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Robustness of the Moore-Greitzer Compressor Model's Surge Subsystem with New Dynamic Output Feedback Controllers ⋆

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Abstract: This work presents an extension of a design procedure for dynamic output feedback design for systems with nonlinearities satisfying quadratic constraints. In this work we used an axial gas compressor model described by the 3-state Moore-Greitzer compressor model (MG) that has some challenges for output feedback control design (Planovsky and Nikolaev 1990), (Rubanova 2013). The more general constraints for the investigation of the robustness with respect to parametric uncertainties and measurement noise are shown.

Keywords: Dynamic output feedback, Compressors, Nonlinear control systems, Stability robustness, Constraints, Nonlinearity, Lyapunov stability

1. INTRODUCTION

A gas compressor is a mechanical device for compressing and supplying the air or other gas under a certain pressure. An axial gas compressor is one of the main components of gas turbines, aircraft jet engines, high-speed ship engines and small-scale power stations (Greitzer 1976). They are also widely used in high-voltage installations in the blast furnaces, in the chemical and petroleum industries (Vedernikov 1974).

In 1986 Moore and Greitzer published a differential equations model describing the airflow through the compression system in turbomachines (such as gas turbines, fans, etc.) (Moore and Greitzer 1986). The Moore-Greitzer model includes the differential equations:

$$
\begin{aligned}
\frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1-(1+\phi)^3}{2} - 3R(1+\phi) \\
\frac{d}{dt}\psi &= \frac{1}{\beta^2}(\phi - u) \\
\frac{d}{dt}R &= -\sigma R^2 - \sigma R(2\phi + \phi^2),\ R(0) \geq 0 \\
y &= \psi
\end{aligned}
\tag{1}
$$

Here, both ϕ and ψ represent the deviations of the averaged flow and pressure from their respective nominal values. The variable u is the control variable and y is the measurement (with only deviation of the averaged pressure available).

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The stall R is the squared amplitude of oscillations of the rotary derangement. An inlet flow deformation may result in the rotating stall, surge (or axi-symmetric stall), or a combination of them (Greitzer 1976). The flow oscillations and strong vibrations of the blades in the machine may cause damage to the complete engine and even flow reversal is possible (Ng 2007). The fully developed surge is not viewed as an unstable state of the compressor but as a set of equilibria along which R is nonzero Paduano et al. (2001).

Subsequently we thus want to control the compressor dynamics to the appropriate set-point $(\phi,\ \psi,\ R) = (0,\ 0,\ 0)$.

The challenges to stabilize the origin of the 3-state Moore-Greitzer model that were presented before in (Moore and Greitzer 1986), (Shiriaev 2009), (Shiriaev 2010), (Rubanova 2013) are:

- the linearized dynamics are not stabilizable;
- the fact that R cannot be measured or used for feedback design;
- the presence of non-globally Lipschitz (cubic) nonlinearity;
- the nonlinearity in ϕ-dynamics is known only approximately.

In this work we discuss the degree of robustness and present the method that will simplify the controller choice based on a specific task. By this method and with the help of the previous project results one can choose the optimal controller for the given model and analyze the quality of the controller design.

2. THE CONTROL SYNTHESIS METHOD (SHIRIAEV 2010)

We will apply this method to the surge subsystem of the MG compressor model (1). The surge subsystem is:

$$\begin{aligned} \frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1-(1+\phi)^3}{2} \\ \frac{d}{dt}\psi &= \frac{1}{\beta^2}(\phi - u) \\ y &= \psi \end{aligned} \tag{2}$$

with the nonlinearity

$$W^{\{\phi\}}(\phi) = 1-(1+\phi)^3 \tag{3}$$

We used the general form of a dynamic output feedback control law (Shiriaev 2010):

$$u = \mathcal{U}(z,y), \quad \dot{z} = \mathcal{F}(z,y) \tag{4}$$

where $\mathcal{U}(\cdot)$ and $\mathcal{F}(\cdot)$ are smooth functions of appropriate dimensions. The family of stabilizing output feedback controllers has the following structure:

$$\begin{aligned} u &= \Lambda_\psi^{\{u\}}\psi + \Lambda_z^{\{u\}}z + \omega_u \cdot W(\psi,z) \\ \frac{d}{dt}z &= \Lambda_\psi^{\{z\}}\psi + \Lambda_z^{\{z\}}z + \omega_z \cdot W(\psi,z) \end{aligned} \tag{5}$$

with $z \in \mathbb{R}$, where $\Lambda^{\{u\}}$, $\Lambda_z^{\{u\}}$, $\Lambda_\psi^{\{z\}}$, $\Lambda_z^{\{z\}}$, ω_u, ω_z are constants to be defined.

The nonlinearities in the controller of Eq. (5) are static nonlinearities and defined as

$$W(\psi,z) = 1-(1+t_\psi\psi+t_z z)^3 \tag{6}$$

where t_ψ, t_z are constants to be defined too.

The closed-loop system with the surge subsystem of Eq. (2) and the controller of Eq. (5) takes the form:

$$\begin{bmatrix}\dot\phi\\ \dot\psi\\ \dot z\end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{\beta^2} & -\frac{\Lambda_\psi^{\{u\}}}{\beta^2} & -\frac{\Lambda_z^{\{u\}}}{\beta^2} \\ 0 & \Lambda_\psi^{\{z\}} & \Lambda_z^{\{z\}} \end{bmatrix}}_{=\mathcal{A}_{cl}} \begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{\omega_u}{\beta^2} \\ 0 & \omega_z \end{bmatrix}}_{\mathcal{B}_{cl}=[\mathcal{B}_{cl_1},\ \mathcal{B}_{cl_2}]} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W(\psi,z)\end{bmatrix} \tag{7}$$

with $z \in \mathbb{R}$ and the output matrix $\mathcal{C}_{cl_2} = [0,\ t_\psi,\ t_z]$.

In (Rubanova 2013) we presented some stabilizing controllers and new constraints for the corresponding parameters.

The task now is to analyze the quality of the set of the stabilizing controllers presented.

3. ROBUSTNESS OF THE CLOSED-LOOP SURGE SUBSYSTEM

The synthesis of stabilizing controllers and their application to the surge subsystem of the MG compressor model was presented in (Rubanova 2013). The alternative proof of stability of the closed-loop system of Eq. (7) is based on the Circle criterion (Yakubovich 2004), (Khalil 2002).

As we already know, in the closed-loop system of Eq. (7) there are two nonlinearities of Eqs. (3) and (6). We will simplify the notation

$$\begin{aligned} W^{\{\phi\}}(\upsilon_1) &= W^{\{\phi\}}(\phi) \\ W(\upsilon_2) &= W(\psi,z) \end{aligned} \tag{8}$$

where

$$\begin{aligned} \upsilon_1 &= \mathcal{C}_{cl_1}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} = [\,1\ \ 0\ \ 0\,]\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} \\ \upsilon_2 &= \mathcal{C}_{cl_2}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} = [\,0\ \ t_\psi\ \ t_z\,]\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} \end{aligned} \tag{9}$$

One of the main parts of the constructive steps in design is that nonlinearities $W^{\{\phi\}}(\upsilon_1)$ and $W(\upsilon_2)$ have to satisfy the quadratic constraints

$$\begin{aligned} -\mathcal{C}_{cl_1}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} W^{\{\phi\}}(\upsilon_1) - \frac{3}{4}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix}^T \mathcal{C}_{cl_1}^T\mathcal{C}_{cl_1}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} \ge 0 \\ -\mathcal{C}_{cl_2}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} W(\upsilon_2) - \frac{3}{4}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix}^T \mathcal{C}_{cl_2}^T\mathcal{C}_{cl_2}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} \ge 0 \end{aligned} \tag{10}$$

By using the simplified notation of Eq. (8) we can rewrite the quadratic constraints of Eq. (10)

$$\begin{aligned} -W^{\{\phi\}}(\upsilon_1)\upsilon_1 - \frac{3}{4}|\upsilon_1|^2 \ge 0 \\ -W(\upsilon_2)\upsilon_2 - \frac{3}{4}|\upsilon_2|^2 \ge 0 \end{aligned} \tag{11}$$

Since the static nonlinearity $W(\upsilon_2)$ is assumed to resemble the original nonlinearity $W^{\{\phi\}}(\upsilon_1)$ of the system of Eq. (2) we need to have one additional constraint that will be connected to both nonlinearities

$$-(W^{\{\phi\}}(\upsilon_1) - W(\upsilon_2))(\upsilon_1 - \upsilon_2) \ge 0 \tag{12}$$

The three quadratic constraints of Eqs. (11-12) should be satisfied $\forall \phi, \psi, z$.

In general we have

$$\dot{x} = \mathcal{A}_{cl}x + [\mathcal{B}_{cl_1},\ \mathcal{B}_{cl_2}]\begin{bmatrix}\tilde\omega_1\\ \tilde\omega_2\end{bmatrix} \tag{13}$$

where $x = [\phi\ \psi\ z]^T$ and $\tilde\omega_1, \tilde\omega_2$ represent the nonlinearities of the form $W^{\{\phi\}}(\phi)$ and $W(\psi,z)$ and satisfy the given conditions of Eqs. (11-12).

To check the stability of the closed-loop system of Eq. (7) we will use the *Circle criterion* (CC) (Yakubovich 2004), (Shiriaev 2010).

Stability conditions by using the Circle criterion Following the CC, to claim stability of the closed-loop system of Eq. (7) it is enough to check the following two conditions:

(1) There are constants $\tau_1 \ge 0,\ \tau_2 \ge 0,\ \tau_3 \ge 0$ such that $\tau_1 + \tau_2 + \tau_3 > 0$ and there are transfer functions

$$\begin{aligned} G_{11}(j\omega) &= \mathcal{C}_{cl_1}(j\omega I_n - \mathcal{A}_{cl})^{-1}\mathcal{B}_{cl_1} \\ G_{12}(j\omega) &= \mathcal{C}_{cl_1}(j\omega I_n - \mathcal{A}_{cl})^{-1}\mathcal{B}_{cl_2} \\ G_{21}(j\omega) &= \mathcal{C}_{cl_2}(j\omega I_n - \mathcal{A}_{cl})^{-1}\mathcal{B}_{cl_1} \\ G_{22}(j\omega) &= \mathcal{C}_{cl_2}(j\omega I_n - \mathcal{A}_{cl})^{-1}\mathcal{B}_{cl_2} \end{aligned} \tag{14}$$

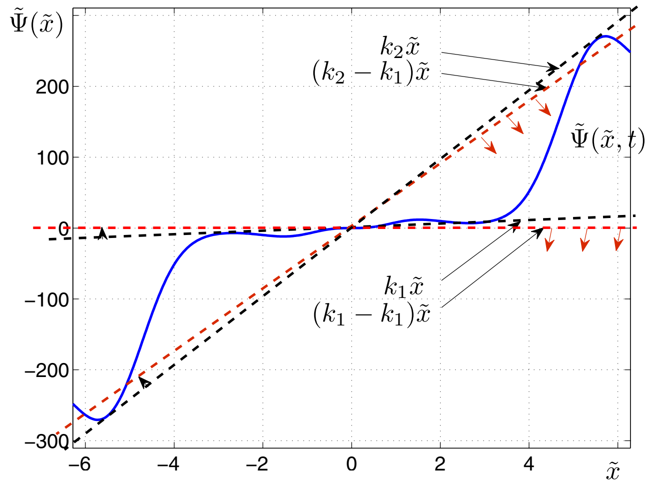

Fig. 1. The example of the possibility to move the sector condition for the nonlinearity (loop-shaping).

such

$$-\tau_1\text{Re}\{\tilde{\omega}_1^*\tilde{\upsilon}_1+\frac{3}{4}|\tilde{\upsilon}_1|^2\}-\tau_2\text{Re}\{\tilde{\omega}_2^*\tilde{\upsilon}_2+\frac{3}{4}|\tilde{\upsilon}_2|^2\} -\tau_3\text{Re}\{(\tilde{\omega}_1-\tilde{\omega}_2)^*(\tilde{\upsilon}_1-\tilde{\upsilon}_2)\}<0 \quad (15)$$

holds $\forall\tilde{\omega}_1\in\mathbb{C},\ \forall\tilde{\omega}_2\in\mathbb{C},\ \forall\omega\in\mathbb{R}$, where

$$\begin{aligned}\tilde{\upsilon}_1&=G_{11}(j\omega)\tilde{\omega}_1+G_{12}(j\omega)\tilde{\omega}_2\\ \tilde{\upsilon}_2&=G_{21}(j\omega)\tilde{\omega}_1+G_{22}(j\omega)\tilde{\omega}_2\end{aligned} \quad (16)$$

(2) There are row matrices K_1 and K_2 such that K_1x and K_2x are linear relations satisfy all the quadtaric constraints and the matrix

$$(\mathcal{A}_{cl}+\mathcal{B}_{cl_1}K_1+\mathcal{B}_{cl_2}K_2) \quad (17)$$

is Hurwitz.

First, in order to be able to use these two conditions we will choose the gains K_1 and K_2. In Fig. 1 the sector condition for some nonlinearity $\tilde{\Psi}(\tilde{x},t)$ is illustrated as an example. The given nonlinearity never leaves the sector area between two lines

$$k_2\tilde{x}\geq\tilde{\Psi}(\tilde{x},t)\geq k_1\tilde{x} \quad (18)$$

We can move the whole sector and the given nonlinearity clockwise on the same angle as the angle between the zero and the line $k_1\tilde{x}$. As a result, we will have a new nonlinearity and a new sector condition for it to simplify the following calculations.

We will rewrite the closed-loop system of Eq. (7) and Eq. (13) as follows

$$\begin{aligned}\dot{x}&=\mathcal{A}_{cl}x+[\mathcal{B}_{cl_1}\mathcal{B}_{cl_2}]\begin{bmatrix}\tilde{\omega}_1+\frac{3}{4}\mathcal{C}_{cl_2}x\\ \tilde{\omega}_2+\frac{3}{4}\mathcal{C}_{cl_2}x\end{bmatrix}\\ &-[\mathcal{B}_{cl_1}\mathcal{B}_{cl_2}]\begin{bmatrix}\frac{3}{4}\mathcal{C}_{cl_2}x\\ \frac{3}{4}\mathcal{C}_{cl_2}x\end{bmatrix}\end{aligned} \quad (19)$$

That gives us

$$\begin{aligned}\dot{x}&=\mathcal{A}_{cl}x-[\mathcal{B}_{cl_1}\ \mathcal{B}_{cl_2}]\begin{bmatrix}\frac{3}{4}\mathcal{C}_{cl_2}x\\ \frac{3}{4}\mathcal{C}_{cl_2}x\end{bmatrix}\\ &+[\mathcal{B}_{cl_1}\ \mathcal{B}_{cl_2}]\begin{bmatrix}\hat{\omega}_1\\ \hat{\omega}_2\end{bmatrix}\\ &=\left[\mathcal{A}_{cl}-\frac{3}{4}\mathcal{B}_{cl_1}\mathcal{C}_{cl_2}-\frac{3}{4}\mathcal{B}_{cl_2}\mathcal{C}_{cl_2}\right]x\\ &+[\mathcal{B}_{cl_1}\ \mathcal{B}_{cl_2}]\begin{bmatrix}\hat{\omega}_1\\ \hat{\omega}_2\end{bmatrix}\end{aligned} \quad (20)$$

where

$$\hat{A}_{cl}=\left[\mathcal{A}_{cl}-\frac{3}{4}\mathcal{B}_{cl_1}\mathcal{C}_{cl_2}-\frac{3}{4}\mathcal{B}_{cl_2}\mathcal{C}_{cl_2}\right] \quad (21)$$

is new state matrix of a form of Eq. (17) and $\hat{\omega}_1,\hat{\omega}_2$ are new nonlinearities such that

$$\hat{\omega}_1=\tilde{\omega}_1+\frac{3}{4}\mathcal{C}_{cl_2}x,\ \hat{\omega}_2=\tilde{\omega}_2+\frac{3}{4}\mathcal{C}_{cl_2}x \quad (22)$$

and satisfy the same conditions of Eqs. (11-12).

The new quadratic constraints are

$$\begin{aligned}\hat{\omega}_2^*\upsilon_2&\leq 0\\ (\hat{\omega}_1-\hat{\omega}_2)^*(\upsilon_1-\upsilon_2)&\leq 0\end{aligned} \quad (23)$$

Second, as presented in (Shiriaev 2010), by choosing $\tau_1,\ \tau_2$ or τ_3 equal to zero we can remove one of the terms of Eq. (25). For a start, we have to introduce the following notations for new transfer functions in order to avoid the presence of long expressions:

$$\begin{aligned}\hat{G}_{11}(j\omega)&=\mathcal{C}_{cl_1}(j\omega I_n-\hat{\mathcal{A}}_{cl})^{-1}\mathcal{B}_{cl_1}\\ \hat{G}_{12}(j\omega)&=\mathcal{C}_{cl_1}(j\omega I_n-\hat{\mathcal{A}}_{cl})^{-1}\mathcal{B}_{cl_2}\\ \hat{G}_{21}(j\omega)&=\mathcal{C}_{cl_2}(j\omega I_n-\hat{\mathcal{A}}_{cl})^{-1}\mathcal{B}_{cl_1}\\ \hat{G}_{22}(j\omega)&=\mathcal{C}_{cl_2}(j\omega I_n-\hat{\mathcal{A}}_{cl})^{-1}\mathcal{B}_{cl_2}\\ G_1(j\omega)&=\hat{G}_{22}(j\omega)+\hat{G}_{21}(j\omega)\\ G_2(j\omega)&=\hat{G}_{11}(j\omega)-\hat{G}_{21}(j\omega)\\ G_3(j\omega)&=\hat{G}_{12}(j\omega)-\hat{G}_{22}(j\omega)\\ G_4(j\omega)&=G_3(j\omega)+G_2(j\omega)\end{aligned} \quad (24)$$

We will use $\tau_1=0$ and $\tau_2=1$ to be able to reduce the inequality of Eq.(25) with the new nonlinearities of Eq. (22) to

$$-Re\{\hat{\omega}_2^*\hat{\upsilon}_2\}-\tau_3Re\{(\hat{\omega}_1-\hat{\omega}_2)^*(\hat{\upsilon}_1-\hat{\upsilon}_2)\}<0 \quad (25)$$

with

$$\begin{aligned}\hat{\upsilon}_1&=\hat{G}_{11}(j\omega)\hat{\omega}_1+\hat{G}_{12}(j\omega)\hat{\omega}_2\\ \hat{\upsilon}_2&=\hat{G}_{21}(j\omega)\hat{\omega}_1+\hat{G}_{22}(j\omega)\hat{\omega}_2\end{aligned} \quad (26)$$

that holds $\forall\hat{\omega}_1\in\mathbb{C},\ \forall\hat{\omega}_2\in\mathbb{C},\ \forall\omega\in\mathbb{R}$.

To present the inequality of Eq. (25) as the matrix form we have to change the variables as follows:

(1) from the first part of the inequality of Eq. (25) we have

$$\begin{aligned}&Re\{\hat{\omega}_2^*\hat{\upsilon}_2\}=\\ &=Re\{\hat{\omega}_2^*(\hat{G}_{21}(j\omega)\hat{\omega}_1+G_1(j\omega)\hat{\omega}_2-\hat{G}_{21}(j\omega)\hat{\omega}_2)\}\\ &=\frac{1}{2}\{\hat{\omega}_2^*\hat{G}_{21}(j\omega)(\hat{\omega}_1-\hat{\omega}_2)\}\\ &+\frac{1}{2}\{(\hat{\omega}_1-\hat{\omega}_2)^*\hat{G}_{21}^*(j\omega)\hat{\omega}_2\}+Re\{\hat{\omega}_2^*G_1(j\omega)\hat{\omega}_2\}\end{aligned} \quad (27)$$

(2) from the second part of the inequality of Eq. (25) we have

$$\begin{aligned}&\tau_3 Re\{(\hat{\omega}_1-\hat{\omega}_2)^*(\hat{v}_1-\hat{v}_2)\} =\\ &= \tau_3 Re\{(\hat{\omega}_1-\hat{\omega}_2)^*(\hat{\omega}_1(\hat{G}_{11}(j\omega)-\hat{G}_{21}(j\omega)))\}\\ &+\tau_3 Re\{(\hat{\omega}_1-\hat{\omega}_2)^*(\hat{\omega}_2(\hat{G}_{12}(j\omega)-\hat{G}_{22}(j\omega)))\}\\ &= \tau_3 Re\{(\hat{\omega}_1-\hat{\omega}_2)^* G_2(j\omega)(\hat{\omega}_1-\hat{\omega}_2)\} \\ &+\frac{1}{2}(\hat{\omega}_1-\hat{\omega}_2)^*\tau_3 G_4(j\omega)\hat{\omega}_2\\ &+\frac{1}{2}\hat{\omega}_2^*\tau_3 G_4^*(j\omega)(\hat{\omega}_1-\hat{\omega}_2)\end{aligned} \quad (28)$$

Now we are able to rewrite the inequality of Eq. (25) in the matrix form

$$\begin{bmatrix}\hat{\omega}_1\\ \hat{\omega}_1-\hat{\omega}_2\end{bmatrix}^* \Pi(j\omega)\begin{bmatrix}\hat{\omega}_1\\ \hat{\omega}_1-\hat{\omega}_2\end{bmatrix} > 0 \quad (29)$$

with

$$\Pi(j\omega)=\begin{bmatrix}\mathrm{Re}\{G_1(j\omega)\} & 0.5G_5(j\omega)\\ 0.5G_5^*(j\omega) & \tau_3\mathrm{Re}\{G_2(j\omega)\}\end{bmatrix} \quad (30)$$

with

$$\begin{aligned}G_1(j\omega)&=\hat{G}_{22}(j\omega)+\hat{G}_{21}(j\omega)\\ G_5(j\omega)&=\hat{G}_{21}(j\omega)+\tau_3 G_4^*(j\omega)\end{aligned} \quad (31)$$

and it should be valid for some $\tau_3 > 0$, $\forall\tilde{\omega}_1,\tilde{\omega}_2 \in \mathbb{C}$ and $\forall\omega\in\mathbb{R}$.

The inequality of Eq. (29) is positive if the matrix $\Pi(j\omega)$ of Eq. (30) is positive definite. The constraint of Eq. (29) includes the both stability conditions of Eq. (15) and Eq. (17).

3.1 Example

To show the benefit of the alternative proof we choose the same numerical values for the linear part for the controller (5) as in (Rubanova 2013):

$$\Lambda_\psi^{\{u\}}=-19,\ \Lambda_z^{\{u\}}=-7,\ \Lambda_\psi^{\{z\}}=-73,\ \Lambda_z^{\{z\}}=-26 \quad (32)$$

and the corresponding parameters of the nonlinear part of the controller are:

$$\begin{aligned}t_\psi&=4.7168,\ t_z=1.4492;\\ \omega_u&=-1,\ \omega_z=-2.9027;\end{aligned} \quad (33)$$

The controller is thus given by

$$\begin{aligned}u&=-19\psi-7z-W(\psi,z)\\ \frac{d}{dt}z&=-73\psi-26z-2.9027W(\psi,z)\end{aligned} \quad (34)$$

where

$$W(\psi,z)=1-(1+4.7168\psi+1.4492z)^3 \quad (35)$$

Now we are able to check the conditions of Eq. (15), (21):

(1) The transfer functions are

$$\begin{aligned}G_1(j\omega)&=\frac{0.5102s+0.8708}{s^2+2.671s+0.8143}\\ G_{21}(j\omega)&=\frac{2.358s+8.423}{s^3+5.883s^2+9.393s+2.615}\\ G_2(j\omega)&=\frac{0.5s+0.1858}{s^2+3.563s+1.127}\\ G_4(j\omega)&=\frac{-0.01021s-7.644\cdot 10^{-4}}{s^2+3.563s+1.127}\end{aligned} \quad (36)$$

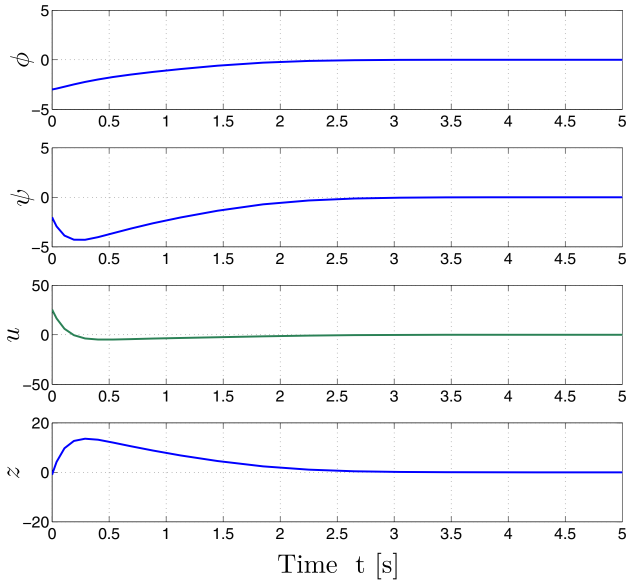

Fig. 2. Simulation results of the closed-loop system with the controller of Eq. (5) and the system of Eq. (2).

We already know that the inequality of Eq. (29) is positive if the matrix $\Pi(j\omega)$ of Eq. (30) is positive definite. In our case we have a 2×2 matrix and it will be sufficiently to show that its determinant and diagonal elements are positive.

The determinant of the matrix of Eq. (30) is positive for

$$\tau_3 > 2.64 \quad (37)$$

The higher tolerance for state elimination or pole-zero cancellation of transfer functions of Eq. (36) forces additional cancellations. For this example we used high tolerance to be able to show the calculations with the lower order polynomials. For higher order polynomials the derivation of the determinant of the matrix $\Pi(j\omega)$ will be complicated, but we will get approximately the same numerical value for τ_3. Hence, the rough approximation was chosen in order to simplify the method presentation in this work.

(2) The eigenvalues of the matrix of Eq. (21) are $[-0.1268;\ -0.8753;\ -7.8730]$, hence it is Hurwitz.

Running the simulation model with the controller of Eq. (5) shows that the closed-loop system of Eq. (7) is stable. The result of the simulation is shown in Fig. 2.

4. THE LYAPUNOV FUNCTION SEARCH METHOD FOR THE KNOWN SUBSYSTEM

In the MG compressor model of Eq. (1) the deviation of the averaged flow ϕ is not available for the measurements. The controller parameters are chosen separately for different subsystems of the closed-loop system or Eq. (7). We refer to (Shiriaev 2010) and (Rubanova 2013) for a more detailed explanation.

Thereby, to simplify the calculations, we will find the quadratic function for the known subsystems from the system of Eq. (7)

$$\begin{bmatrix}\dot{\psi}\\ \dot{z}\end{bmatrix}=\mathcal{A}_1\begin{bmatrix}\psi\\ z\end{bmatrix}+\mathcal{B}_{cl_2}W(\psi,z) \quad (38)$$

First, we choose the quadratic function as

$$V(\psi, z) = \frac{1}{2}\begin{bmatrix}\psi \\ z\end{bmatrix}^T P \begin{bmatrix}\psi \\ z\end{bmatrix} \tag{39}$$

with a matrix $P^T = P > 0$ such that the time-derivative of $V(\psi, z)$ is non positive.

The quadratic constraint of Eq. (10) is valid for the static nonlinearity $W(\psi, z)$ from the closed-loop system with the subsystem of Eq. (38) and the controller of Eq. (5).

Then the time-derivative of the quadratic function of Eq. (39) is

$$\begin{aligned}
&\frac{d}{dt}V(\psi, z) = \\
&= \frac{1}{2}\begin{bmatrix}\psi \\ z\end{bmatrix}^T \left[\mathcal{A}_1^T P + P\mathcal{A}_1\right]\begin{bmatrix}\psi \\ z\end{bmatrix} + \begin{bmatrix}\psi \\ z\end{bmatrix}^T P\mathcal{B}_{cl_2}W(\psi, z) \\
&\leq \frac{1}{2}\begin{bmatrix}\psi \\ z\end{bmatrix}^T \left[\mathcal{A}_1^T P + P\mathcal{A}_1\right]\begin{bmatrix}\psi \\ z\end{bmatrix} + \begin{bmatrix}\psi \\ z\end{bmatrix}^T P\mathcal{B}_{cl_2}W(\psi, z) \\
&- \mathcal{C}_{cl_2}\begin{bmatrix}\psi \\ z\end{bmatrix} W(\psi, z) - \frac{3}{4}\begin{bmatrix}\psi \\ z\end{bmatrix}^T \mathcal{C}_{cl_2}^T\mathcal{C}_{cl_2}\begin{bmatrix}\psi \\ z\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}\psi \\ z\end{bmatrix}^T \left[\mathcal{A}_1 P + P\mathcal{A}_1 - \frac{3}{2}\mathcal{C}_{cl_2}^T\mathcal{C}_{cl_2}\right]\begin{bmatrix}\psi \\ z\end{bmatrix} \\
&+ \begin{bmatrix}\psi \\ z\end{bmatrix}^T \left[P\mathcal{B}_{cl_2} - \mathcal{C}_{cl_2}^T\right] W(\psi, z)
\end{aligned} \tag{40}$$

where matrices $\mathcal{B}_{cl_2}$, $\mathcal{C}_{cl_2}$ are the same as in Eq. (7).

A sufficient condition of the existence of the negative time derivative of the quadratic function of Eq. (39) can be written as

$$\begin{cases} \mathcal{A}_1^T P + P\mathcal{A}_1 - \frac{3}{2}\mathcal{C}_{cl_2}^T\mathcal{C}_{cl_2} < 0 \\ P\mathcal{B}_{cl_2} = \mathcal{C}_{cl_2}^T \end{cases} \tag{41}$$

To find the the matrix P for the expression of Eq. (41) we solve the convex optimization problem by using CVX—a software tool for convex optimization by (Boyd and Vandenberghe 2004). The specification for CVX looks like:

```
cvx_begin sdp
variable P1(2,2) symmetric
minimize(0)
subject to
P>0;
A1'*P+P*A1-3/2* C2'*C2<0
P*B2==C2';
cvx_end
```

For the example in the subsection 3.1 we get

$$P = \begin{bmatrix} 31.5657 & 10.0621 \\ 10.0621 & 3.2168 \end{bmatrix} \tag{42}$$

In Fig. 3 the time derivative of the quadratic function of Eq. (39) is presented.

The matrix P of Eq. (42) is an approximate solution and it is depending on the chosen accuracy for the calculations. For example, the presence of the equality in the expression of Eq. (41) is making this mathematical problem complicated. In addition the CVX solvers will be not able to find the solution for the similar condition as in Eq. (41) but for the closed-loop surge subsystem of Eq. (7). That is why for the higher order calculations we suggest to use the condition of Eq. (29) to investigate the robustness of the controller of Eq. (5).

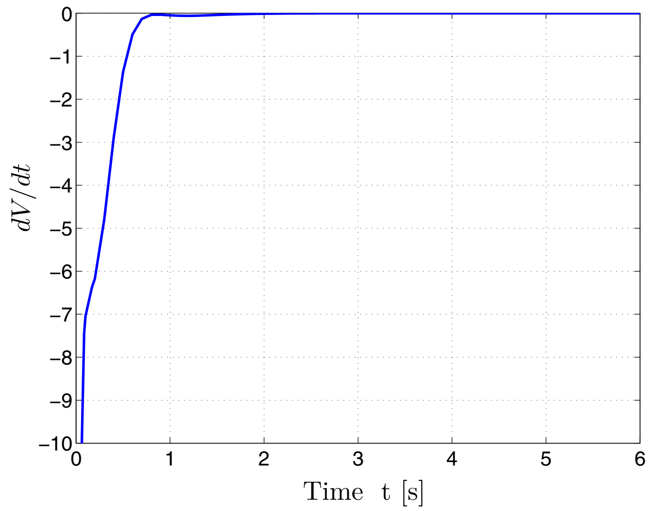

Fig. 3. The time derivative of the quadratic function of Eq. (39).

5. CONCLUSIONS

In control theory it is very important to investigate the robustness of the stabilizing controllers. We presented an extension of a previous research based on a procedure for dynamic output feedback design for systems with nonlinearities satisfying quadratic constraints (Shiriaev 2009), (Shiriaev 2010), (Rubanova 2013). The new constraint for the robustness investigation were presented as an inequality of Eq. (29).

The coefficient τ_3 belongs to the condition for the nonlinearities of Eq. (12) as shown in the inequality of Eq. (15). It is possible to find some positive τ_3 for the controllers of Eq. (5). In other words, the controllers have a certain degree of robustness and there is a possibility to resist some of the parametric uncertainty and measurement noise.

The alternative proof of stability of the closed-loop system of Eq. (7) uses less number of conditions, then in (Rubanova 2013). The results are applicable for dynamic output feedback design for systems with nonlinearities satisfying quadratic constraints. Also, there is a possibility to include a new quadratic constraint in the analysis.

The inequality of Eq. (29) is a more general stability condition than presented in the previous part of the research in (Rubanova 2013). The matrix $\Pi(j\omega)$ of Eq. (29) is shown.

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