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Controllability of the space semi-discrete approximation for the beam equation ⋆

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Abstract: The aim of this work is to study of the numerical approximation of the controls for the hinged beam equation. A consequence of the numerical spurious high frequencies is the lack of the uniform controllability property of the semi-discrete model for the beam equation, in the classical setting. We solve this deficiency by adding a vanishing numerical viscosity term, which will damp out these high frequencies. An approximation algorithm based on the conjugate gradient method and some numerical experiments are presented.

Keywords: beam equation, numerical methods, controllability, vanishing viscosity, discrete model.

1. INTRODUCTION

We consider the controlled transversal vibrations of a beam with hinged ends, for which the model are given by the following system

$$\begin{cases} u''(t,x) + u_{xxxx}(t,x) = 0 & (t,x) \in (0,T)\times(0,1) \\ u(t,0) = u(t,1) = u_{xx}(t,0) = 0 & t \in (0,T) \\ u_{xx}(t,1) = v(t) & t \in (0,T) \\ u(0,x) = u^0(x) & x \in (0,1) \\ u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{1}$$

where by $'$ we denote the derivative in time and v is the control acting on the extremity $x = 1$ of the beam.

By using multipliers techniques or Fourier series, the following controllability property holds *given* $T > 0$ *and an initial data* $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H} = H_0^1(0,1) \times H^{-1}(0,1)$*, then there exists a control* $v \in L^2(0,T)$ *with the property that the solution of* (1) *verifies*

$$u(T,x) = u'(T,x) = 0 \quad (x \in (0,1)). \tag{2}$$

For more details, see Komornik and Loreti (2005).

In this paper we study the controllability of the semi–discrete space approximation of (1). Let us consider the equidistant partition of the interval $(0,1)$ with $N+2$ nodes, for $N \in \mathbb{N}^*$, $x_0 = 0 < x_1 = h < ... < x_k = kh < ... < x_{N+1} = 1$, where the mesh-size is $h = \frac{1}{N+1}$. In order to discretize the boundary conditions, we consider two additional points $x_{-1} = x_0 - h$ and $x_{N+2} = x_{N+1} + h$. We consider finite differences in order to obtain the classical semi–discretization of (1), given by

$$\begin{cases} u_k'' = -\dfrac{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}}{h^4} \\ \qquad\qquad 1 \le k \le N, \ t \in (0,T) \\ u_0(t) = 0, \quad u_{N+1}(t) = 0 & t \in (0,T) \\ u_{-1}(t) = -u_1(t), & t \in (0,T) \\ u_{N+2}(t) = h^2 v_h(t) - u_N(t), & t \in (0,T) \\ u_k(0) = u_k^0, \ u_k'(0) = u_k^1 & 1 \le k \le N. \end{cases} \tag{3}$$

We remark that, our problem consists of solving N linear equations with N unknowns $u_1, u_2, \dots, u_N$. More precisely, $u_k(t)$ is the approximation of $u(t,x_k)$, the solution of (1), if $\begin{pmatrix} u_k^0 \\ u_k^1 \end{pmatrix}_{0\le k\le N+1}$ approximates the initial data of (1). If we consider initial data which are sufficiently regular, we shall choose

$$u_k^0 = u^0(kh), \quad u_k^1 = u^1(kh) \qquad (0 \le k \le N+1). \tag{4}$$

In this paper we address the following controllability property for (3): for $T > 0$ and $\begin{pmatrix} u_k^0 \\ u_k^1 \end{pmatrix}_{1\le k\le N} \in \mathbb{C}^{2N}$, we look for a control $v_h \in L^2(0,T)$ such that the solution $\begin{pmatrix} u_k \\ u_k' \end{pmatrix}_{1\le k\le N}$ of (3) verifies

$$u_k(T) = u_k'(T) = 0 \qquad (1 \le k \le N). \tag{5}$$

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If (5) is verified for every initial data $\begin{pmatrix} u_k^0 \\ u_k^1 \end{pmatrix}_{1\le k\le N} \in \mathbb{C}^{2N}$, we will say that (3) is *null-controllable in time* T. Note that the above controllability problem is not a difficult task and it can be easily constructed a sequence of discrete controls $(v_h)_{h>0}$. Of course, it is difficult to prove that the controls sequence $(v_h)_{h>0}$ is convergent to a control of a continuous beam equation (1).

In fact, as it was shown in Leon and Zuazua (2002), this is not necessarily true in the case of discretization (3). Indeed, for any $h > 0$ there exist initial data $\begin{pmatrix} u_k^0 \\ u_k^1 \end{pmatrix}_{1\le k\le N} \in \mathbb{C}^{2N}$ (converging to an initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H}$ when h goes to zero) such that any sequence of discrete controls $(v_h)_{h>0}$ for (3) diverges in $L^2(0,T)$.

This deficiency appears because, the dynamics of the semi-discrete model generate bad high frequencies oscillations. In order to eliminate this deficiency two possibilities have been proposed and analyzed in Leon and Zuazua (2002):

(a) to project the solution of (3) over a space in which the high frequencies have been eliminated and this projection is controlled to zero;
(b) to add an extra boundary control, which vanishes in limit, appears in (3).

In this article, we study a third possibility, to introduce a numerical viscosity term which vanishes in the limit. Since this term damps out high frequencies we can expect that it will improve the desired convergence properties of the discrete controls. More precisely, we will prove the null–controllability of the following different discretization of (1)

$$\begin{cases} u_k'' + \dfrac{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}}{h^4} - \\ -\varepsilon \dfrac{u_{k+1}' - 2u_k' + u_{k-1}'}{h^2} = 0 \quad 1 \le k \le N,\ t \in (0,T) \\ u_0(t) = 0, \quad u_{N+1}(t) = 0 \qquad t \in (0,T) \\ u_{-1}(t) = -u_1(t), \qquad t \in (0,T) \\ u_{N+2}(t) = h^2 v_h(t) - u_N(t) \qquad t \in (0,T) \\ u_k(0) = u_k^0, \quad u_k'(0) = u_k^1 \qquad 1 \le k \le N. \end{cases} \tag{6}$$

In (6), the ratio $\frac{u_{k+1}'(t) - 2u_k'(t) + u_{k-1}'(t)}{h^2}$ represents a viscous term and the parameter ε which multiplies it depends on the step size h as follows

$$\lim_{h\to 0} \varepsilon(h) = 0. \tag{7}$$

We will choose the parameter ε sufficiently small in order to preserve the convergence and the precision of the numerical scheme but sufficiently large to improve the observability properties.

The vanishing viscosity method was used in control problems for the wave equation in Micu (2008). Note that, we want to obtain an uniform controllability result in arbitrary small time. This is not possible for the wave equation but it is perfectly realistic for the beam equation.

We are able to obtain a uniformly bounded family of controls for the perturbed problem (8) with the property that any weak limit of it is a control for the continuous problem. More precisely, the following result holds (see Bugariu et al. (2013)).

Theorem 1. Let $T > 0$ and $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H}$. There exist $h_0, c_0 > 0$ such that for any $h \in (0, h_0)$, $\varepsilon \in \left(c_0 \frac{1}{T^2} h^2 \ln \frac{1}{h}, c_0 h\right)$ and any initial data $\begin{pmatrix} u_k^0 \\ u_k^1 \end{pmatrix}_{1\le k\le N} \in \mathbb{C}^{2N}$ (which weakly converges in ℓ^2 to $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H}$), there exists a family of controls $(v_h)_{h>0} \subset L^2(0,T)$ for problem (6) which converges to a null-control v for (1) in $L^2(0,T)$.

Remark 2. To obtain the convergent result from Theorem 1, the parameter ε has to be large enough (greater than $c_0 h^2 \ln(\frac{1}{h})$). This key condition makes the dissipation mechanism efficient in our control problem but also introduces an unbounded perturbation in the system which is more difficult to analyse.

2. THE DISCRETE CONTROL PROBLEM AND THE HUM APPROACH

In this section, we give the equivalent vectorial form of (6) and we present our controllability problem as a minimization problem. Firstly, we write (6) as an abstract Cauchy form by using the matrices A_h, $B_h \in \mathcal{M}_{N\times N}(\mathbb{R})$ given by

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}, \qquad B_h = A_h^2.$$

If we denote by

$$U_h^0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_N^0 \end{pmatrix}, \qquad U_h^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ u_N^1 \end{pmatrix},$$

$$U_h(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{pmatrix}, \; F_h(t) = \frac{1}{h^2} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -v_h(t) \end{pmatrix},$$

then system (6) may be written vectorially as follows:

$$\begin{cases} U_h''(t) + B_h U_h(t) + \varepsilon A_h U_h'(t) = F_h(t) \; t \in (0,T) \\ U_h(0) = U_h^0, \quad U_h'(0) = U_h^1. \end{cases} \tag{8}$$

The system (8) is *null–controllable in time* $T > 0$ if for any initial data $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$, there exists a control $v_h \in L^2(0,T)$ such that the corresponding solution $\begin{pmatrix} U_h \\ U_h' \end{pmatrix}$ of (8) verifies

$$U_h(T) = U_h'(T) = 0. \tag{9}$$

For $l = (l_k)_{1\le k\le N} \in \mathbb{C}^N$, $d = (d_k)_{1\le k\le N} \in \mathbb{C}^N$ we define the following discrete inner products

$$\langle l, d\rangle = h\sum_{k=1}^{N} l_k\overline{d_k}, \tag{10}$$

$$(l,\, d)_1 = \langle A_h l,\, d\rangle, \tag{11}$$

with the corresponding norm $\|\cdot\|_1$ and

$$(l,\, d)_{-1} = \langle A_h^{-1} l,\, d\rangle, \tag{12}$$

with the corresponding norm $\|\cdot\|_{-1}$.

Finally, we consider the discrete inner product in $\mathbb{C}^{2N}$

$$\big\langle (l^1, l^2),\, (d^1, d^2)\big\rangle_{1,-1} = (l^1, d^1)_1 + (l^2, d^2)_{-1} \tag{13}$$

with the corresponding norm $\|\cdot\|_{1,-1}$.

In order to prove the controllability properties of (8) we study the properties of the homogeneous "adjoint" backward problem

$$\begin{cases} W_h''(t) - \varepsilon A_h W_h'(t) + A_h^2 W_h(t) = 0 \;\; t\in(0,T) \\ W_h(T) = W_h^0, \quad W_h'(T) = W_h^1, \end{cases} \tag{14}$$

where the initial data $\begin{pmatrix} W_h^0 \\ W_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ are given. Multiplying system (14) with U_h, the solution of system (8) and if we integrate in time we deduce the characterization of the controllability property, given in the following theorem.

Theorem 3. Given $T>0$, system (8) *is null–controllable in time T if and only if, for any initial data $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$, there exists $v_h \in L^2(0,T)$ which verifies*

$$\int_0^T v_h(t)\,\overline{\frac{W_N(t)}{h}}\,dt \tag{15}$$
$$= \left\langle U_h^1, W_h(0)\right\rangle - \left\langle U_h^0, W_h'(0) - \varepsilon A_h W_h(0)\right\rangle$$

where $\begin{pmatrix} W_h^0 \\ W_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ and $\begin{pmatrix} W_h \\ W_h' \end{pmatrix}$ is the solution of (14).

For simplicity, we denote by

$$\langle (U_h^0, U_h^1), (W_h(0), W_h'(0))\rangle_D$$
$$= -\left\langle U_h^1, W_h(0)\right\rangle + \left\langle U_h^0, W_h'(0) - \varepsilon A_h W_h(0)\right\rangle.$$

Given $T>0$, let $q \in C^\infty[0,T]$ be a cut-off function such that there exists a positive real number $\delta < \frac{T}{2}$ with the following properties

$$\begin{aligned} &\text{(i) } \operatorname{supp}(q) \subset \left(\tfrac{\delta}{2}, T - \tfrac{\delta}{2}\right), \\ &\text{(ii) } 0 \le q(t) \le 1 \text{ for all } t\in[0,T], \\ &\text{(iii) } q(t) \ge 1/2 \text{ for all } t \in [\delta, T-\delta]. \end{aligned} \tag{16}$$

The function q has the role to improve the numerical approximations of the controls, avoiding incompatibility between the initial data and the nonhomogeneous term at the origin.

Let us consider a functional $J : \mathbb{C}^{2N} \to \mathbb{C}$ given by

$$\begin{aligned} L(W_h^0, W_h^1) = \frac{1}{2}\int_0^T q(t)\left|\frac{W_N(t)}{h}\right|^2 dt + \\ + \langle (U_h^0, U_h^1), (W_h(0), W_h'(0))\rangle_D, \end{aligned} \tag{17}$$

where (W_h, W_h') is the solution of the system (14).

Theorem 4. Given any $T>0$ and $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$, then there exists a unique $(\widehat{W}_h^0, \widehat{W}_h^1) \in \mathbb{C}^{2N}$ which is the unique minimizer of the functional L, given by (17). If we consider $v_h \in H_0^1(0,T)$ defined by

$$v_h = q\frac{\widehat{W}_N}{h}, \tag{18}$$

where $(\widehat{W}_h, \widehat{W}_h')$ is the solution of (14) with initial data $(\widehat{W}_h^0, \widehat{W}_h^1)$, then v_h is a control for (8).

Proof. It is easy tot show that the map J from (18) is continuous, strictly convex and coercive. Indeed, there exists a constant $C = C(T,h)$ such that

$$\|(W_h(0), W_h'(0))\|_{1,-1}^2 \le C\int_0^T q(t)\left|\frac{W_N(t)}{h}\right|^2 dt. \tag{19}$$

It follows that J has a unique minimizer. The optimality condition for the minimizer and (15) show that v_h is the control for (8). □

Remark 5. Obsrevability inequality (19) holds for (3) and (6). If we want to obtain the result from Theorem 1, the constant C should be uniformly bounded in h. This property is true for (6) but, as shown in Leon and Zuazua (2002), fails to hold for (3).

We define

$$\begin{aligned} &b : \mathbb{C}^{2N} \times \mathbb{C}^{2N} \to \mathbb{C}, \\ &b((W_h^0, W_h^1), (\xi_h^0, \xi_h^1)) = \int_0^T q(t)\left(\frac{W_N}{h}\right)\left(\frac{\xi_N}{h}\right) dt, \end{aligned} \tag{20}$$

$$\begin{aligned} &\mathcal{T} : \mathbb{C}^{2N} \to \mathbb{C}, \\ &\mathcal{T}(W_h^0, W_h^1) = \langle (U_h^0, U_h^1), (W_h(0), W_h'(0))\rangle_D \end{aligned} \tag{21}$$

where (W, W') and (ξ, ξ') verifies (14) with initial data (W_h^0, W_h^1) and (ξ_h^0, ξ_h^1) respectively.

Hence, we deduce that

$$L(W_h^0, W_h^1) = \frac{1}{2} b((W_h^0, W_h^1), (W_h^0, W_h^1)) + \mathcal{T}(W_h^0, W_h^1).$$

In the following sentences we present some remarks which give the main ideas about the computation algorithm. We consider the following steps.

i) *We compute $\mathcal{T}(W_h^0, W_h^1)$.*
   Let (τ, τ') such that

$$\begin{cases} \tau''(t) + A_h^2\tau(t) + \delta A_h \tau'(t) = 0, \; t\in(0,T), \\ \tau(0) = U_h^0, \; \tau'(0) = U_h^1 \end{cases} \tag{22}$$

where (W, W') verifies (14) with initial data (W_h^0, W_h^1). Then we have that

$$\langle (U_h^0, U_h^1), (W_0, W_0')\rangle_D =$$
$$= \langle (\tau(T), \tau'(T)), (W_h^0, W_h^1)\rangle_D.$$

Moreover, for any (ξ_h^0, ξ_h^1), we have that

$$\langle (\tau(T), \tau'(T)), (\xi_h^0, \xi_h^1)\rangle_D = \langle ((f_h^0, f_h^1), (\xi_h^0, \xi_h^1))\rangle_{1,-1},$$

where (f_h^0, f_h^1) are given by

$$\begin{cases} f_h^1 = -A_h \tau(T), \\ f_h^0 = A_h^{-1} \left(\tau'(T) + \delta A_h \tau(T) \right). \end{cases} \quad (23)$$

Hence,

$$\mathcal{T}(W_h^0, W_h^1) = \langle ((f_h^0, f_h^1), (W_h^0, W_h^1)) \rangle_{1,-1}, \quad (24)$$

where (f_h^0, f_h^1) is given by (22)-(23).

ii) *We compute* $b((W_h^0, W_h^1), (\xi_h^0, \xi_h^1))$.
For any (W_h^0, W_h^1) and (ξ_h^0, ξ_h^1), we obtain that

$$\int_0^T q(t) \left(\frac{W_N}{h} \right) \left(\frac{\xi_N}{h} \right) dt = \\ = \langle (z(T), z'(T)), (\xi_h^0, \xi_h^1) \rangle_D,$$

where (z, z') verifies

$$\begin{cases} z'' + A_h^2 z + \delta A_h z' = F, \\ z(0) = 0, \ z'(0) = 0, \end{cases} \quad (25)$$

and F verifies

$$F(t) = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ \frac{1}{h^3} q W_N \end{pmatrix}.$$

Moreover,

$$\langle (\tau(T), \tau'(T)), (\xi_h^0, \xi_h^1) \rangle_D = \langle ((\tilde{f}_h^0, \tilde{f}_h^1), (\xi_h^0, \xi_h^1)) \rangle_{1,-1},$$

where $(\tilde{f}_h^0, \tilde{f}_h^1)$ is given by

$$\begin{cases} \tilde{f}_h^1 = -A_h z(T), \\ \tilde{f}_h^0 = A_h^{-1} \left(z'(T) + \delta A_h z(T) \right). \end{cases} \quad (26)$$

Hence,

$$b((W_h^0, W_h^1), (\xi_h^0, \xi_h^1)) = \langle ((\tilde{f}_h^0, \tilde{f}_h^1), (\xi_h^0, \xi_h^1)) \rangle_{1,-1}, \quad (27)$$

where $(\tilde{f}_h^0, \tilde{f}_h^1)$ verifies (25)-(26).

iii) *We compute the gradient of* L.
Firstly, we obtain that

$$\nabla L(W_h^0, W_h^1)(\xi_h^0, \xi_h^1) = \\ = b((W_h^0, W_h^1), (\xi_h^0, \xi_h^1)) + T(\xi_h^0, \xi_h^1) = \\ = \langle ((f_h^0 + \tilde{f}_h^0, f_h^1 + \tilde{f}_h^1), (\xi_h^0, \xi_h^1)) \rangle_{1,-1}.$$

Hence,

$$\nabla L(W_h^0, W_h^1) = (l^0, l^1), \quad (28)$$

where (l^0, l^1) verifies

$$\begin{cases} l^1 = -A_h(\tau(T) + z(T)), \\ l^0 = A_h^{-1} \left(\tau'(T) + z'(T) + \delta A_h(\tau(T) + z(T)) \right). \end{cases} \quad (29)$$

where (τ, τ') and (z, z') are the solutions of (22) and (25) respectively.

3. CONJUGATE GRADIENT METHOD

We consider H a Hilbert space endowed with the inner product $(\cdot, \cdot)$ and the norm $|| \, ||$.

In this show section we present the main steps of the conjugate gradient method in H, in order to solve a variational problem (see Glowinski et al. (1990)).

We try to solve the general variational problem: find $v \in H$ which verifies

$$b(v, \psi) + \mathcal{T}(\psi) = 0, \quad \forall \psi \in H, \quad (30)$$

where $b : H \times H \to \mathbb{R}$ is bilinear, continuous, symmetric and coercive and $\mathcal{T} : H \to \mathbb{R}$ is linear and continuous.

The above assumptions gives the existence and the uniqueness of the solution $v \in H$ of (30).

Let us consider $L : H \to \mathbb{R}$, defined as

$$L(\psi) = \frac{1}{2} b(\psi, \psi) + \mathcal{T}(\psi).$$

Hence, for the problem

$$L(\widehat{\psi}) = \min_{\psi \in H} L(\psi) \quad (31)$$

we infer the existence and the uniqueness of the solution $\widehat{\psi} \in H$, which is in fact v, the solution of (30).

Now, we need to approximate the solution $\widehat{\psi}$ for the problem (31). In order to do that, we will use the conjugate gradient method as in the following steps:

0. We consider $\psi_0 \in H$.
1. We approximate the gradient $l_0 = \nabla L(\psi_0) \in H$ by using

$$(l_0, \varphi) = (\nabla L(\psi_0), \varphi) = \lim_{h \to 0} \frac{L(\psi_0 + h\varphi) - L(\psi_0)}{h} \\ = b(\psi_0, \varphi) + \mathcal{T}(\varphi), \quad \forall \varphi \in H.$$

2. For $||l_0|| \leq \delta$ we choose $\widehat{\psi} = \psi_0$ and stop.
3. For $||l_0|| > \delta$ we choose the descent direction $d_0 = -l_0$.

Now, we assume the ψ_n, $l_n = \nabla L(\psi_n)$ and d_n are known and we want to compute ψ_{n+1}, l_{n+1} and d_{n+1}, as in the following steps:

4. We evaluate

$$q_n = -\frac{(l_n, d_n)}{b(d_n, d_n)}$$

By using the fact that $(l_n, l_j) = 0$, $0 \leq j \leq n-1$, we obtain that

$$q_n = \frac{(l_n, l_n)}{b(d_n, d_n)}.$$

5. Nextly, we approximate

$$\psi_{n+1} = \psi_n + q_n d_n.$$

6. The new gradient is evaluated as $l_{n+1} = \nabla L(\psi_{n+1})$ by taking into account the fact that

$$(l_{n+1}, \varphi) = b(\psi_{n+1}, \varphi) + \mathcal{T}(\varphi), \quad \forall \varphi \in H$$

or, by using the fact that $\psi_{n+1} = \psi_n + q_n d_n$

$$(l_{n+1}, \varphi) = (l_n, \varphi) + q_n b(d_n, \varphi), \quad \forall \varphi \in H.$$

7. For $||l_{n+1}|| \leq \delta$ we choose $\widehat{\psi} = \psi_{n+1}$ and stop.
8. For $||l_{n+1}|| > \delta$ we consider

$$d_{n+1} = -l_{n+1} + \frac{||l_{n+1}||^2}{||l_n||^2} d_n.$$

9. Consider $n = n + 1$ and go back to step 4.

Remark 6. To evaluate $b(\psi, \varphi)$ it can be used the fact that, for each $\psi \in H$ there exists a unique $\zeta = \zeta(\psi) \in H$ which verifies

$$b(\psi, \varphi) = (\zeta(\psi), \varphi), \quad \forall \varphi \in H. \quad (32)$$

In order to evaluate q_n at the step 4, note that

$$q_n = \frac{(l_n, l_n)}{(d_n, \zeta(d_n))}$$

and in order to evaluate l_{n+1} at the step 6 note that

$$l_{n+1} = l_n + q_n \zeta(d_n).$$

Remark 7. By using the Riesz identification we take into account that $\nabla L(\psi_0) \in H$ and $\nabla L(\psi_0)(\varphi) = (\nabla L(\psi_0), \varphi)$.

4. A NUMERICAL ALGORITHM

By using the conjugate gradient method as in Glowinski et al. (1990) and Carthel et al. (1994) we present a numerical algorithm which approximates the minimizer of L given by (17) and the approximate control v_h.

0. We consider $(\psi_0^0, \psi_0^1) \in \mathbb{R}^N \times \mathbb{R}^N$.
1. We compute the gradient

$$(l_0^0, l_0^1) = \nabla L(\psi_0^0, \psi_0^1) \in \mathbb{R}^N \times \mathbb{R}^N.$$

We consider the equations:

$$\begin{cases} W'' + A_h^2 W - \delta A_h W' = 0 \text{ in } (0,T) \\ W(T, \cdot) = W_0^0, \quad W'(T, \cdot) = W_0^1, \end{cases} \tag{33}$$

$$\begin{cases} z'' + A_h^2 z + \delta A_h z' = F \text{ in } (0,T) \\ z(0, \cdot) = z'(0, \cdot) = 0, \end{cases} \tag{34}$$

$$\begin{cases} \tau'' + A_h^2 \tau + \delta A_h \tau = 0 \text{ in } (0,T) \\ \tau(0, \cdot) = u^0, \quad \tau'(0) = u^1, \end{cases} \tag{35}$$

$$\begin{cases} l_0^1 = -A_h(\tau(T) - w(T)), \\ l_0^0 = A_h^{-1}\left(\tau'(T) + z'(T) + \delta A_h(\tau(T) + z(T))\right) \end{cases} \tag{36}$$

with

$$F = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ \frac{1}{h^3} q W_N \end{pmatrix}.$$

2. For $||(l_0^0, l_0^1)||_{1-1} \leq \delta$ we choose $(\widehat{\psi}^0, \widehat{\psi}^1) = (\psi_0^0, \psi_0^1)$ and stop.
3. For $||(l_0^0, l_0^1)||_{1,-1} > \delta$ we choose

$$(d_0^0, d_0^1) = -(l_0^0, l_0^1).$$

Suppose that we have (W_n^0, W_n^1), $(l_n^0, l_n^1) = \nabla L(W_n^0, W_n^1)$ and (d_n^0, d_n^1). Compute (W_{n+1}^0, W_{n+1}^1), (l_{n+1}^0, l_{n+1}^1) and (d_{n+1}^0, d_{n+1}^1) as it follows:

4. Consider the equations:

$$\begin{cases} W'' + A_h^2 W - \delta A_h W' = 0 \text{ in } (0,T) \\ W(T, \cdot) = d_n^0, \quad W'(T, \cdot) = d_n^1, \end{cases} \tag{37}$$

$$\begin{cases} z'' + A_h^2 z + \delta A_h z' = F \text{ in } (0,T) \\ z(0, \cdot) = z'(0, \cdot) = 0, \end{cases} \tag{38}$$

$$\begin{cases} \tilde{l}_n^1 = -A_h z(T), \\ \tilde{l}_n^0 = A_h^{-1}\left(z'(T) + \delta A_h z(T)\right), \end{cases} \tag{39}$$

where

$$F = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ \frac{1}{h^3} q W_N \end{pmatrix}.$$

5. We compute the step descent,

$$q_n = -\frac{\langle((l_n^0, l_n^1), (d_n^0, d_n^1))\rangle_{1,-1}}{b((d_n^0, d_n^1), (d_n^0, d_n^1))} =$$
$$= \frac{||(l_n^0, l_n^1)||_{1,-1}^2}{b((d_n^0, d_n^1), (d_n^0, d_n^1))} = \frac{||(l_n^0, l_n^1)||_{1,-1}^2}{\langle((\tilde{l}_n^0, \tilde{l}_n^1), (d_n^0, d_n^1))\rangle_{1,-1}}.$$

6. We compute the next approximation

$$(\psi_{n+1}^0, \psi_{n+1}^1) = (\psi_n^0, \psi_n^1) + q_n(d_n^0, d_n^1).$$

7. We consider

$$(l_{n+1}^0, l_{n+1}^1) = \nabla L(\psi_{n+1}^0, \psi_{n+1}^1),$$

taking into account the fact that

$$(l_{n+1}^0, l_{n+1}^1) = (l_n^0, l_n^1) + q_n(\tilde{l}_n^0, \tilde{l}_n^1).$$

8. For $||(l_{n+1}^0, l_{n+1}^1)||_{1,-1} \leq \delta$ we choose $(\widehat{\psi}^0, \widehat{\psi}^1) = (\psi_{n+1}^0, \psi_{n+1}^1)$ and stop.
9. For $||(l_{n+1}^0, l_{n+1}^1)||_{1,-1} > \delta$ we evaluate

$$(d_{n+1}^0, d_{n+1}^1) = -(l_{n+1}^0, l_{n+1}^1) + $$
$$+ \frac{||(l_{n+1}^0, l_{n+1}^1)||_{1,-1}^2}{||(l_n^0, l_n^1)||_{1,-1}^2}(d_n^0, d_n^1).$$

10. Consider $n = n + 1$ and go back to step 4.

5. NUMERICAL RESULTS

We compute the approximation of the control of (1), by using two numerical experiments based on (6) and the algorithm from the above section.

In the algorithm several beam equations have to be solved. The differential equations in t are solved by using Newmark Method when the parameters $\gamma = 0.5$ and $\beta = 0.25$ (see Hughes (1987)).

Numerical example: In this example we take $T = 1.5$, $\varepsilon = h$ and the initial data to be controlled

$$u^0(x) = 1 - |2x - 1|, \quad u^1(x) = 0 \qquad (x \in (0, 1)).$$

The approximations of the control andof the corresponding controlled solution, for $N = 100$ and $\varepsilon = h$, are presented in Figures 1 and 2 . We remark that, in this case, we have obtained a good approximationof the control and the solution goes to 0 at time T.

In Figure 3 we present the evolution of the error in the conjugate gradient method for $N = 100$ with and without viscosity ($\varepsilon = h$ and $\varepsilon = 0$ respectively). We remark that the algorithm is clearly convergent only in the case in which a viscosity is added.

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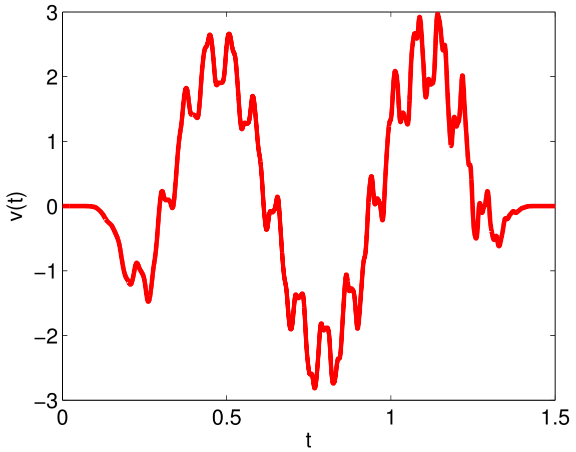

Fig. 1. Approximations of the control $\widehat{v}_h$

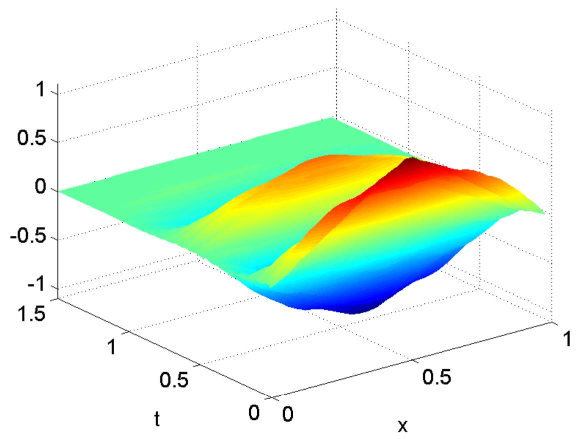

Fig. 2. The controlled solution U_h for $N = 100$ and $\varepsilon = h$.

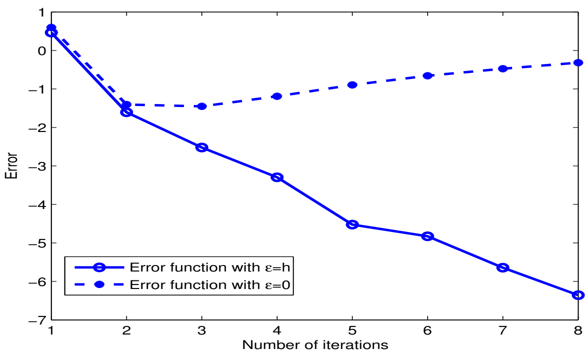

Fig. 3. Error evolution in the conjugate gradient method for $N = 100$, with and without viscosity.

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