 h_o u \\ \quad = Bx_2 x_4^2 - Bgx_4\cos x_3 + 2Bx_1 x_4 u \end{cases} \tag{7}$$

Then the controller is obtained as

$$u = \frac{-L_{f_o}^3 h_o}{L_{g_o} L_{f_o}^2 h_o} + \frac{1}{L_{g_o} L_{f_o}^2 h_o} v,$$

where v is linear controller for the linearized system of (6). But the relative degree is 3 except at $x_1 x_4 = 0$ and the relative degree is not well defined at $x_1 x_4 = 0$. The zero dynamics of unobservable state is stable at $x(0) = 0$.

The approximation I/O linearization methods for (6) were proposed by Hauser et al. [1989]. One of the approximation system is obtained as

$$\begin{cases} y = h_o = x_1, \\ \dot{y} = L_{f_o} h_o = x_2, \\ \ddot{y} = L_{f_o}^2 h_o = B(x_1 x_4^2 - g\sin x_3), \\ y^{(3)} = L_{f_o}^3 h_o \equiv -Bgx_4\cos x_3, \\ y^{(4)} = L_{f_o}^4 h_o + L_{g_o} L_{f_o}^3 h_o u \\ \quad = Bx_4^2 \sin x_3 - Bg\cos x_3 u \end{cases} \tag{8}$$

The approximation is to drop $x_1 x_4^2$ of $L_{f_o}^2 h_o$. Then the controller is obtained as

$$u = \frac{-L_{f_o}^4 h_o}{L_{g_o} L_{f_o}^3 h_o} + \frac{1}{L_{g_o} L_{f_o}^3 h_o} v, \tag{9}$$

where v is linear controller for the linearized system of (6).

Switching control of the ball and beam system was presented by Sastry [1999]. This control scheme switches between the approximate tracking law in a neighborhood of the singularity $x_1 x_4 = 0$.

4. SWITCHING CONTROL FOR NONLINEAR SYSTEMS WITH SINGULARITIES BASED ON PIECEWISE BILINEAR MODELS

We construct two PB models of the ball and beam system (6). One is with respect to the neighborhood of the singularities. The second one is the other regions. We divide state space of the ball and beam system as

$$\begin{cases} x_1 \in \{-3,\ -1.5,\ -\delta_1,\ 0, \delta_1,\ 1.5,\ 3\}, \\ x_2 \in \{-2,\ -1,\ 0,\ 1,\ 2\} \\ x_3 \in \{-\pi/2,\ -\pi/4,\ 0,\ \pi/4,\ \pi/2\} \\ x_4 \in \{-2,\ -1,\ -\delta_2,\ 0, \delta_2,\ 1,\ 2\}, \end{cases} \tag{10}$$

where $\|x_1\| \le \delta_1$ and $\|x_4\| \le \delta_2$ are the piecewise regions with the singularities as shown in Section 3. The PB model is constructed as

$$\begin{cases} \dot{x} = f + gu = (f_1,\ f_2,\ f_3,\ 0)^T + (0,\ 0,\ 0,\ 1)^T u, \\ y = h = x_1, \end{cases} \tag{11}$$

where

$$f_1 = \sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_2^{i_2}(x_2) f_{o_1}(i_2),\ f_3 = \sum_{i_4=\sigma_4}^{\sigma_4+1} \omega_4^{i_4}(x_4) f_{o_3}(i_4),$$

$$f_2 = \sum_{i_1=\sigma_1}^{\sigma_1+1} \sum_{i_4=\sigma_4}^{\sigma_4+1} \omega_1^{i_1}(x_1)\omega_4^{i_4}(x_4) f_{o_{21}}(i_1, i_4) + \sum_{i_3=\sigma_3}^{\sigma_3+1} \omega_3^{i_3}(x_3) f_{o_{22}}(i_3),$$

$$h = \sum_{i_1=\sigma_1}^{\sigma_1+1} \omega_1^{i_1}(x_1) d_1(i_1).$$

The vertex values of $f_{o_1}(i_2)$, $f_{o_{21}}(i_1, i_4)$, $f_{o_{22}}(i_3)$ and $f_{o_3}(i_4)$ are calculated by substituting values of (10) into the system (6). Table 1 shows the vertex values of $f_{o_1}(i_2)$ and $f_{o_{22}}(i_3)$. Table 2 is about the vertex values of $f_{o_{21}}(i_1, i_4)$

Note that the PB models can be constructed using only the vertex values. The internal models are obtained by the convex combinations of the vertices for $\omega_j^i(x_i)$'s as shown in Fig. 1.

Table 1. Vertex values of $f_{o_1}(i_2)$ and $f_{o_{22}}(i_3)$

$f_{o_1}(1)$	$f_{o_1}(2)$	$f_{o_1}(3)$	$f_{o_1}(4)$	$f_{o_1}(5)$
-2	-1	0	1	2

$f_{o_{22}}(1)$	$f_{o_{22}}(2)$	$f_{o_{22}}(3)$	$f_{o_{22}}(4)$	$f_{o_{22}}(5)$
-1	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	1

4.1 Controller design in the regions with singularities

First, we design the PB controller in the neighborhood of the singularities. We calculate the time derivatives of the output y until the input u appears.

$$\begin{cases} y = h, \\ \dot{y} = L_f h = f_1, \\ \ddot{y} = L_f^2 h = f_2, \\ y^{(3)} = L_f^3 h = \dfrac{\partial f_2'}{\partial x} f = \dfrac{\partial f_2'}{\partial x_3} f_3, \\ y^{(4)} = L_f^4 h + L_g L_f^3 h u \ = \dfrac{\partial}{\partial x_4}\left\{\dfrac{\partial f_2'}{\partial x_3} f_3\right\} u \end{cases}$$

Table 2. Vertex values of $f_{o_{21}}(i_1, i_4)$

$f_{o_{21}}(i_1,i_4)$	$i_1=1$	$i_1=2$	$i_1=3$	$i_1=4$
$i_4=1$	4.29	2.14	$1.43\delta_1$	0
$i_4=2$	2.14	1.07	$0.71\delta_1$	0
$i_4=3$	$2.14\delta_2$	$1.07\delta_2$	$0.71\delta_1\delta_2$	0
$i_4=4$	0	0	0	0
$i_4=5$	$-2.14\delta_2$	$-1.07\delta_2$	$-0.71\delta_1\delta_2$	0
$i_4=6$	-2.14	-1.07	$-0.71\delta_1$	0
$i_4=7$	-4.29	-2.14	$-1.43\delta_1$	0
$f_{o_{21}}(i_1,i_4)$	$i_1=5$	$i_1=6$	$i_1=7$	
$i_4=1$	$-1.43\delta_1$	-2.14	-4.29	
$i_4=2$	$-0.71\delta_1$	-1.07	-2.14	
$i_4=3$	$-0.71\delta_1\delta_2$	$-1.07\delta_2$	$-2.14\delta_2$	
$i_4=4$	0	0	0	
$i_4=5$	$0.71\delta_1\delta_2$	$1.07\delta_2$	$2.14\delta_2$	
$i_4=6$	$0.71\delta_1$	1.07	2.14	
$i_4=7$	$1.43\delta_1$	2.14	4.29	

where

$$
\begin{aligned}
f_2' =& \sum_{i_1=\sigma_1}^{\sigma_1+1} \sum_{i_4=\sigma_4}^{\sigma_4+1} \omega_1^{i_1}(0)\omega_4^{i_4}(0) f_{o_{21}}(i_1,i_4) \\
&+ \sum_{i_3=\sigma_3}^{\sigma_3+1} \omega_3^{i_3}(x_3) f_{o_{22}}(i_3), \\
\frac{\partial f_2'}{\partial x_3} f_3 =& \frac{f_{o_{22}}(\sigma_3+1) - f_{o_{22}}(\sigma_3)}{\Delta d_3} \sum_{i_4=\sigma_4}^{\sigma_4+1} \omega_4^{i_4}(x_4) f_{o_3}(i_4), \\
\frac{\partial}{\partial x_4}\left\{\frac{\partial f_2'}{\partial x_3} f_3\right\} =& \frac{f_{o_3}(\sigma_4+1) - f_{o_3}(\sigma_4)}{\Delta d_4} \\
&\times \frac{f_{o_{22}}(\sigma_3+1) - f_{o_{22}}(\sigma_3)}{\Delta d_3} \\
\Delta d_3 =& d_3(\sigma_3+1) - d_3(\sigma_3), \\
\Delta d_4 =& d_4(\sigma_4+1) - d_4(\sigma_4).
\end{aligned}
$$

f_2' is to drop $x_1 x_4^2$ of f_2 in the same manner as the previous section. The relative degree is 4. The input-output linearized system is formulated as

$$
\begin{cases} \dot{z}_s = A_s z_s + B_s v, \\ \;y = C_s z_s, \end{cases}
$$

where $z_s = (h,\ L_f h,\ L_f^2 h,\ L_f^3 h)^T \in \Re^4$,

$$
A_s = \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix},\ B_s = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ C_s = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}^T.
$$

Then the controller is obtained as

$$
u = \frac{1}{L_g L_f^3 h} v, \tag{12}
$$

where $v = -F_s z_s$ is linear controller for the linearized system of (6). Note that the controller (12) is linear because the denominator $L_g L_f^3 h$ is constant.

4.2 Controller design in the other regions

Next, we design the PB controller in the other piecewise regions. We calculate the time derivatives of the output y until the input u appears.

$$
\begin{cases}
\;\;\; y =h, \\
\;\;\; \dot{y} =L_f h = f_1, \\
\;\;\; \ddot{y} =L_f^2 h = f_2, \\
y^{(3)} =L_f^3 h + L_g L_f^2 h u = \dfrac{\partial f_2}{\partial x} f + \dfrac{\partial f_2}{\partial x_4} u \\
\qquad = \dfrac{\partial f_2}{\partial x_1} f_1 + \dfrac{\partial f_2}{\partial x_3} f_3 + \dfrac{\partial f_2}{\partial x_4} f_4 + \dfrac{\partial f_2}{\partial x_4} u
\end{cases}
$$

where

$$
\begin{aligned}
\frac{\partial f_2}{\partial x_1} =& \sum_{i_4=\sigma_4}^{\sigma_4+1} \omega_4^{i_4}(x_4) \frac{f_{o_{21}}(\sigma_1+1, i_4) - f_{o_{21}}(\sigma_1, i_4)}{\Delta d_1}, \\
\frac{\partial f_2}{\partial x_3} =& \frac{f_{o_{22}}(\sigma_3+1) - f_{o_{22}}(\sigma_3)}{\Delta d_3}, \\
\frac{\partial f_2}{\partial x_4} =& \sum_{i_1=\sigma_1}^{\sigma_1+1} \omega_1^{i_1}(x_1) \frac{f_{o_{21}}(i_1, \sigma_4+1) - f_{o_{21}}(i_1, \sigma_4)}{\Delta d_4}, \\
\Delta d_1 =& d_1(\sigma_1+1) - d_1(\sigma_1), \\
\Delta d_3 =& d_3(\sigma_3+1) - d_3(\sigma_3), \\
\Delta d_4 =& d_4(\sigma_4+1) - d_4(\sigma_4).
\end{aligned}
$$

The relative degree is 3 except at $x_1 = 0$ and the relative degree is not well defined at $x_1 = 0$. Note that the singularity is not the same as the conventional I/O linearization (7). The I/O linearized system is formulated as

$$
\begin{cases} \dot{z} = Az + Bv, \\ y = Cz, \end{cases} \tag{13}
$$

where $z = (h,\ L_f h,\ L_f^2 h)^T \in \Re^3$,

$$
A = \begin{pmatrix} 0&1&0\\ 0&0&1\\ 0&0&0 \end{pmatrix},\ B = \begin{pmatrix}0\\0\\1\end{pmatrix},\ C = \begin{pmatrix}1\\0\\0\end{pmatrix}^T.
$$

The zero dynamics of unobservable state is also stable at $x(0) = 0$. Then the controller is obtained as

$$
u = \frac{-L_f^3 h}{L_g L_f^2 h} + \frac{1}{L_g L_f^2 h} v, \tag{14}
$$

where $v = -Fz$ is linear controller for the linearized system of (13). The controller can not be applied to piecewised regions in the neighborhood of singularities.

4.3 Stability using the switching controller

We design the stabilizing controller which consists of the controllers (12) and (14) using the following theorem proposed by Pettersson and Lennartson [1997].

Assume that the hybrid state space is partitioned into l disjoint regions Ω_q, $q \in I_l$, where $I_l = \{1, \ldots, l\}$. In each region Ω_q, $q \in I_l$, a scalar function $V_q(x,t)$ is used as a measure of the hybrid system's energy. Let the overall energy be defined by

$$
V(x,t) = V_q(x,t) \quad \text{when } (x,m) \in \Omega_q
$$

which in general is a discontinuous function. Let Ω_q^x denote the continuous states in Ω_q. Let the switch region Λ_{qr} be the set of continuous states for which the trajectories $x(t)$, with initial states $(x_0, m_0) \in I_0$, pass from Ω_q to Ω_r, i.e.:

$$
\Lambda_{qr} = \{x \in \Re^n \mid \exists t \geq t_0 \text{ such that } x(t^-) \in \Omega_q,\ x(t) \in \Omega_r\}
$$

Theorem 1. If there exists scalar functions $V_q : \Omega_q^x \times \Re \to \Re$, $q \in I_l$, and class K functions $\alpha : \Re^+$ and $\beta : \Re^+ \to \Re^+$ such that

- $\forall(x,m) \in \Omega_q^x$, $\alpha(\|x\|) \leq V_q(x,t) \leq \beta(\|x\|)$, $q \in I_l$
- $\forall(x,m) \in \Omega_q$, $\dot{V}_q(x) \leq 0$, $q \in I_l$
- $\forall x \in \Lambda_{qr}$, $V_r(x) \leq V_q(x)$, $(q,r) \in I_\Lambda$

then 0 is uniformly stable in the sense of Lyapunov.

In this theorem, state space is divided into l regions and Lyapunov-like function V_i is considered for each region. This theorem has the following features:

- This theorem can be applied to $f(x)$ which is enough to be locally Lipschitz.
- Origin is not necessary to be all equilibrium points of $f(x)$.

5. EXAMPLES

We construct Lyapunov-like functions based on Theorem 1. In the neighborhood of singularities, the Lyapunov-like function and linear controller are designed as

$$\begin{aligned} V_s(x) =& z_s^T P_s z_s, \ \dot{V}_s(x) < 0, \\ P_s =& \begin{pmatrix} 3.078 & 4.236 & 3.078 & 1.000 \\ 4.236 & 9.960 & 8.472 & 3.078 \\ 3.078 & 8.472 & 9.960 & 4.236 \\ 1.000 & 3.078 & 4.236 & 3.078 \end{pmatrix} > 0, \\ F_s =& (1.000, \ 3.078, \ 4.236, \ 3.078), \end{aligned}$$

where $\|x_1\| \leq \delta_1 = 0.15$ and $\|x_4\| \leq \delta_2 = 0.2$.

In the other regions, the Lyapunov-like function and linear controller are designed as

$$\begin{aligned} V_o(x) =& z_o^T P_o z_o, \ \dot{V}_o(x) < 0, \\ P_o =& \begin{pmatrix} 10.48 & 8.477 & 2.236 \\ 8.477 & 15.53 & 4.685 \\ 2.236 & 4.685 & 3.791 \end{pmatrix} > 0, \\ F_o =& (2.236, \ 4.685, \ 3.791), \end{aligned}$$

where $\|x_1\| > \delta_1 = 0.15$ and $\|x_4\| > \delta_2 = 0.2$. Finally, we design the following Lyapunov-like function.

$$V(x) = \begin{cases} V_s(x), x \in \{x \mid \|x_1\| \leq \delta_1, \|x_4\| \leq \delta_2\}, \\ V_o(x), x \in \{x \mid \|x_1\| > \delta_1, \|x_4\| > \delta_2\}. \end{cases}$$

We apply the switching controller to the ball and beam system (6). Fig. 4 shows the state response x from the initial value $x(0) = (2.4, \ -0.1, \ 0.6, \ 0.1)^T$. Fig. 3 shows the Lyapunov function. In this case, the switching controller switches from the controller (12) to controller (14). Then the controller switches from the controller (14) to controller (12). Fig. 5 shows the state response x from the same initial value using the conventional I/O linearizing controller (9).

Figs. 4 and 5 show the overshoot responses using the switching controller are lower than the responses of conventional I/O linearizing controller. Fig. 6 shows the control inputs with respect to the switching controller and the conventional I/O linearizing controller. The results confirm that the proposed switching controller can easily adjust the control performance. Although the PB model (11) and controllers (12) and (14) are simpler than the model (6) and the conventional I/O linearizing controller (9), the control performance is better than the conventional I/O linearization.

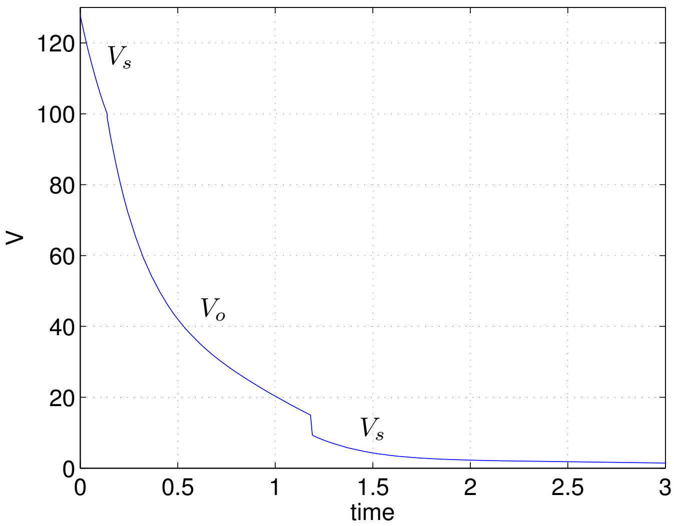

Fig. 3. Lyapunov functions of the switching controller

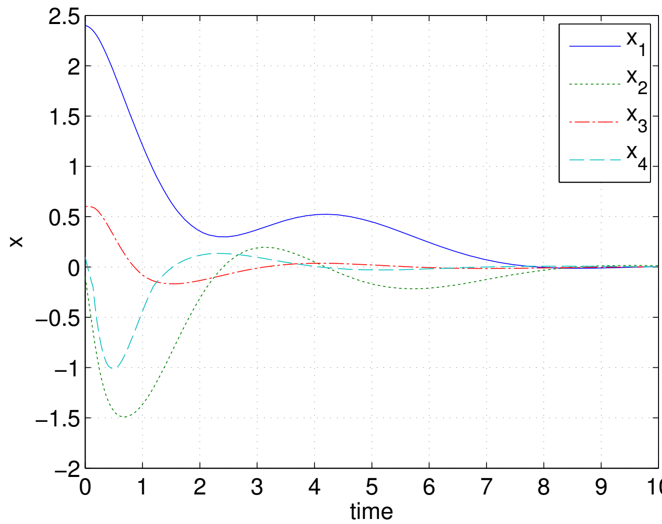

Fig. 4. State responses x_i using the switching controller

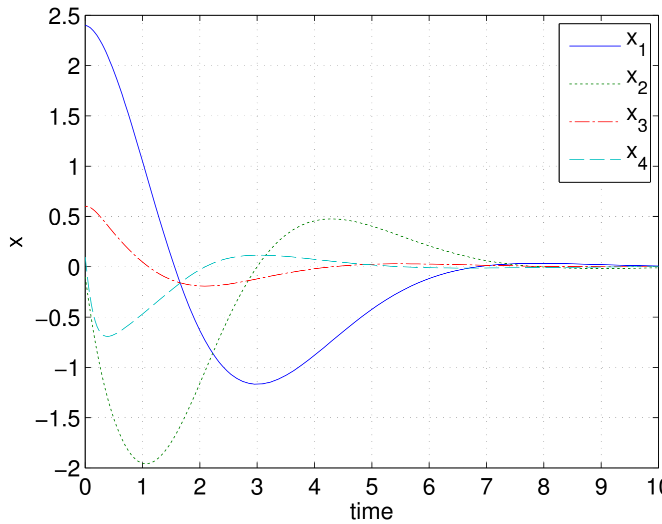

Fig. 5. State responses x_i using the conventional I/O linearizing controller

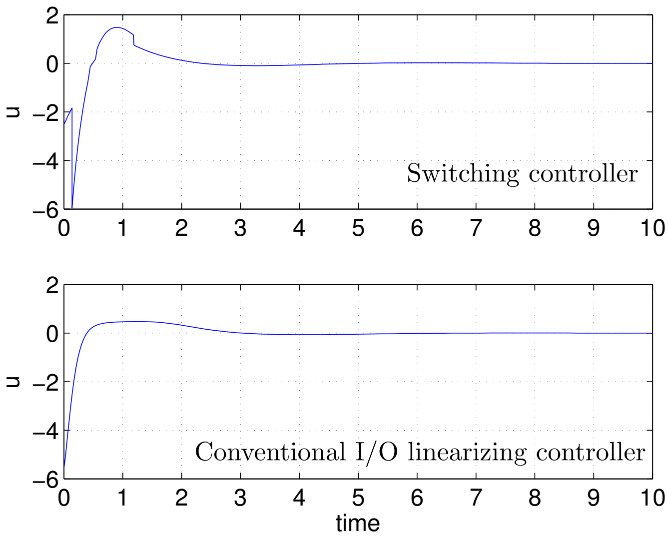

Fig. 6. Control inputs using the switching controller and the conventional I/O linearizing controller

6. CONCLUSION

We have proposed the stabilization of nonlinear systems with singularities via switching piecewise bilinear control. The approximated model is fully parametric. Input-output feedback linearization is applied to stabilize PB control systems. We have constructed piecewise Lyapunov-like function for each piecewise region and stabilized the state space using a hybrid control method. Illustrative examples have been given to show the feasibility of proposed methods.

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