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Passivity Based Control of a Quadrotor UAV

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Abstract: This work presents an IDA-PBC control methodology for a quadrotor helicopter to perform path tracking. The control law is designed considering the trajectory tracking formulation for underactuated systems. The goal is to regulate the controlled degrees of freedom (DOF), the translational position, x, y and z, and the yaw angle, ψ, and stabilize the remaining DOF, the roll and pitch angles, ϕ and θ, respectively, using only one control loop. This formulation leads to a set of partial differential equations constraints due to the underactuation degree of the system. The IDA-PBC controller performance is corroborated through simulation results, being compared with a backstepping controller.

Keywords: Passivity based control, underactuated mechanical systems, quadrotor helicopter, port-controlled hamiltonian.

1. INTRODUCTION

Autonomous or semi-autonomous unmanned aerial vehicles (UAVs) are increasingly common and have been the subject of several research and academic works due to their broad applicability and the challenge offered in the control area. UAVs exist in several configurations, but one stands out, the quadrotor, due to its ability to take off and land vertically, perform stationary flight, high manoeuvrability, low cost and maintenance (Erginer and Altug, 2007).

Quadrotor is a rotary-wing aircraft that makes use of four rotors in which movement is performed by accelerating and decelerating them. This kind of helicopter tries to achieve flight stability using equilibrium forces produced by its four rotors (Raffo et al., 2010). Despite all the advantages that quadrotors present, there are some limiting factors in their application: the high energy consumption and difficulty in controlling them, since they have highly nonlinear behaviour, high degree of coupling, underactuated and unstable dynamics, and are constantly affected by external disturbances.

The classic control strategy for helicopters assumes a linear model obtained in a particular operating point, however the application of the nonlinear modern control theory can improve the system performance and allow tracking of aggressive trajectories (Castillo et al., 2005).

In Bouabdallah and Siegwart (2005), the UAV dynamic modeling was performed and two controllers, based on backstepping and sliding mode techniques, were compared. It was showed that the sliding mode control introduces high frequency vibrations of low amplitude due to their switching behavior, while the backstepping control proved to be able to control the rotational motion in the presence of large disturbances. Several variations of control laws using backstepping technique have been proposed in recent years to control the quadrotor helicopter (Bouabdallah and Siegwart, 2007; Al-Younes and Jarrah, 2008; Fang and Gao, 2011; Saif et al., 2012).

Other control techniques have also been used to improve the performance of the quadrotor in the path tracking task. In Das et al. (2009), a control strategy with two feedback loops was proposed. A control technique based on the inverse dynamics is applied to the internal loop to perform hovering. As the quadrotor dynamics are unstable, a robust control law was applied to the outer loop to stabilize the residual dynamics. A predictive control with integral action and a robust nonlinear control have been developed in Raffo et al. (2010) and Raffo et al. (2011b). The predictive control is used to track the reference trajectory while a nonlinear controller $\mathscr{H}_\infty$ stabilizes the rotational movements. Both consider the integral of the position error to reject disturbances. In Raffo et al. (2011b) was used the same predictive controller for the control of X and Y, whereas a nonlinear controller $\mathscr{H}_\infty$ for underactuated systems was developed for altitude and attitude control.

In Raffo et al. (2011a) a control law based on the nonlinear $\mathscr{H}_\infty$ theory for underactuated mechanical systems was proposed for guidance of a modified model of quadrotor. This model considers a tilt angle of the rotors to create coupling between longitudinal and lateral movements and pitch and roll motion. This procedure increases the controllability of the quadrotor helicopter avoiding the necessity of using cascade controllers or augmented space state.

The objective of this work consists in exploring the application of a nonlinear control technique to solve the path tracking problem for a mechanical system with underactuation degree two, the quadrotor helicopter. The control technique used is the IDA-PBC. The Interconnection and

Damping Assignment (IDA) is a Passivity Based Control (PBC) methodology, presented in Ortega et al. (2002b), which is based on systems described by the Port Controlled Hamiltonian (PCH) model. This methodology consists in assigning a new PCH model in closed loop through the interconnection and damping assignment.

In Ortega et al. (2002a), the IDA-PBC methodology is applied to the problem of stabilizing underactuated mechanical systems, which leads to restrictions in the form of partial differential equations to ensure asymptotic stability. In this work it was proved that the controlled Lagrangian method can be viewed as a special case of IDA-PBC methodology. In Wang and Goldsmith (2008), a modified IDA-PBC methodology was presented to solve the trajectory tracking problem of systems described by Euler-Lagrange equations. This new approach gives a new interpretation to the well known controllers as computed torque, PD + (Proportional Derivative) control and Slotine-Li. The new equations allow the application of IDA-PBC to trajectory tracking to a wider range of HPC systems and some underactuated non-passive systems.

Considering the quadrotor context, the possibility to incorporate information about the underactuated system structure and to deal with the concept of energy in the control strategy, and considering the lack of works involving passivity based control for this type of aircraft to perform path tracking, motivated the research in this area.

This paper is organized as follows: in section 2 the model used in the control design is developed; section 3 presents the IDA-PBC control law design for the path tracking problem; the simulation results obtained with the quadrotor helicopter model are illustrated in section 4; finally, section 5 concludes this paper.

2. QUADROTOR MODELLING

The quadrotor dynamic equations can be easily found through Euler-Lagrange formulation and expressed as follows:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{B}(\mathbf{q})\mathbf{\Gamma} + \mathbf{\Gamma}_d \tag{1}$$

$$\begin{bmatrix} m\mathbb{1}_{3\times 3} & -m\mathbf{R}_{\mathscr{I}}\mathbf{S}(\mathbf{r})\mathbf{W}_{\boldsymbol{\eta}} \\ -m\mathbf{W}_{\boldsymbol{\eta}}'\mathbf{S}(\mathbf{r})'\mathbf{R}_{\mathscr{I}}' & \mathbf{J}(\eta) \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\xi}} \\ \ddot{\boldsymbol{\eta}} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{\xi\dot{\xi}}(\mathbf{q},\dot{\mathbf{q}}) & \mathbf{C}_{\xi\dot{\eta}}(\mathbf{q},\dot{\mathbf{q}}) \\ \mathbf{C}_{\eta\dot{\xi}}(\mathbf{q},\dot{\mathbf{q}}) & \mathbf{C}_{\eta\dot{\eta}}(\mathbf{q},\dot{\mathbf{q}}) \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\xi}} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{\xi}(\mathbf{q}) \\ \mathbf{G}_{\eta}(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\xi}(\mathbf{q}) \\ \mathbf{B}_{\eta}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} T \\ \boldsymbol{\tau}_{\mathbf{a}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\delta}_{\xi}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},f_{\xi_{\mathbf{d}}}) \\ \boldsymbol{\delta}_{\eta}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\boldsymbol{\tau}_{\eta_{\mathbf{d}}}) \end{bmatrix} \tag{2}$$

The inertia matrix, $\mathbf{M}(\mathbf{q})$, is composed by the helicopter total mass, m, the Euler matrix, $\mathbf{W}_{\boldsymbol{\eta}}$, the skew symmetric matrix, $\mathbf{S}(\mathbf{r})$, which is defined through the coordinates of the vector $\mathbf{r}$, the rotational matrix $\mathbf{R}_{\mathscr{I}}$ and the rotational subsystem inertia matrix $\mathbf{J}(\boldsymbol{\eta})$. The vector $\mathbf{r}$ is the distance between the center of the mass and the center of rotation, as can be seen in Figure 1. The matrix $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ represents the Coriolis and centrifugal term, which can be obtained through the Christoffel symbols of the first kind, $\mathbf{G}(\mathbf{q})$ is the gravitational force vector, $\mathbf{B}(\mathbf{q}) = [\mathbf{b_1}(\mathbf{q}),...,\mathbf{b_m}\mathbf{q}] \in \Re^{n\times m}$ is the input matrix, $\mathbf{\Gamma} \in U$ is the control vector composed by the total thrust T and the torques $\boldsymbol{\tau}_{\mathbf{a}}$, being U the m-dimensional actuation space, and $\mathbf{\Gamma}_d$ represents the total effect of modeling errors and external disturbances. The generalized coordinates vector $\mathbf{q}$ is defined as follows:

$$\mathbf{q} = [\boldsymbol{\xi}' \ \ \boldsymbol{\eta}']' = [x \ \ y \ \ z \ \ \phi \ \ \theta \ \ \psi]', \tag{3}$$

that express the quadrotor position and orientation in Euclidean space.

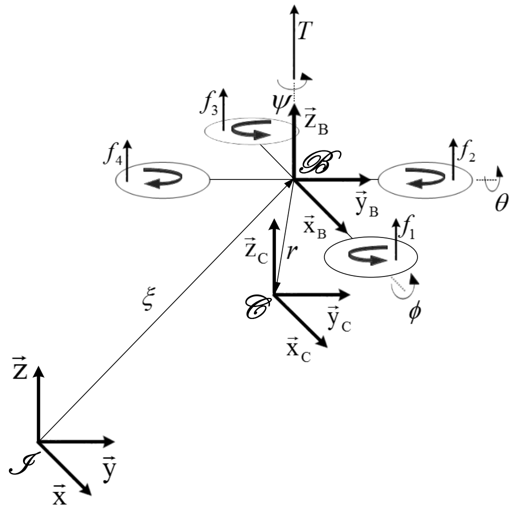

Fig. 1. Quadrotor helicopter scheme.

The model used to design the controller is a simplified one, which the center of rotation and the center of mass are assumed to be congruent, and the mechanical structure is assumed to be symmetrical, resulting in a diagonal inertia tensor matrix (see Figure 2). Additionally, a tilt of the rotors toward the center of the quadrotor, originally proposed in Raffo et al. (2011a), is adopted with the purpose of increasing the controllability and the stability of the model. This structural change, which is considered only in the controller design, creates a force concentration point above the helicopter and coupling in the control input vector, causing the quadrotor to behave like a pendulum and avoiding the need to use an augmented state space or cascade control strategies.

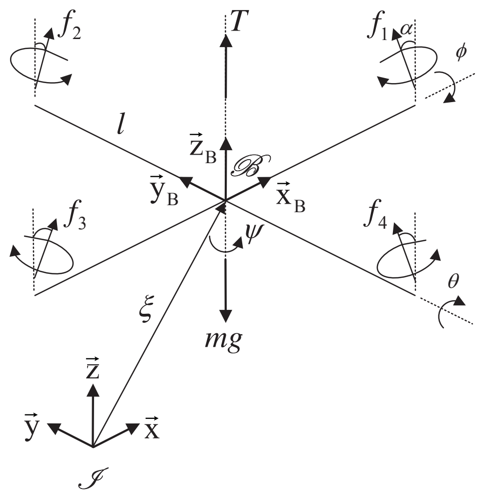

Fig. 2. Quadrotor helicopter scheme for control purposes.

The force of each rotor depends directly on the rotational speed of the propeller, that can be approximated according to the following function, $f_i = b\Omega_i^2$, where b is the thrust coefficient of each engine. The forces projected into each quadrotor axis are given by:

$$f_a = \begin{bmatrix} f_{x_a} \\ f_{y_a} \\ f_{z_a} \end{bmatrix} = \begin{bmatrix} \sin(\alpha)(f_3 - f_1) \\ \sin(\alpha)(f_4 - f_2) \\ \sum_{i=1}^{4} \cos(\alpha) f_i \end{bmatrix}, \tag{4}$$

being α the tilt angle of the rotor.

The applied torques with respect to the body fixed frame are obtained as follows:

$$\boldsymbol{\tau_a} = \begin{bmatrix} \tau_{xa} \\ \tau_{ya} \\ \tau_{za} \end{bmatrix} = \begin{bmatrix} \cos(\alpha)(f_2 - f_4)l \\ \cos(\alpha)(f_3 - f_1)l \\ \sum_{i=1}^{4} \cos(\alpha)\tau_{Mi} \end{bmatrix}, \tag{5}$$

where l is the distance between the center of the rotors and the quadrotor rotation center and τ_{Mi} is the torque generated by each rotor, which can be approximated in steady state by $\tau_{Mi} = k_\tau \Omega_i^2$, with $k_\tau > 0$ being the rotor drag constant.

The input matrix can be rewritten as function of the square of each rotor angular speed:

$$\mathbf{B}(\mathbf{q})\boldsymbol{\Gamma} = \mathbf{B}(\mathbf{q})\mathbf{B_M}\mathbf{u_M} = \mathbf{B}_{\mathscr{I}}(\mathbf{q})\mathbf{u_M} =$$

$$= \begin{bmatrix} \mathbf{R}_{\mathscr{I}} & 0 \\ 0 & \mathbf{W}'_{\boldsymbol{\eta}} \end{bmatrix} \begin{bmatrix} -b\mathbf{S}(\alpha) & 0 & b\mathbf{S}(\alpha) & 0 \\ 0 & -b\mathbf{S}(\alpha) & 0 & b\mathbf{S}(\alpha) \\ b\mathbf{C}(\alpha) & b\mathbf{C}(\alpha) & b\mathbf{C}(\alpha) & b\mathbf{C}(\alpha) \\ 0 & lb\mathbf{C}(\alpha) & 0 & -lb\mathbf{C}(\alpha) \\ -lb\mathbf{C}(\alpha) & 0 & lb\mathbf{C}(\alpha) & 0 \\ k\tau\mathbf{C}(\alpha) & -k\tau\mathbf{C}(\alpha) & k\tau\mathbf{C}(\alpha) & -k\tau\mathbf{C}(\alpha) \end{bmatrix} \begin{bmatrix} \Omega_1^2 \\ \Omega_2^2 \\ \Omega_3^2 \\ \Omega_4^2 \end{bmatrix},$$

where $\mathbf{W}_{\boldsymbol{\eta}}$ is given by:

$$\mathbf{W}_{\boldsymbol{\eta}} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}.$$

Given the simplifications and the proposed structural modification of the quadrotor, the following model is obtained:

$$\begin{bmatrix} m\mathbb{1}_{3\times3} & \mathbb{0}_{3\times3} \\ \mathbb{0}_{3\times3} & \mathscr{J}(\boldsymbol{\eta}) \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\xi}} \\ \ddot{\boldsymbol{\eta}} \end{bmatrix} + \begin{bmatrix} \mathbb{0}_{3\times3} & \mathbb{0}_{3\times3} \\ \mathbb{0}_{3\times3} & \mathbf{C}_{\boldsymbol{\eta}\dot{\boldsymbol{\eta}}}(\mathbf{q},\dot{\mathbf{q}}) \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\xi}} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} + \begin{bmatrix} mg\mathbf{e_3} \\ \mathbb{0}_{3\times1} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{B}_{\mathscr{I}_\xi}(\mathbf{q}) \\ \mathbf{B}_{\mathscr{I}_\eta}(\mathbf{q}) \end{bmatrix} \mathbf{u_M} + \begin{bmatrix} \boldsymbol{\delta}_\xi(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},f_{\xi_\mathbf{d}}) \\ \boldsymbol{\delta}_{\boldsymbol{\eta}}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\boldsymbol{\tau}_{\boldsymbol{\eta}_\mathbf{d}}) \end{bmatrix}. \tag{6}$$

To design controllers based on the IDA methodology, the Port Controlled Hamiltonian (PCH) model is needed. PCH represents a wide range of systems, including systems described by Euler-Lagrange equations. These models are characterized by individually represent the interconnection portions and the dissipative elements revealing, in addition, the role played by the energy function in the system dynamics. Consequently, it enables the incorporation of information about the system. The representation of the systems through the PCH is essential in the IDA-PBC methodology and is given as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= [\mathbf{J}(\mathbf{x}) - \mathbf{R}(\mathbf{x})]\nabla_{\mathbf{x}}H(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \\ \mathbf{y} &= \mathbf{B}'(\mathbf{x})\nabla_{\mathbf{x}}H(\mathbf{x}). \end{aligned} \tag{7}$$

The new state vector is choosen as $\mathbf{x} = [\mathbf{q}',\ \mathbf{p}']'$, where $\mathbf{q}$ represents the generalized coordinates, while $\mathbf{p} = \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ is called generalized momentum. $\mathbf{J}(\mathbf{x}) = -\mathbf{J}'(\mathbf{x})$ is a skew symmetric matrix that represents the system interconnection. The semi-defined positive matrix $\mathbf{R}(\mathbf{x}) = \mathbf{R}'(\mathbf{x}) \geq 0$ possesses the system dissipative elements and is called damping matrix. $H(\mathbf{x})$ is the Hamiltonian, which in this case has the total energy accumulation function interpretation, $\boldsymbol{u} \in \Re^n$ and $\mathbf{y} \in \Re^n$ are the power ports variables, since their product defines the exchanged power flux with the environment (Ortega et al., 2002b). The total stored energy contained in a mechanic system is given by the sum of kinetic and potential energies yielding:

$$\begin{aligned} H(\mathbf{q},\mathbf{p}) &= \mathscr{K}(\mathbf{q},\mathbf{p}) + \mathscr{U}(\mathbf{q}) \\ &= \frac{1}{2}\mathbf{p}'\mathbf{M}^{-1}(\mathbf{q})\mathbf{p} + \mathscr{U}(\mathbf{q}), \end{aligned} \tag{8}$$

in which $\mathscr{K}(\mathbf{q},\mathbf{p})$ represents the kinetic energy and $\mathscr{U}(\mathbf{q}) = m \cdot g_a \cdot z$ the potential energy, being g_a the gravitational acceleration.

Therefore, the quadrotor model (6) can be expressed in terms of equation (7) with:

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{R}(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3. PASSIVITY BASED CONTROL FOR PATH TRACKING

3.1 Preliminaries about PBC Theory

Let us consider a system S, and its two power ports variables $u \in \Re^n$ and $y \in \Re^n$, whose internal energy can only be increased through external supply (Ortega et al., 2002b). Associated to this system there is a function $w : \Re^n \times \Re^n \to \Re$ called supply rate, which is locally integrable for $u \in U$, that is, satisfies:

$$\int_{t_0}^{t_1} w[u(t), y(t)]dt < \infty,\ \forall\ t_0 \leq t_1\,. \tag{9}$$

Assuming that $\mathbf{X}$ is a subset of $\Re^n$ containing the origin, it is said that S is dissipative in $\mathbf{X}$ with the following supply rate $w(u,y)$ if there is a function $H(x), H(0) = 0$, such that for all $x \in \mathbf{X}$,

$$H(x) > 0$$
$$H[x(t_1)] - H[x(t_0)] = \int_{t_0}^{t_1} w[u(t), y(t)]dt - d(t)\,, \tag{10}$$

for all $u \in U$ and all $t_1 \geq 0$ such that $x(t) \in \mathbf{X}$ for all $t \in [t_0, t_1]$. $H(x)$ is called storage function and $d(t)$ is the function that represents the dissipated energy through resistance and friction.

It is know from Kalman-Yakubovich-Popov lemma that for systems described by:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}, \tag{11}$$

passivity is equivalent to the existence of a scalar function $H(x)$ such that (Ortega et al., 2001):

$$\left[\frac{\partial H(x)}{\partial x}\right]^T f(x) \leq 0 \quad h(x) = g^T(x)\frac{\partial H(x)}{\partial x}.$$

Thus, if exists, for systems described by (11), a vector function $\beta(x)$ such that the PDE:

$$\left[\frac{\partial H_a(x)}{\partial x}\right]^T [f(x) + g(x)\beta(x)] = -h^T(x)\beta(x) \tag{12}$$

is solved for an additional energy function $H_a(x)$, and the desired energy function $H_d(x) \doteq H(x) + H_a(x)$ has a minimum in x_*, thus, $u = \beta(x)$ estabilizes x_* with the difference between stored and supplied energy constituting a Lyapunov function.

3.2 IDA-PBC for path tracking

The application of the IDA-PBC methodology to systems described by EL equations normally leads to the following matching equation:

$$[\mathbf{J_d}(\mathbf{x}) - \mathbf{R_d}(\mathbf{x})]\nabla_{\mathbf{x}}H_a(\mathbf{x}) = [-\mathbf{J_a}(\mathbf{x}) + \mathbf{R_a}(\mathbf{x})]\nabla_{\mathbf{x}}H(\mathbf{x}) + g(\mathbf{x})\mathbf{u}. \tag{13}$$

However, the regulation and trajectory tracking applied to some underactuated systems requires modification of the kinetic energy. The Equation (13) is not usable in these cases (Wang and Goldsmith, 2008). Therefore, to solve the problem of path tracking for the quadrirrotor, the modified matching equations proposed in Wang and Goldsmith (2008) will be used.

Assume a system as (7) whose desired dynamic is given by:

$$\dot{\bar{\mathbf{x}}} = [\mathbf{J_d}(\bar{\mathbf{x}}) - \mathbf{R_d}(\bar{\mathbf{x}})]\nabla_{\bar{\mathbf{x}}}H_d(\bar{\mathbf{x}}),$$

in which $\bar{\mathbf{x}} = [\bar{\mathbf{q}}', \quad \bar{\mathbf{p}}']'$. The equation (13) can be rewritten as follows:

$$[\mathbf{J}(\mathbf{x}) - \mathbf{R}(\mathbf{x})]\nabla_{\mathbf{x}}H(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} - \dot{\mathbf{x}} + \dot{\bar{\mathbf{x}}} = [\mathbf{J_d}(\bar{\mathbf{x}}) - \mathbf{R_d}(\bar{\mathbf{x}})]\nabla_{\bar{\mathbf{x}}}H_d(\bar{\mathbf{x}}),$$

Further, it yields to:

$$\mathbf{g}(\mathbf{x})\mathbf{u} = [\mathbf{J_d}(\bar{\mathbf{x}}) - \mathbf{R_d}(\bar{\mathbf{x}})]\nabla_{\bar{\mathbf{x}}}H_d(\bar{\mathbf{x}}) - [\mathbf{J}(\mathbf{x}) - \mathbf{R}(\mathbf{x})]\nabla_{\mathbf{x}}H(\mathbf{x}) + \dot{\mathbf{x}} - \dot{\bar{\mathbf{x}}}.$$

If the position and velocity errors are taken as the new set of state variables, it leads to:

$$\bar{\mathbf{x}} = [\mathbf{q_e}', \quad \mathbf{p_e}']', \quad \mathbf{q_e} = \mathbf{q} - \mathbf{q_r}, \quad \mathbf{p_e} = \mathbf{M_d}(\mathbf{q_e})\dot{\mathbf{q}}_\mathbf{e},$$

in which $\mathbf{q_e}$ are the position errors and the new generalized momentum is described by $\mathbf{p_e} = \mathbf{M_d}(\mathbf{q_e})\dot{\mathbf{q}}_\mathbf{e}$, with $\dot{\mathbf{q}}_\mathbf{e} = [\dot{\mathbf{q}} - \dot{\mathbf{q}}_\mathbf{r}]'$, and $\mathbf{q_r}$ represents the references of x, y, z and yaw angle, ψ.

The new energy function is given by:

$$H_d(\mathbf{q_e}, \mathbf{p_e}) = \frac{1}{2}\mathbf{p_e}'\mathbf{M_d}^{-1}(\mathbf{q_e})\mathbf{p_e} + \mathscr{U}_d(\mathbf{q_e}),$$

where $H_d(\mathbf{q_e}, \mathbf{p_e})$ is required to has its minimum at $\mathbf{q} = \mathbf{q_r}$.

In the PBC theory is possible to separate the control action into two parts (Ortega et al., 2002a):

$$\mathbf{u} = \mathbf{u_{me}} + \mathbf{u_{aa}}, \tag{14}$$

where $\mathbf{u_{me}}$ is the portion of the control action responsible by the energy shaping, while $\mathbf{u_{aa}}$ assigns damping. The desired interconnection and damping matrices are defined as follows:

$$\begin{aligned} \mathbf{J_d}(\mathbf{x}) = -\mathbf{J_d}(\mathbf{x})' &= \begin{bmatrix} 0 & \mathbf{J_{a_{12}}} \\ -\mathbf{J_{a_{12}}}' & \mathbf{J_{d_{22}}} \end{bmatrix} \\ \mathbf{R_d}(\mathbf{x}) = \mathbf{R_d}(\mathbf{x})' &= \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{R_{d_{22}}} \end{bmatrix} \geq 0. \end{aligned} \tag{15}$$

The therm $\mathbf{R_{d_{22}}}$ is introduced to create damping in the system. This is achieved through the negative feedback of the new passive output $\mathbf{y} = \mathbf{B}_{\mathscr{I}}'\nabla_{\mathbf{p}}H_d(\mathbf{q},\mathbf{p})$. Thus, it leads to:

$$\mathbf{u_{aa}} = -\mathbf{K_v}\mathbf{B}_{\mathscr{I}}'\nabla_{\mathbf{p}}H_d(\mathbf{q},\mathbf{p}), \tag{16}$$

being $\mathbf{K_v}$ a diagonal and positive gain matrix. Replacing equations (15) and (14) in (13), the following is obtained:

$$\begin{cases} \nabla_{\mathbf{p}}H - \dot{\mathbf{q}}_\mathbf{r} = \mathbf{J_{d_{22}}}\nabla_{\mathbf{p_e}}H_d \\ -\nabla_{\mathbf{q}}H + \mathbf{B}_{\mathscr{I}}(\mathbf{u_{me}} - \mathbf{K_v}\mathbf{B}_{\mathscr{I}}'\nabla_{\mathbf{p_e}}H_d) = \\ \qquad -\mathbf{J_{d_{22}}}'\nabla_{\mathbf{q_e}}H_d + (\mathbf{J_{d_{22}}} - \mathbf{R_{d_{22}}})\nabla_{\mathbf{p_e}}H_d + \dot{\mathbf{p}}_\mathbf{r}. \end{cases}$$

Solving $\nabla_{\mathbf{p}}H$ and $\nabla_{\mathbf{p_e}}H_d$, and making $\mathbf{p} = \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ and $\mathbf{p_e} = \mathbf{M_d}(\mathbf{q_e})\dot{\mathbf{q}}_\mathbf{e}$ it follows that $\mathbf{J_{a_{12}}}\dot{\mathbf{q}}_\mathbf{e} = \dot{\mathbf{q}} - \dot{\mathbf{q}}_\mathbf{r}$ thus $\mathbf{J_{a_{12}}} = \mathbb{1}_{n\times n}$. From the second equation, the expression is extracted:

$$\mathbf{B}_{\mathscr{I}}\mathbf{u_{me}} = \nabla_{\mathbf{q}}H - \nabla_{\mathbf{q_e}}H_d + \mathbf{J_{d_{22}}}\mathbf{M_d}^{-1}\mathbf{p_e} + \dot{\mathbf{p}}_\mathbf{r}. \tag{17}$$

Multiplying both sides of (17) by $\begin{bmatrix} \mathbf{B}_{\mathscr{I}}{}^{\#} \\ \mathbf{B}_{\mathscr{I}}{}^{\perp} \end{bmatrix}$ by the left side, gives:

$$\begin{bmatrix} \mathbb{1}_{4\times 4} \\ \mathbb{0}_{2\times 4} \end{bmatrix}\mathbf{u_{me}} = \begin{bmatrix} \mathbf{B}^{\#}_{\mathscr{I}} \\ \mathbf{B}^{\perp}_{\mathscr{I}} \end{bmatrix}\mathbf{L_d}, \tag{18}$$

where $\mathbf{L_d}$ is the right side of (17), and $\mathbf{B}^{\#}_{\mathscr{I}}$ and $\mathbf{B}^{\perp}_{\mathscr{I}}$ are the full rank pseudo-inverse and the left annihilator of $\mathbf{B}_{\mathscr{I}}$, respectively, and satisfies the next equality:

$$\mathbf{B}^{\perp}_{\mathscr{I}}\mathbf{B}_{\mathscr{I}} = 0.$$

From the last equation of (18) we obtain the control component $\mathbf{u_{me}}$ and the following set of constraint equations:

$$\mathbf{B}^{\perp}_{\mathscr{I}}\left\{\nabla_{\mathbf{q}}(\mathbf{p}'\mathbf{M}^{-1}\mathbf{p}) - \nabla_{\mathbf{q_e}}(\mathbf{p_e}'\mathbf{M_d}^{-1}\mathbf{p_e}) + 2\mathbf{J_{d_{22}}}\mathbf{M_d}^{-1}\mathbf{p_e} + \dot{\mathbf{p}}_\mathbf{r}\right\} = 0 \tag{19}$$

$$\mathbf{B}^{\perp}_{\mathscr{I}}\left\{\nabla_{\mathbf{q}}\mathscr{U} - \nabla_{\mathbf{q_e}}\mathscr{U}_d\right\} = 0. \tag{20}$$

The second equation, in comparison with the first one, is a simple PDE that must be solved for $\mathscr{U}_d$. The challenge for solving these constraints is to find the solution of the PDE (19) for the elements of the matrices $\mathbf{M_d}$ and $\mathbf{J_{d_{22}}}$. To simplify this task, the following equality is used:

$$\nabla_{\mathbf{q}}[\mathbf{s}'\mathbf{P}(\mathbf{q})\mathbf{s}] = \{\nabla_{\mathbf{q}}[\mathbf{P}(\mathbf{q})\mathbf{s}]\}'\mathbf{s} \tag{21}$$

for all $\mathbf{s} \in \Re^n$ and all simetric matrix $\mathbf{P} \in \Re^{n\times n}$. By this way, the constraint equation (19) can be simplified as follows:

$$\mathbf{B}^{\perp}_{\mathscr{I}}\left\{\left[\nabla_{\mathbf{q}}(\mathbf{M}^{-1}\mathbf{p})\right]'\mathbf{p} - \left[\nabla_{\mathbf{q_e}}(\mathbf{M_d}^{-1}\mathbf{p_e})\right]'\mathbf{p_e} + 2\mathbf{J_{d_{22}}}\mathbf{M_d}^{-1}\mathbf{p_e} + \dot{\mathbf{p}}_\mathbf{r}\right\} = 0. \tag{22}$$

3.3 IDA-PBC applied to the quadrotor

The control methodology presented in this work to solve the path tracking of underactuated mechanical systems problem led to a set of constraint equations of complex solution, composed by (20) and (22). The control action $\mathbf{u_{me}}$ is isolated in the equation (17) using the pseudo inverse of $\mathbf{B}_{\mathscr{I}}$, once $\mathbf{B}_{\mathscr{I}}$ is a non invertible matrix with full rank:

$$\mathbf{u_{me}} = \mathbf{B}_{\mathscr{I}}{}^{\#}(\nabla_{\mathbf{q}}H - \nabla_{\mathbf{q_e}}H_d + \mathbf{J_{d_{22}}}\mathbf{M_d}^{-1}\mathbf{p_e} + \dot{\mathbf{p}}_\mathbf{r}). \tag{23}$$

In this work, the control law (14), composed by (23) and (16), is applied to the quadrotor.

Disregarding the constraints, since the solution of these equations is a problem that remains outstanding, the candidate solutions have been proposed assuming the matrix $\mathbf{J_{d_{22}}}$ equal to zero, while the matrix $\mathbf{M_d}$ is considered as free parameters for tuning the control law. Furthermore, since the goal of the controller is to take the states x, y, z and ψ error position to zero, the following function $\mathscr{U}_d$ is proposed:

$$\mathscr{U}_d = \frac{1}{2}\Big[kp_1(x-x_d)^2 + kp_2(y-y_d)^2 + kp_3(z-z_d)^2 + kp_4(\psi-\psi_d)^2\Big] \quad (24)$$

The kp gains weigh the position errors, and the diagonal matrix $\mathbf{K_v} \in \Re^{n\times n}$ weighs the velocity errors. It is important to remark that the proposed control law has shown to guarantee closed-loop tracking property regardless of the solution of (22) and (20).

4. SIMULATION RESULTS

In this section we present the results obtained with the controller developed in the previous section using Matlab©/Simulink© simulation environment. To this end, the model (2) was used to emulate the quadrotor helicopter with the model parameters presented in Table 1, while the equations (6), represented through the PCH, are used to design the IDA-PBC controller.

Table 1. Quadrotor model parameters.

Description	Parameter	Value
Quadrotor mass	m	$2.24kg$
Distance between rotor and center of mass	l	$0.332m$
Rotor thrust coefficient	b	$9.5e-6Ns^2$
Rotor drag coefficient	k_t	$1.7e-7Nms^2$
Gravity acceleration	g	$9.81m/s^2$
Axis x inertia	I_{xx}	$0.0363kg.m^2$
Plane xy inertia	I_{xy}	$-0.88307e-3kg.m^2$
Plane xz inertia	I_{xz}	$-0.20331e-3kg.m^2$
Axis y inertia	I_{yy}	$0.0363kg.m^2$
Plane yz inertia	I_{yz}	$-0.41908e-3kg.m^2$
Axis z inertia	I_{zz}	$0.0615kg.m^2$
Distance between $\mathscr{B}$ and center of mass on x	r_x	$-0.00069m$
Distance between $\mathscr{B}$ and center of mass on y	r_y	$-0.0014m$
Distance between $\mathscr{B}$ and center of mass on z	r_z	$-0.0311m$

The trajectory used in the simulations is illustrated in Figure 3. The quadrotor starts from the initial position $(1,1,0)$ with the angles $(0,0,0)$ and null speed. In order to evaluate the results obtained from the IDA-PBC controller presented here, a backstepping controller, which is a well known cascade control strategy, is used and the results of both controllers are compared. It was introduced a white noise on the system acceleration outputs and a perturbation of impulse type on the system translational forces inputs. These impulses were introduced at 4, 48 and 80 sec on X, Y and Z respectively. The controller gains are adjusted according to the Table 2.

Through Figure 3 one can observe the path tracking on the tridimensional space, while Figures 4 and 5 shown that the IDA-PBC controller tracks the controlled degrees of freedom x, y, z and ψ, while stabilizes roll and pitch motion.

Table 2. Weights for the controllers.

IDA-PBC		*Backstepping*	
Parameter	Value	Parameter	Value
kp_1	10	c_1	10
kp_2	11	c_2	10
kp_3	45	c_3	10
kp_4	0.4	c_4	10
kv_1	50×10^{10}	c_5	2
kv_2	60×10^{10}	c_6	2
kv_3	3.75×10^{10}	c_7	20
kv_4	5×10^{10}	c_8	5
kv_5	5×10^{10}	c_9	3
kv_6	20×10^{10}	c_{10}	3
m_d	m	c_{11}	3
I_{xxd}	I_{xx}	c_{12}	3
I_{yyd}	I_{yy}		
I_{zzd}	I_{zz}		
α	4°		
$\mathbf{J_{d_{22}}}$	0		

Despite the control strategy presented in this work, based on the IDA-PBC methodology, uses only one control loop, it presented similar performance during simulation results when compared with the cascaded backstepping control approach. It should be noted that the proposed control law, in comparison with the backstepping controller, possesses less degrees of freedom to adjust the entire control loop to perform path tracking of the quadrotor. It is due to the proposed goal, which pursue only the stabilization of the roll and pitch angles, while in a cascade strategy they are regulated around a reference value gives by the outer loop.

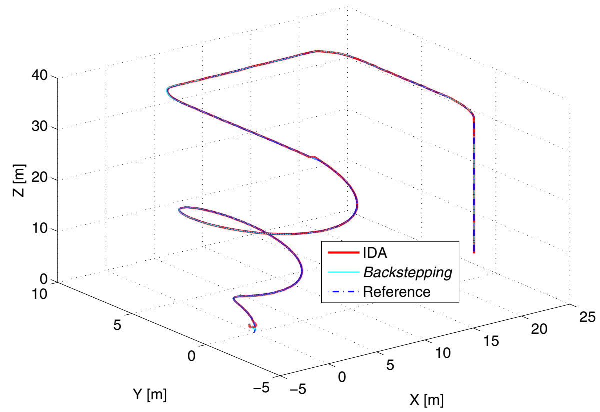

Fig. 3. Path tracking.

5. CONCLUSION

This work presented the design of an IDA-PBC strategy to solve the path tracking problem for a quadrotor, that avoids the use of cascaded controllers and augmented state space. This is achieved due to a change in the helicopter mechanical model, which is used only for the controller design purposes. Despite this mechanical change decreases the effectiveness of the actuators, since it is obtained considering that the rotors are tilted toward the

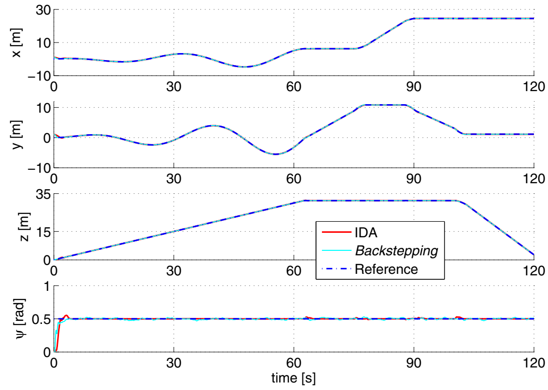

Fig. 4. Position of the controlled DOF (x, y, z, ψ).

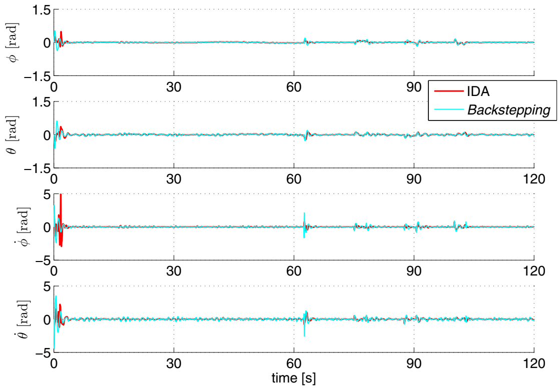

Fig. 5. Position and velocity of the remaining DOF $(\phi, \theta, \dot{\phi}, \dot{\theta})$.

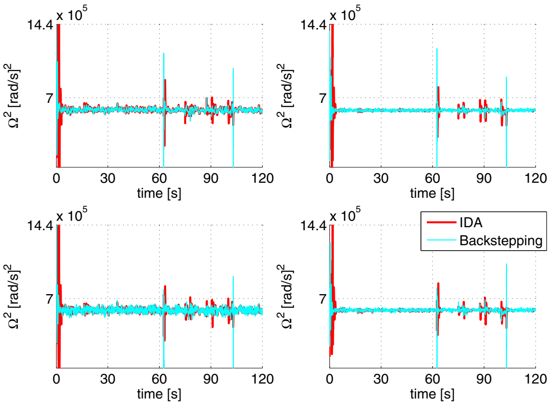

Fig. 6. Control signal.

center of the quadrotor, it increases the controllability and stability of the quadrotor helicopter model. Furthermore, the constraint equations solution problem, which arise due to the fact that the system is underactuated, is presented.

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