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Backstepping design for parabolic systems with in-domain actuation and Robin boundary conditions

Frank Woittennek [*,**] **Siqian Wang** [*] **Torsten Knüppel** [*]

[*] *Institute of Control Theory, Technische Universität Dresden, 01069 Dresden, Germany*
[**] *e-mail: Frank.Woittennek@tu-dresden.de*

Abstract: State feedback design for linear parabolic systems with in-domain actuation and general Robin boundary conditions is considered. To this end the system is shown to be state equivalent to a boundary controlled system. By means of the well established backstepping transformation this latter system is feedback equivalent to a stable parabolic equation. Within the contribution previous results concerning systems with Neumann boundary conditions are generalized by means of functional analytic methods. Existence of the involved transformations is discussed by means of the Fredholm theory while a late lumping approach is proposed for the numerical implementation.

Keywords: distributed-parameter system; parabolic system; state feedback; backstepping

1. INTRODUCTION

Over the past decade the so-called backstepping method has proven to be an efficient tool for feedback design for distributed parameter systems with boundary control (cf. Krstic and Smyshlyaev, 2008, and the contained references). Originally developed for spatially one-dimensional linear parabolic systems the approach has been generalized to a broader class of distributed parameter systems comprising particular higher-dimensional equations (cf. Meurer, 2012, and the contained references), nonlinear equations (see, e.g., Vazquez and Krstic, 2008), and hyperbolic equations (Smyshlyaev et al., 2010). However, results concerning systems with more general actuation are very rare: Systems with a (very particular) distributed control operator are discussed by Tsubakino et al. (2012), while interior point control for parabolic 1D systems with constant coefficients has been emphasized by Wang and Woittennek (2013). The latter contribution relied on the equivalence of a system with in-domain control and a boundary controlled system under certain additional assumptions. This equivalence was established using algebraic methods relying on the parametrization of the solution by a flat output (cf. Woittennek and Mounier (2010)). Contrary to the hyperbolic case, where flatness can be immediately used in order compute a transformation to the hyperbolic controller form [1], these methods allow the consideration of (particular) smooth solutions of the considered systems only [2]. Therefore, the equivalence of the given interior controlled system and the boundary controlled system has to be carefully checked when considering the usual spaces of square integrable functions as the state space. In the above cited reference this equivalence has been discussed for Neumann boundary conditions while the case of the more general Robin boundary conditions was only partially treated. Aside from the presentation of the numerical approximation scheme the main result of the present contribution lies in bridging this gap.

The paper is organized as follows. In the following section an outline of the proposed method is given. This includes a more detailed sketch of previously obtained results. In Section 3 the main results of the present contribution are presented and proven. Section 4 is devoted to the numerical approximation of the transformation and the stabilizing feedback controller with modal analysis.

2. A SHORT SUMMARY OF THE METHOD

2.1 Models considered and design goal

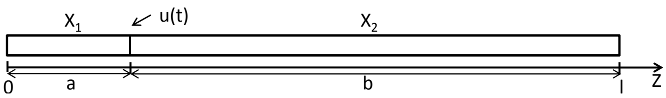

Fig. 1. The parabolic system with in-domain control

Parabolic boundary value problems involving a distributed variable $x(z,t)$ defined on the spatially one-dimensional domain $\Omega = [0,\ell]$ and a lumped control variable $u(t)$ acting at $a \in \Omega$ are considered (cf. Fig. 1). Dividing Ω into the disjoint subdomains $\Omega_1 = [0,a]$ and $\Omega_2 = [a,\ell]$ the mathematical model reads:

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[1] See Russell (1991) for the introduction of the hyperbolic controller form and Woittennek and Rudolph (2012), Woittennek (2013) for the relation of the controller form with flatness

[2] Note, that such time domain interpretations of the obtained results have not been emphasized to in Wang and Woittennek (2013). Instead these results rely on formal computations in the Laplace domain. A detailed discussion of these computations in the time-domain is postponed to a forthcoming publication.

$$\frac{\partial x_1}{\partial t}(z,t) = \frac{\partial^2 x_1}{\partial z^2}(z,t) + c x_1(z,t), \quad z \in \Omega_1 \tag{1a}$$
$$\frac{\partial x_2}{\partial t}(z,t) = \frac{\partial^2 x_2}{\partial z^2}(z,t) + c x_2(z,t), \quad z \in \Omega_2 \tag{1b}$$

with c an arbitrary constant parameter and the variables x_1 and x_2 corresponding to the restrictions of x to the intervals Ω_1 and Ω_2, respectively:

$$x(z,t) = \begin{cases} x_1(z,t) & z \in \Omega_1 \\ x_2(z,t) & z \in \Omega_2. \end{cases}$$

The model is completed by the compatibility conditions

$$\frac{\partial x_1}{\partial z}(a,t) = \frac{\partial x_2}{\partial z}(a,t) + u(t), \tag{2a}$$
$$x_1(a,t) = x_2(a,t), \tag{2b}$$

and the boundary conditions

$$\beta x_1(0,t) = \frac{\partial x_1}{\partial z}(0,t), \tag{2c}$$
$$-\alpha x_2(\ell,t) = \frac{\partial x_2}{\partial z}(\ell,t). \tag{2d}$$

Depending on the values of the constant parameters α, β Neumann ($\alpha, \beta = 0$) or Robin boundary conditions ($\alpha, \beta \neq 0$) can be obtained at $z = 0$ and $z = \ell$, respectively.

The goal of the design process lies in the computation of a continuously invertible linear transformation $w = \Phi x$ and a state feedback $v = u - Kx$ such that that the transformed state satisfies the differential equation

$$\frac{\partial w}{\partial t}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) - \hat{c} w(z,t), \quad z \in \Omega, \tag{3a}$$

with boundary conditions

$$\frac{\partial w}{\partial z}(0,t) = \beta w(0,t), \quad \frac{\partial w}{\partial z}(\ell,t) = v(t) \tag{3b}$$

which is exponentially stable for $v(t) = 0$ when choosing the constant design parameter $\hat{c} \geq 0$.

2.2 Design method

The idea proposed in Wang and Woittennek (2013) basically consists in splitting up the calculation of the state transformation Φ into two steps: Firstly, a transformation $x = T\bar{x}$ is computed in such a way that, in the new coordinates, the system (1), (2) appears as

$$\frac{\partial \bar{x}}{\partial t}(z,t) = \frac{\partial^2 \bar{x}}{\partial z^2}(z,t) + c\bar{x}(z,t), \quad z \in \Omega \tag{4a}$$

with boundary conditions

$$\frac{\partial \bar{x}}{\partial z}(0,t) = \beta \bar{x}(0,t), \tag{4b}$$
$$\frac{\partial \bar{x}}{\partial z}(\ell,t) = u(t) - \alpha \bar{x}(\ell,t). \tag{4c}$$

As already pointed out in the introduction a candidate for the transformation T has been derived in Wang and Woittennek (2013) by means of algebraic computations in the Laplace domain. However, the invertibility of this mapping has been roughly examined for the case $\alpha = \beta = 0$ only. The detailed discussion of the properties of T in section 3 constitutes one of the main results of the present contribution and is omitted for the moment.

Now assume that the transformation T exists and is known. Then the "classical" backstepping approach developed for boundary controlled systems can be applied in the second step: The main ingredient of this second step is the application of the Volterra integral transformation $w = V\bar{x}$ defined for $z \in \Omega$ by

$$w(z,t) = \bar{x}(z,t) - \int_0^z \kappa(z,\zeta)\bar{x}(\zeta,t)\,d\zeta \tag{5}$$

along with the feedback

$$v(t) = u(t) - (\kappa(\ell,\ell) + \alpha)\bar{x}(\ell,t) - \int_0^\ell \frac{\partial \kappa}{\partial z}(\ell,\zeta)\bar{x}(\zeta,t)\,d\zeta. \tag{6}$$

The computation of the kernel function $(z,\zeta) \mapsto \kappa(z,\zeta)$ defined on the triangle $\{(z,\zeta) \in \Omega^2 | \zeta \leq z\}$ has been extensively studied in the cited literature (cf., e.g., Krstic and Smyshlyaev, 2008; Smyshlyaev and Krstic, 2004) and is omitted for brevity.

The feedback law for the original system with in-domain control and the associated state transform to the target system (3) are obtained from (5) and (6) by expressing $\bar{x}$ in terms of the original coordinates x:

$$(\Phi x)(z,t) = (T^{-1}x)(z,t) - \int_0^z \kappa(z,\zeta)(T^{-1}x)(\zeta,t)\,d\zeta$$

and the feedback operator K defined by

$$(Kx)(t) = (\kappa(\ell,\ell) + \alpha)(T^{-1}x)(\ell,t) + \int_0^\ell \frac{\partial \kappa}{\partial z}(\ell,\zeta)(T^{-1}x)(\zeta,t)\,d\zeta. \tag{7}$$

3. STATE EQUIVALENCE TO A BOUNDARY CONTROLLED SYSTEM

This section is devoted to the careful analysis of the following candidate for the transformation T piecewise defined by (cf. Wang and Woittennek, 2013)

$$x(z) = \frac{1}{2}\left[\bar{x}(z+b) + \bar{x}(b-z)\right] - \frac{\alpha}{2}\int_{b+z}^{b-z} \bar{x}(\zeta)\,d\zeta + (\alpha-\beta)\int_0^{b-z} e^{\beta(z+\zeta-b)}\bar{x}(\zeta)\,d\zeta, \quad z \in [0,a] \tag{8a}$$

$$x(z) = \frac{1}{2}\left[\bar{x}(b-z) + \bar{x}(\ell+a-z)\right] - \frac{\alpha}{2}\int_{\ell+a-z}^{b-z} \bar{x}(\zeta)\,d\zeta + (\alpha-\beta)\int_0^{b-z} e^{\beta(z+\zeta-b)}\bar{x}(\zeta)\,d\zeta, \quad z \in [a,b] \tag{8b}$$

$$x(z) = \frac{1}{2}\left[\bar{x}(z-b) + \bar{x}(\ell+a-z)\right] - \frac{\alpha}{2}\int_{\ell+a-z}^{z-b} \bar{x}(\zeta)\,d\zeta, \quad z \in [b,\ell]. \tag{8c}$$

which aims to establish the equivalence of the system of p.d.e. (1) with b.c. (2) and the b.v.p. (4). Therein, without loss of generality $a < l - a =: b$ is assumed.

3.1 Problem formulation

In order to proof the main result the boundary value problem (1), (2) is rewritten as an abstract differential equation on the Hilbert space $X = L_2(\Omega)$:

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{9a}$$

Therein, the unbounded self-adjoint system operator A is given by

$$A = \tfrac{\partial^2}{\partial z^2} + c : D(A) \to X, \tag{9b}$$
$$D(A) = \left\{x \in H^2(\Omega) | \beta x(0) = \tfrac{dx}{dz}(0),\ -\alpha x(\ell) = \tfrac{dx}{dz}(\ell)\right\}$$

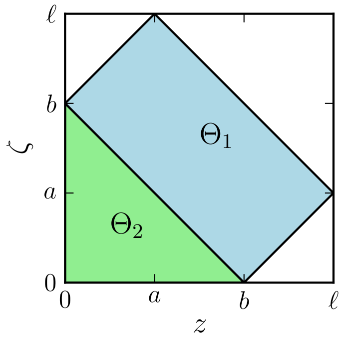

Fig. 2. Supporting sets $\Theta_1, \Theta_2 \subset \Omega^2$ of the integral kernel k of the operator T_c.

and the unbounded input operator B reads

$$B = \delta_a : \mathbb{R} \to D(A)'.$$

Above, $H^2(\Omega)$ corresponds to the Sobolev space of twice (weakly) differentiable functions with second derivative in $L^2(\Omega)$ and δ_a is the Dirac distribution centered at $z = a$, i.e., $\delta_a \varphi = \varphi(a)$ for any $\varphi \in D(A)$.

Similarly, the target differential equation (4) to be satisfied by the transformed state can be written as

$$\dot{\bar{x}}(t) = A\bar{x}(t) + \bar{B}u(t), \tag{10a}$$

with $\bar{B} = \delta_\ell : \mathbb{R} \to D(A)'$. Since the ordinary boundary value problem $Ax = y$ has a unique solution in $D(A)$ for all $y \in X$, A is onto.

A linear continuous transformation

$$T : X \to X, \quad x = T\bar{x} \tag{11}$$

associating the solutions of (9) with those of (10) must satisfy the well known conditions

$$TA = AT, \tag{12a}$$
$$T\bar{B} = B. \tag{12b}$$

This can be easily seen by substituting (11) into (9a)

$$T\dot{\bar{x}}(t) = AT\bar{x}(t) + Bu(t) = TA\bar{x}(t) + T\bar{B}u(t).$$

Moreover, in order obtain a state transform T has to be a bijection.

3.2 Transformation from target to original coordinates

In this section it will be shown that the map (11) defined by (8) or, equivalently, by

$$T\bar{x} = T_0\bar{x} + T_c\bar{x} \tag{13}$$

with

$$(T_0\bar{x})(z) = \frac{1}{2}\Big(h(a-z)\bar{x}(z+b) + h(b-z)\bar{x}(b-z) + h(z-a)\bar{x}(\ell+a-z) + h(z-b)\bar{x}(z-b)\Big) \tag{14}$$

and

$$(T_c\bar{x})(z) = \int_\Omega k(z,\zeta)\bar{x}(\zeta)d\zeta \tag{15}$$

satisfies (12). Therein, h is the Heaviside function and the integral kernel of T_c is given by

$$k(z,\zeta) = \frac{\alpha}{2}\chi_{\Theta_1}(z,\zeta) + (\alpha-\beta)\chi_{\Theta_2}(z,\zeta)\mathrm{e}^{\beta(z+\zeta-b)},$$

where χ_Θ denotes the characteristic function of the set Θ. Moreover, Θ_1 is the rectangle

$$\Theta_1 = \big\{(z,\zeta) \in \Omega^2 : b < \zeta + z < \ell + a\big\} \cap \big\{(z,\zeta) \in \Omega^2 : -b < \zeta - z < b\big\}$$

and Θ_2 is the triangle

$$\Theta_2 = \big\{(z,\zeta) \in \Omega^2 : \zeta + z < b\big\}.$$

Lemma 1. The operator T defined by (13) is self-adjoint.

Proof. The compact integral operator T_c is clearly self-adjoint by the symmetry of its integral kernel. Moreover, for $y, \bar{x} \in X$

$$\begin{aligned}
2\langle y, T_0\bar{x}\rangle_X &= \int_0^a y(z)\bar{x}(z+b)dz + \int_0^b y(z)\bar{x}(b-z)dz + \\
&\quad \int_a^\ell y(z)\bar{x}(\ell+a-z)dz + \int_b^\ell y(z)\bar{x}(z-b)dz \\
&= \int_b^\ell y(z-b)\bar{x}(z)dz + \int_0^b y(b-z)\bar{x}(z)dz + \\
&\quad \int_a^\ell y(\ell+a-z)\bar{x}(z)dz + \int_0^a y(z+b)\bar{x}(z)dz \\
&= 2\langle T_0 y, \bar{x}\rangle_X.
\end{aligned}$$

Therefore, T_0 is self-adjoint as well and so is $T = T_0 + T_c$

Theorem 2. The restriction of the map $T \in \mathcal{L}(L^2(\Omega))$ defined by (13) to $D(A)$ belongs to $\mathcal{L}(D(A))$ where $D(A)$ is equipped with the graph norm $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$. Moreover, $AT = TA$ and $T\bar{B} = B$.

Proof. Firstly, it has to be checked that $TD(A) \subset D(A)$. To this end define

$$\bar{X} = \Big\{\bar{x} \in C^2(\Omega) : \tfrac{d\bar{x}}{dz}(0) = \beta\bar{x}(0),\ \tfrac{d\bar{x}}{dz}(\ell) = -\alpha\bar{x}(0)\Big\}.$$

and assume that $\bar{x} \in \bar{X}$. It will be shown that $T : \bar{X} \to \bar{X}$. The piecewise definition (8) of T immediately shows that $T\bar{x}$ is twice differentiable on each of the sections $(0,a)$, (a,b) and (b,ℓ) (cf. (13)). The corresponding differentiability properties at a and b as well as the fulfilment of the boundary conditions at 0 and ℓ follow by evaluating the corresponding derivatives of (8) at 0, a, b and ℓ, respectively. It is not hard to show that T is continuous on $\bar{X}$ with respect to the graph norm of A. Since, moreover, $\bar{X}$ is a dense subspace of $D(A)$, T can be uniquely extended as a continuous operator on $D(A)$. Now the equality $AT = TA$ can be verified by substituting (13) or (8) into the claim. These simple but tedious computations are omitted for brevity.

In order to check the equality of $T\bar{B} \in D(A)'$ and $B \in D(A)'$ both operators have to be applied to an arbitrary element $\varphi \in D(A)$. By the fact that T is self-adjoint, evaluating (8c) at $z = \ell$ (with $\bar{x} = \varphi$) immediately shows that

$$\langle T\delta_\ell, \varphi\rangle_{D(A)',D(A)} = \langle \delta_\ell, T\varphi\rangle_{D(A)',D(A)} = \varphi(a),$$

with $\langle\cdot,\cdot\rangle$ the duality pairing in $D(A)' \times D(A)$.

The above calculations show that T is an endomorphism on $L^2(\Omega)$ satisfying (12). In order to show that it is indeed a state transformation its invertibility remains to be proven.

3.3 Spectral decomposition and injectivity

Injectivity of T is a necessary condition for the invertibility of T. This property is discussed on the basis of the spectral decomposition of the operator T.

Lemma 3. Assume that no eigenfunction of A vanishes at $z = a$. Then the map $T \in \mathcal{L}(L^2(\Omega))$ defined by (13) is a continuous injection.

Proof. Since the system operator A in (9b) is self-adjoint its spectrum contains only isolated real eigenvalues λ_k, $k \in \mathbb{N}$ with corresponding orthogonal eigenfunctions $\varphi_k \in X$ constituting an orthogonal basis of X. Since T commutes with A it maps eigenfunctions of A to eigenfunctions of A (up to a scaling):

$$\lambda_k \varphi_k = A\varphi_k \;\Rightarrow\; \lambda_k(T\varphi_k) = A(T\varphi_k) \;\Rightarrow\; T\varphi_k = c_k\varphi_k.$$

By the continuity of T one obtains for an arbitrary $\bar{x} \in X$

$$T\bar{x} = T\sum_{k\in\mathbb{N}} \bar{x}_k\varphi_k = \sum_{k\in\mathbb{N}} \bar{x}_k T\varphi_k = \sum_{i\in\mathbb{N}} \bar{x}_k c_k \varphi_k.$$

Therefore, assuming $T\bar{x} = 0$ for some $\bar{x} \neq 0$ yields for all $j \in \mathbb{N}$

$$\langle T\bar{x}, \varphi_j\rangle_X = \bar{x}_j c_j = 0.$$

Since $\bar{x} \neq 0$ not all of the coordinates $\bar{x}_j$ are zero. As a consequence, the non-injectivity of T implies $T\varphi_j = c_j\varphi_j = 0$ and therefore $c_j = 0$ for some j. Since the converse is obvious, T is an injection iff $c_j \neq 0$ for all $j \in \mathbb{N}$. To check this condition the scalar parameters $c_j \in \mathbb{R}$, $j \in \mathbb{N}$ will be determined from the condition $T\bar{B} = B$. To this end the operator B is applied to $\varphi_k \in D(A)$ which, thanks to self-adjointness of T, leads to

$$\langle B, \varphi_k\rangle = \langle T\bar{B}, \varphi_k\rangle = \langle \bar{B}, T\varphi_k\rangle = \langle \bar{B}, c_k\varphi\rangle = c_k\langle \bar{B}, \varphi_k\rangle.$$

with $\langle\cdot,\cdot\rangle$ the duality pairing in $D(A)' \times D(A)$. As a result c_k vanishes iff

$$\langle B, \varphi_k\rangle = \langle \delta_a, \varphi_k\rangle = \varphi_k(a) = 0,$$

which completes the proof.

Lemma 4. Let $\alpha = \beta = 0$. Assume that ℓ/a is rational, i.e., $\ell/a = n/d$ with $n, d \in \mathbb{N}$ co-prime. Then T is an injection iff n is odd.

Proof. For $\alpha = \beta = 0$ the eigenfunctions of A are given by

$$\varphi_k(z) = \cos(\omega_k z), \quad \omega_k = k\frac{\pi}{\ell}, \quad k \in \mathbb{N}$$

The zeros $z_{k,i}$, $0 \leq i < k$ of φ_k satisfy $(i + 1/2) = k\frac{z_{k,i}}{\ell}$ with $i \in \mathbb{N}$. Consequently, in order to ensure that φ_k vanishes at $z = a$, $2k/2i+1 = n/d$ must hold for some integer i. If n is odd this can never happen which implies the injectivity of T by Lemma 3. Moreover, since n even implies d odd by the coprimeness of n and d the condition n even implies the existence of zeros $z_{k,i} = a$ for $k = mn/2$ and $i = (md-1)/2$ for arbitrary odd m which finally shows the non-injectivity of T for even n.

3.4 Invertibility

The main result of the contribution is the invertibility of the transformation T defined by (13) under some additional assumptions:

Theorem 5. Assume that ℓ/a is a rational number satisfying the assumption formulated in Lemma 4. Then the kernel of T is of finite dimension $\mu \in \mathbb{N}$. Moreover, if $\mu = 0$ then the map T defined by (1) is a continuous bijection on X.

Before the main result is proven a similar result for the operator T_0 has to be obtained.

Lemma 6. Under the assumptions of Theorem 5 T_0 is a continuous bijection on X.

Proof. Since T_0 coincides with the operator T obtained for $\alpha = \beta = 0$, T_0 is injective by Lemma 4. Defining $(Rx)(z) = x(\ell - z)$ the operator T_0 corresponds to the restriction of the injective operator

$$T_0^2 : X^2 \to X^2, \quad T_0^2(x_1, x_2) = (T_0x_1, RT_0Rx_2)$$

to the closed T_0^2-invariant subspace

$$X_{\text{sym}} = \{(x_1, x_2) \in X^2 : x_2 = Rx_1\} \subset X^2.$$

On X_{sym} one has $T_0^2(x_1, x_2) = \tilde{T}_0^2(x_1, x_2)$ where

$$\tilde{T}_0^2 : X^2 \to X^2, \quad (\bar{x}_1, \bar{x}_2) \mapsto \tilde{T}_0^2(\bar{x}_1, \bar{x}_2) = (x_1, x_2)$$

is pointwise defined by

$$\begin{aligned} x_1(z) = \frac{1}{2}\Big(&h(a-z)\bar{x}_1(z+b) + h(b-z)\bar{x}_2(z+a) \\ &+ h(z-a)\bar{x}_2(z-a) + h(z-b)\bar{x}_1(z-b)\Big) \end{aligned} \tag{16a}$$

$$\begin{aligned} x_2(z) = \frac{1}{2}\Big(&h(z-b)\bar{x}_2(z-b) + h(z-a)\bar{x}_1(z-a) \\ &+ h(b-z)\bar{x}_1(z+a) + h(a-z)\bar{x}_2(z+b)\Big). \end{aligned} \tag{16b}$$

In order to proceed it is advantageous to decompose the elements of X into n pieces of length ℓ_0. To this end let $\Xi = L^2([0, \ell_0])$ and define the bijective operator

$$\begin{aligned} D : X^2 \to \Xi^n \times \Xi^n, \quad &(x_1, x_2) \mapsto \boldsymbol{\xi} = (\boldsymbol{\xi}_1^T, \boldsymbol{\xi}_2^T)^T \\ &\boldsymbol{\xi}_i = (\xi_{i,0}, \ldots, \xi_{i,n-1})^T, \quad i = 1, 2. \end{aligned}$$

by ($i = 1, 2$, $j = 0, \ldots, n-1$)

$$\xi_{i,j}(z) = x_i(j\ell_0 + z), \quad z \in [0, \ell_0].$$

This way according to the definition (16) of $\tilde{T}_0^2$ the operator $S = D \circ \tilde{T}_0^2 \circ D^{-1}$, $\Xi^{2n} \to \Xi^{2n}$ can be written by means of a real matrix $M \in \mathbb{R}^{2n\times 2n}$

$$\boldsymbol{\xi} = M\bar{\boldsymbol{\xi}}, \quad M = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}, \quad M_1, M_2 \in \mathbb{R}^{n\times n}.$$

$$(M_1)_{i,j} = \tfrac{1}{2}(\delta_{i,j+b} + \delta_{i,j-b})$$
$$(M_2)_{i,j} = \tfrac{1}{2}(\delta_{i,j+a} + \delta_{i,j-a}).$$

Here, $\delta_{i,j}$ denotes the Kronecker symbol. It will be proven that M is invertible. To this end assume the contrary and chose some real vector $(\boldsymbol{v}_1^T, \boldsymbol{v}_2^T)^T \in \ker M$. By the particular structure of M

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{v}_1 + R_M\boldsymbol{v}_2 \\ R_M\boldsymbol{v}_1 + \boldsymbol{v}_2 \end{pmatrix} \in \ker M,$$

$$R_M\boldsymbol{v}_i = (v_{i,n-1}, \ldots, v_{i,0})^T, \quad i = 1, 2.$$

Thus, $\boldsymbol{\xi}_v \in \Xi^2$ defined by $\boldsymbol{\xi}_v(z) = \boldsymbol{v}$ belongs to DX_{sym} while $M\boldsymbol{\xi}_v = 0$ and, therefore, $\boldsymbol{\xi}_v \in \ker S$. Consequently, the intersection of the kernel of $\tilde{T}_0^2$ with X_{sym} is non-empty. Since, T_0^2 and $\tilde{T}_0^2$ coincide on X_{sym}, T_0^2 and, therefore, T is not an injection which contradicts Lemma 4. Now, the invertibility of T_0 follows from that of M.

Proof of Theorem 5. Having shown, that T_0 is a bijection on X, T can be rewritten as $T = T_0 \circ \bar{T}$ where

$\bar{T} = 1 + T_0^{-1} \circ T_c$. Consequently, the equation $T_0^{-1}x = \bar{x} + T_0^{-1} \circ T_c\bar{x}$ is a Fredholm equation of the second kind. As a consequence the kernel of $\bar{T}$ is finite dimensional and the invertibility of $\bar{T}$ immediately follows from its injectivity thanks to the celebrated Fredholm alternative (cf. any text book on functional analysis, e.g., Heuser (1982)) provided $\bar{T}_c = T_0^{-1} \circ T_c$ is compact. The compactness of this latter operator is an immediate consequence of the (obvious) square integrability of the integral kernel k of T which also implies the square integrability of the kernel of $\bar{T}_c$ (cf., e.g., Heuser (1982)).

4. NUMERICAL IMPLEMENTATION

4.1 General approximation scheme

Start with a sequence of N-dimensional subspaces $X_N \subset X$ with basis $(\varphi_1^N, \ldots, \varphi_N^N)$, $\varphi_k^N \in X$ such that for each $x \in X$ there is a sequence $(x^N)_{N\in\mathbb{N}}$ of finite dimensional approximations

$$x^N = \sum_{k=1}^{N} x_k^N \varphi_k^N \tag{17}$$

converging to $x \in X$. An approximation of the derived transformation $\Phi = V \circ T^{-1}$ with V and T defined by[3] (5) and (13) can be easily given provided the transformed basis

$$(\eta_1^N, \ldots, \bar{\eta}_N^N), \quad \eta_k^N = \Phi\varphi_k^N, \quad k = 1, \ldots, N$$

of the N^{th} order approximation is known. By the continuity and the linearity of Φ

$$\begin{aligned}\Phi x &= \Phi \lim_{N\to\infty} \textstyle\sum_{k=1}^{N} x_k^N \varphi_k^N \\ &= \lim_{N\to\infty} \textstyle\sum_{k=1}^{N} x_k^N \Phi\varphi_k^N = \lim_{N\to\infty} \textstyle\sum_{k=1}^{N} x_k^N \eta_k^N.\end{aligned}$$

However, such approximations cannot directly used in the feedback law (3), since the latter feedback law (7) is unbounded and will, therefore, in general not commute with the limit[4]. Therefore, it is advantageous to decompose the feedback law into an unbounded and a continuous part. To this end start with the relation (6) obtained via the backstepping approach applied to the boundary controlled system. The new input v can be expressed as

$$\begin{aligned}v(t) &= \frac{\partial w}{\partial z}(\ell,t) = \left[\frac{\partial}{\partial z} V\bar{x}(z,t)\right]_{z=\ell} = \\ &= \frac{\partial \bar{x}}{\partial z}(\ell,t) - \kappa(\ell,\ell)\bar{x}(\ell,t) - \int_0^\ell \frac{\partial \kappa}{\partial z}(\ell,\zeta)\bar{x}(\zeta,t)d\zeta\end{aligned} \tag{18}$$

Observe that the Volterra integral operator on the right-hand-side of this equation corresponds to a continuous linear operator $K_1 : X \to \mathbb{R}$ which alternatively can be written as

$$K_1\bar{x}(t) = \frac{\partial \bar{x}}{\partial z}(\ell,t) - \kappa(\ell,\ell)\bar{x}(\ell,t) - \left[\frac{\partial}{\partial z} V\bar{x}(z,t)\right]_{z=\ell}.$$

Eliminating the boundary gradient of $\bar{x}$ by means of the boundary condition (4c) and expressing $\bar{x}$ by means of $T^{-1}x$ (18) can be rewritten as

$$v(t) = u(t) + (-\alpha - \kappa(\ell,\ell))(T^{-1}x)(\ell,t) - K_1 \circ T^{-1}x(t). \tag{19}$$

Therein the operator T^{-1} can be decomposed as

$$T^{-1} = T_0^{-1} + T_c^{\text{inv}}$$

where the compact part T_c^{inv} of T^{-1} is again a Fredholm integral operator. As a consequence K_2 defined by

$$K_2x(t) = (T_c^{\text{inv}}x)(\ell,t) = (T^{-1}x - T_0^{-1}x)(\ell,t)$$

is a continuous operator $X \to \mathbb{R}$ and (19) rereads with $\gamma = \alpha + \kappa(\ell,\ell)$

$$v(t) = u(t) - \gamma(T_0^{-1}x)(\ell,t) - (\gamma K_2 + K_1 \circ T^{-1})x(t). \tag{20}$$

This way the feedback law can be divided into the unbounded part which is rather simple to evaluate and, therefore, does not need to be approximated and the continuous part which is accessible for an approximation. The latter reads:

$$(\gamma K_2 + K_1 \circ T^{-1})x(t) = \lim_{N\to\infty} \sum_{k=1}^{N} r_k^N x_k^N(t),$$

$$r_k^N = -\frac{\partial \eta_k^N}{\partial z}(\ell) + \frac{\partial \bar{\varphi}_k^N}{\partial z}(\ell) + \alpha\bar{\varphi}_k^N(\ell) - \gamma(T_0^{-1}\varphi_k^N)(\ell).$$

Above $\bar{\varphi}_k^N = T^{-1}\varphi_k^N$, $k = 1, \ldots, N$, $N \in \mathbb{N}$.

4.2 Modal approximation

For the sake of feedback design the triples $(\varphi_k^N, \bar{\varphi}_k^N, \eta_k^N)$ can be computed offline. Therefore, this (possibly) computational problem won't be an obstacle from an implementation point of view. However, further simplifications are possible, if the approximation is based on a spectral expansion associated with the problem under consideration. To this end assume that, independently of N, $\varphi_k^N = \varphi_k$, $k \in \mathbb{N}$ corresponds to the eigenfunction of A associated with the eigenvalue λ_k.

Now, reconsider the results presented in the proof of Lemma 3 which immediately delivers the desired transformation of the eigenfunctions φ_k, $k \in \mathbb{N}$ to the eigenfunctions $\bar{\varphi}_k = T^{-1}\varphi_k$ of the associated boundary controlled system (4) (resp.):

$$\bar{\varphi}_k = \frac{\langle \bar{B}, \varphi_k \rangle}{\langle B, \varphi_k \rangle}\varphi_k = \frac{\varphi_k(\ell)}{\varphi_k(a)}\varphi_k.$$

The subsequent transformation of the eigenfunctions of the boundary controlled problem to the target coordinates is achieved in a similar way. As the transformation T has to commute with the system operator A the Volterra transformation V satisfies

$$VA = \tilde{A}V,$$

where $\tilde{A} : D(\tilde{A}) \to X$ defined by

$$\begin{aligned}\tilde{A} &= \frac{\partial^2}{\partial z^2} - \hat{c}, \\ D(A) &= \Big\{ w \in H^2(\Omega) | \beta w(0) = \tfrac{dw}{dz}(0), \\ &\qquad \big(\alpha V^{-1}w + \tfrac{dV^{-1}w}{dz}\big)(\ell) = 0 \Big\}\end{aligned}$$

is the system operator associated with the target system (3). From

$$\lambda_k(V\bar{\varphi}_k) = VA\bar{\varphi}_k = \tilde{A}(V\bar{\varphi}_k), \quad k \in \mathbb{N}$$

it follows that eigenfunctions $\bar{\varphi}_k$, $k \in \mathbb{N}$ of A transform to eigenfunctions η_k, $k \in \mathbb{N}$ of $\tilde{A}$. On the other hand, in view

[3] Invertibility of T will be assumed all over this section.

[4] See also Woittennek (2013) for a discussion of a similar issue occurring in connection with flatness based control design for hyperbolic boundary value problems.

of the explicit relation (5), the eigenfunctions transform according to

$$\eta_k(z) = \bar{\varphi}_k(z) - \int_0^z \kappa(z,\zeta)\bar{\varphi}_k(\zeta)d\zeta.$$

Therefore, $\eta_k(0) = \bar{\varphi}_k(0)$ and the eigenfunctions of the target operator are given as the unique solution of the initial value problem

$$\tilde{A}\eta_k = \lambda_k\eta_k, \quad \eta_k(0) = \bar{\varphi}_k(0), \quad \frac{d\eta_k}{dz}(0) = \beta\bar{\varphi}_k(0).$$

5. CONCLUSION AND OUTLOOK

This contribution supplements the results presented in Wang and Woittennek (2013) concerning the extension of the so-called backstepping method to parabolic systems with constant coefficients and pointwise interior actuation. In combination with the proposed numerical control scheme the obtained results allow for the systematic and efficient design of exponentially stabilizing feedback control laws for these systems. In contrast to the "classical" backstepping approach a Volterra integral transformation is not sufficient to obtain the desired control law. Instead more involved Fredholm transformations come into play. Moreover, the obtained results show that the invertibility of such transformations are not guaranteed and have to be carefully checked.

Directions for further research are manifold. The possibly most interesting of them will be the generalization of the obtained results to systems with spatially dependent parameters, i.e., parabolic equations of the form (9) with A given by

$$(Ax)(z) = g_2(z)\frac{\partial^2 x}{\partial z^2}(z,t) + g_1(z)\frac{\partial x}{\partial z}(z,t) + g_0(z)x(z,t).$$

Moreover, even more general input operators B could be considered. As the results of Section 4 suggest this would not essentially complicate the numerical implementation, as long as A is a Riesz spectral operator A. However, the discussion of the existence and invertibility of the involved transformations will most likely turn out to be much more involved. A possible prerequisite for such extensions is possibly a deeper and rigorous discussion of the computations leading to the Fredholm transformation (13) as in Wang and Woittennek (2013). Other important models possibly accessible by the proposed method are wave equations with Kelvin-Voight damping.

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