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Output Feedback Model Predictive Control: a probabilistic approach

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Abstract: In this paper a novel approach for output-feedback model predictive control of discrete-time linear stochastic systems is presented, allowing for the presence of unbounded noise. The satisfaction of probabilistic constraints and the stability properties of the method are guaranteed by imposing suitable constraints on the mean and the variance of the state variable. A simulation example is presented, illustrating the effectiveness of the proposed control scheme.

1. INTRODUCTION

Robust model predictive control (MPC) algorithms have been thoroughly studied in the past years (see, e.g., Magni and Scattolini (2007); Raimondo et al. (2009); Mayne et al. (2005), and the references therein), based on different rationales, e.g., min-max and tube-based methods. These methods prove to be very effective in case of bounded deterministic disturbances and for deterministic constraints. However, in case stochastic disturbances affect the system and chance constraints are required (i.e., constraints must be satisfied in probability) different algorithms are called for. Recently, two approaches have been explored in the framework of probabilistic MPC. The first approach, investigated, e.g., in Calafiore and Fagiano (2013); Batina (2004); Blackmore et al. (2010); Ono (2012) is based on randomized and scenario-based methods, which are nowadays rather computationally demanding for practical implementations. The second approach is characterized by the fact that the stochastic control problem and the probabilistic constraints are formulated as deterministic ones by exploiting the a priori known statistical description of the noise or of the model uncertainty. This is investigated, e.g., in Primbs and Sung (2007); Cannon et al. (2011, 2010); Hokayem et al. (2012). Interestingly, many of these methods (e.g., the ones studied in Cannon et al. (2010); Hokayem et al. (2012)) have been developed in an output-feedback framework.
For systems affected by additive noise, a simple algorithm has been proposed in Farina et al. (2013), whose main features are: (a) the computational burden is only slightly heavier than the one required by stabilizing MPC methods for undisturbed linear systems, (b) the possibility to consider unbounded noises, and (c) guaranteed recursive feasibility and convergence under mild conditions. This method relies on the Cantelli inequality to properly reformulate chance-constraints into deterministic ones, and feasibility is guaranteed by imposing suitable constraints to the mean and the variance of the state variable. In this work we extend the state-feedback approach taken in Farina et al. (2013) to the output-feedback case, where the outputs are assumed to be affected by stochastic noise.
The paper is organized as follows. In Section 2 the control problem is defined in the stochastic framework and the probabilistic constraints are defined. In Section 3 we introduce the main ingredients of the Stochastic MPC optimization problem and we state the main stability result. Finally, in Section 4 we present an application example and in Section 5 we draw some conclusions. For clarity of exposition, the proof of the main theoretical result is postponed to an Appendix.

2. PROBLEM STATEMENT

2.1 Stochastic systems and probabilistic constraints

The following discrete-time linear system is considered

$$\begin{cases} x_{t+1} = Ax_t + Bu_t + Fw_t \quad t \geq 0 \\ y_t = Cx_t + v_t \end{cases} \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the input, $y_t \in \mathbb{R}^p$ is the measured output and $w_t \in \mathbb{R}^{n_w}, v_t \in \mathbb{R}^p$ are two independent, zero-mean, white noises with covariance matrices $W \succeq 0$ and $V \succ 0$, respectively, and a-priori unbounded support.
We assume that the pair (A, C) is observable, and that the pairs (A, B) and $(A, \tilde{F})$ are reachable, where matrix $\tilde{F}$ satisfies $\tilde{F}\tilde{F}^T = FWF^T$.
Constraints on state and input variables of system (1) are imposed in a probabilistic way, i.e., at time t it is required that

$$\mathcal{P}\{b_r^T x_{t+k} \geq x_r^{max}\} \leq p_r^x \quad \forall k > 0 \quad r = 1, \dots, n_r \tag{2}$$

$$\mathcal{P}\{c_s^T u_{t+k} \geq u_s^{max}\} \leq p_s^u \quad \forall k \geq 0 \quad s = 1, \dots, n_s \tag{3}$$

where $\mathcal{P}(\phi)$ denotes the probability of ϕ, b_r, c_s are constant vectors, x_r^{max}, u_s^{max} are bounds for the state and control variables and p_r^x, p_s^u are design parameters. It is also assumed that the set of relations $b_r^T x \leq x_r^{max}$, $r = 1, \dots, n_r$ (respectively, $c_s^T u_{t+k} \leq u_s^{max}$, $s = 1, \dots, n_s$), defines a convex and compact set $\mathbb{X}$ (respectively, $\mathbb{U}$) containing the origin in its interior.

2.2 The observer and the control law

Denoting by $\hat{x}_t$ the estimate of variable x_t, we first define an observer for (1), i.e.,

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + L_{t|t}(y_t - C\hat{x}_t) \tag{4}$$

where $L_{t|t}$ is a time-varying gain whose choice is discussed later in the paper. We want to design the control law

$$u_t = \bar{u}_{t|t} - K_{t|t}(\hat{x}_t - \bar{x}_{t|t}) \tag{5}$$

where the feed-forward term $\bar{u}_{t|t}$ and the time-varying gain $K_{t|t}$ are defined in the sequel as the solutions to a suitable MPC optimization problem. In (5), $\bar{x}_{t|t}$ denotes the expected value of the state x_t at time t, i.e., $\bar{x}_{t|t} = \mathbb{E}\{x_t\}$. Assuming $\bar{x}_{t|t}$ as *a priori* information and recalling that $\mathbb{E}\{w_t\} = 0$ for all t, the predicted values $\bar{x}_{t+k|t} = \mathbb{E}\{x_{t+k}\}$, $k = 1, 2, \ldots$ can be computed using the equation

$$\bar{x}_{t+k+1|t} = A\bar{x}_{t+k|t} + B\bar{u}_{t+k|t} \quad k \geq 0 \tag{6}$$

where also $\bar{u}_{t+k|t}$ is known and will be clarified later on. Letting

$$\delta x_{t+k|t} = x_{t+k} - \bar{x}_{t+k|t} \quad k \geq 0 \tag{7a}$$
$$e_{t+k} = x_{t+k} - \hat{x}_{t+k} \tag{7b}$$
$$\varepsilon_{t+k|t} = \hat{x}_{t+k} - \bar{x}_{t+k|t} \tag{7c}$$

from (7) we obtain that

$$\delta x_{t+k|t} = e_{t+k} + \varepsilon_{t+k|t} \quad k \geq 0 \tag{8}$$

Define also the vector $\sigma_{t+k|t} = \begin{bmatrix} e_{t+k}^T & \varepsilon_{t+k|t}^T \end{bmatrix}^T$. According to (1)-(5), we obtain that

$$\sigma_{t+k+1|t} = \begin{bmatrix} A - L_{t+k|t}C & 0 \\ L_{t+k|t}C & A - BK_{t+k|t} \end{bmatrix} \sigma_{t+k|t} + \begin{bmatrix} F & -L_{t+k|t} \\ 0 & L_{t+k|t} \end{bmatrix} \begin{bmatrix} w_{t+k} \\ v_{t+k} \end{bmatrix} \tag{9}$$

Denote by $\Sigma_{t+k|t} = \mathbb{E}\left\{\sigma_{t+k|t}\sigma_{t+k|t}^T\right\}$ the covariance matrix of the zero-mean vector $\sigma_{t+k|t}$. Assume that at time t the matrix $\Sigma_{t|t}$ is known. Then, according to (9) it is possible to compute its evolution as

$$\begin{aligned} \Sigma_{t+k+1|t} &= \mathbb{E}\left\{\sigma_{t+k+1|t}\sigma_{t+k+1|t}^T\right\} \\ &= \begin{bmatrix} A - L_{t+k|t}C & 0 \\ L_{t+k|t}C & A - BK_{t+k|t} \end{bmatrix} \Sigma_{t+k|t} \\ &\times \begin{bmatrix} A - L_{t+k|t}C & 0 \\ L_{t+k|t}C & A - BK_{t+k|t} \end{bmatrix}^T \\ &+ \begin{bmatrix} F & -L_{t+k|t} \\ 0 & L_{t+k|t} \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} F & -L_{t+k|t} \\ 0 & L_{t+k|t} \end{bmatrix}^T \end{aligned} \tag{10}$$

By definition, also the variable $\delta x_{t+k|t}$, $k \geq 0$, has zero mean. Its covariance matrix is denoted by $X_{t+k|t}$ and is derived from $\Sigma_{t+k+1|t}$ as follows

$$X_{t+k|t} = \mathbb{E}\left\{\delta x_{t+k|t}\delta x_{t+k|t}^T\right\} = [I \ I]\, \Sigma_{t+k|t} \begin{bmatrix} I \\ I \end{bmatrix} \tag{11}$$

Finally, define $\delta u_{t+k|t} = u_{t+k} - \bar{u}_{t+k|t} = -K_{t+k|t}(\hat{x}_{t+k} - \bar{x}_{t+k|t})$, $k \geq 0$, which is also a zero-mean variable, with covariance $U_{t+k|t}$, derived from $\Sigma_{t+k+1|t}$, i.e.

$$\begin{aligned} U_{t+k|t} =& \mathbb{E}\left\{K_{t+k|t}\varepsilon_{t+k|t}\varepsilon_{t+k|t}^T K_{t+k|t}^T\right\} \\ =& \begin{bmatrix} 0 & K_{t+k|t} \end{bmatrix} \Sigma_{t+k|t} \begin{bmatrix} 0 \\ K_{t+k|t}^T \end{bmatrix} \end{aligned} \tag{12}$$

3. MPC ALGORITHM: FORMULATION AND PROPERTIES

According to the standard procedure in MPC, at any time instant t a future prediction horizon of length N is considered and a suitable optimization problem is iteratively solved. In the remainder of the paper, the variables $\bar{x}_{t+k}$, $\bar{u}_{t+k}$, and σ_{t+k} will be used to denote the expected values of x_{t+k} and σ_{t+k} starting from arbitrary values $\bar{x}_t$, $\bar{u}_{t\ldots t+k}$, and σ_t while, in the previous section, $\bar{x}_{t+k|t}$ and $\sigma_{t+k|t}$ are computed starting from the values $\bar{x}_{t|t}$, $\bar{u}_{t\ldots t+k|t}$, and $\sigma_{t|t}$ (optimal with respect to the MPC problem defined below), respectively. Similarly, X_{t+k}, U_{t+k}, and Σ_{t+k} will be used to denote the covariances of x_{t+k}, u_{t+k}, and σ_{t+k}, starting from arbitrary initial values X_t, U_t, and Σ_t.
The main ingredients of the optimization problem are now introduced.

3.1 Cost function

Our aim is to minimize a cost function of the type

$$J = \mathbb{E}\left\{\sum_{i=t}^{t+N-1} \|x_i\|_Q^2 + \|u_i\|_R^2 + \|x_{t+N}\|_S^2\right\} \tag{13}$$

where Q, R, and S are positive definite, symmetric matrices. Define the nominal input sequence $\bar{u}_{t\ldots t+N-1} = \{\bar{u}_t, \ldots, \bar{u}_{t+N-1}\}$ and the gain sequences $K_{t\ldots t+N-1} = \{K_t, \ldots, K_{t+N-1}\}$ and $L_{t\ldots t+N-1} = \{L_t, \ldots, L_{t+N-1}\}$. The evolution of the expected value of x is computed as

$$\bar{x}_{t+k+1} = A\bar{x}_{t+k} + B\bar{u}_{t+k} \tag{14}$$

Also, let $u_{t+k} = \bar{u}_{t+k} - K_k(\hat{x}_{t+k} - \bar{x}_{t+k})$, and $\Sigma_{t+k} = \mathbb{E}\left\{\sigma_{t+k}\sigma_{t+k}^T\right\}$, which evolves according to (10). In view of these terms, the cost function (13) results to be the sum of two components accounting for the expected value and the variance of the state variable, respectively

$$J = J_m(\bar{x}_t, \bar{u}_{t\ldots t+N-1}) + J_v(\Sigma_t, K_{t\ldots t+N-1}, L_{t\ldots t+N-1}) \tag{15}$$

where

$$J_m = \sum_{i=t}^{t+N-1} \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_r^2 + \|\bar{x}_{t+N}\|_S^2 \tag{16}$$

$$\begin{aligned} J_v = \mathbb{E}\Big\{ & \sum_{i=t}^{t+N-1} \|x_i - \bar{x}_i\|_Q^2 + \|u_i - \bar{u}_i\|_R^2 \\ & + \|x_{t+N} - \bar{x}_{t+N}\|_S^2 \Big\} \end{aligned} \tag{17}$$

In this paper we adopt the following approximation of J_v, obtained by formally separating the components related to the estimation error e_t and the control error ε_t, i.e.,

$$\begin{aligned} J_v = & \mathbb{E}\left\{\sum_{i=t}^{t+N-1} \|x_i - \hat{x}_i\|_Q^2 + \|x_{t+N} - \hat{x}_{t+N}\|_{S_L}^2\right\} + \\ & \mathbb{E}\left\{\sum_{i=t}^{t+N-1} \|\hat{x}_i - \bar{x}_i\|_Q^2 + \|u_i - \bar{u}_i\|_R^2 + \|\hat{x}_{t+N} - \bar{x}_{t+N}\|_S^2\right\} \end{aligned} \tag{18}$$

where different weights S_L and S have been introduced for adding more degrees of freedom in the problem formulation. In particular, they must be selected in order to satisfy the following inequality

$$\mathrm{diag}(Q, Q + \bar{K}^T R \bar{K}) - \mathrm{diag}(S_L, S) + \Phi^T \mathrm{diag}(S_L, S)\Phi \preceq 0 \tag{19}$$

where

$$\Phi = \begin{bmatrix} A - \bar{L}C & 0 \\ \bar{L}C & A - B\bar{K} \end{bmatrix}$$

and $\bar{K}$, $\bar{L}$ are stabilizing gains, whose choice is discussed more in details in the sequel.
Remark that equation (19) is a rather standard Lyapunov-like inequality where the unknown matrix has a fixed block-diagonal structure. Therefore, it can be solved using standard tools for the solution of LMIs. Also note that, in principle, the choice $S_L = S$ is more consistent with the original cost function (13), but it is not strictly required in the following. Equation (18) results in

$$\begin{aligned} J_v = \textstyle\sum_{i=t}^{t+N-1} \mathrm{tr}\left\{ \begin{bmatrix} Q & 0 \\ 0 & Q + K_i^T R K_i \end{bmatrix} \Sigma_i \right\} \\ + \mathrm{tr}\left\{ \begin{bmatrix} S_L & 0 \\ 0 & S \end{bmatrix} \Sigma_{t+N} \right\} \end{aligned} \tag{20}$$

Recalling (14) and (10), note that J_m depends on $\bar{u}_{t\ldots t+N-1}$ and on the initial condition $\bar{x}_t$ of the mean value, while J_v depends on $K_{t\ldots t+N-1}, L_{t\ldots t+N-1}$, and on the initial condition Σ_t of the variance. Therefore, when minimizing (15), the aim is to drive the mean value of the state to zero in an optimal way by acting on the nominal input component $\bar{u}_{t\ldots t+N-1}$; on the other hand, the variance of x_t is minimized by acting on the free gains $K_{t\ldots t+N-1}$ and $L_{t\ldots t+N-1}$. Furthermore, also the pair $(\bar{x}_t, \Sigma_t)$ will be accounted for as an argument of the MPC optimization, as later discussed.

3.2 Reformulation of the constraints

In this section we derive deterministic, but tighter, constraints from the probabilistic ones (2) and (3), with the aim of including them in the MPC optimization problem defined below. By resorting to the Cantelli lemma and to a linearization procedure, according to Farina et al. (2013), one obtains the linear state constraints, for all $k \geq 0$

$$b_r^T \bar{x}_{t+k} \leq (1 - 0.5\alpha_x) x_r^{max} - \frac{1 - p_r^x}{2\alpha_x x_r^{max} p_r^x} b_r^T X_{t+k} b_r \tag{21}$$

$$c_s^T \bar{u}_{t+k} \leq (1 - 0.5\alpha_u) u_s^{max} - \frac{1 - p_s^u}{2\alpha_u u_s^{max} p_s^u} c_s^T U_{t+k} c_s \tag{22}$$

with $r = 1, \ldots, n_r$ and $s = 1, \ldots, n_s$, where $\alpha_x \in [0, 1]$ and $\alpha_u \in [0, 1]$ are design parameters.

3.3 Terminal constraints

As usual in MPC with guaranteed stability, see e.g. Mayne et al. (2000), also in the algorithm proposed here some constraints must be imposed at the end of the prediction horizon for both the mean value $\bar{x}_{t+N}$ and the variance Σ_{t+N}. Specifically, the terminal constraints are

$$\bar{x}_{t+N} \in \bar{\mathbb{X}}_f \tag{23}$$

$$\Sigma_{t+N} \preceq \bar{\Sigma} \tag{24}$$

The set $\bar{\mathbb{X}}_f$ is a positively invariant set satisfying

$$(A - B\bar{K})\bar{x} \in \bar{\mathbb{X}}_f \qquad \forall \bar{x} \in \bar{\mathbb{X}}_f \tag{25}$$

while $\bar{\Sigma}$ is the steady-state solution of the Lyapunov equation (10), i.e.,

$$\bar{\Sigma} = \Phi \bar{\Sigma} \Phi^T + \Psi \bar{\Omega} \Psi^T \tag{26}$$

where

$$\Psi = \begin{bmatrix} F & -\bar{L} \\ 0 & \bar{L} \end{bmatrix}$$

and $\bar{\Omega} = \mathrm{diag}(\bar{W}, \bar{V})$, obtained by considering noise variances $\bar{W} \succeq W$ and $\bar{V} \succeq V$, and assuming constant gains $\bar{K}$ and $\bar{L}$.
The following coupling conditions include both $\bar{\mathbb{X}}_f$ and $\bar{\Sigma}$ at the same time, i.e.,

$$b_r^T \bar{x} \leq (1 - 0.5\alpha^x) x_r^{max} - \frac{1 - p_r^x}{2\alpha^x x_r^{max} p_r^x} b_r^T \bar{X} b_r \tag{27a}$$

$$-c_s^T \bar{K} \bar{x} \leq (1 - 0.5\alpha^u) u_s^{max} - \frac{1 - p_s^u}{2\alpha^u u_s^{max} p_s^u} c_s^T \bar{U} c_s \tag{27b}$$

for all $r = 1 \ldots n_r$, $s = 1 \ldots n_s$, and for all $\bar{x} \in \bar{\mathbb{X}}_f$, where

$$\bar{X} = [I \ I]\, \bar{\Sigma} \begin{bmatrix} I \\ I \end{bmatrix} \tag{28a}$$

$$\bar{U} = \begin{bmatrix} 0 & \bar{K} \end{bmatrix} \bar{\Sigma} \begin{bmatrix} 0 \\ \bar{K}^T \end{bmatrix} \tag{28b}$$

3.4 Statement of the stochastic MPC (S-MPC) problem

In this section we formulate the main S-MPC problem, to be solved online according to the receding horizon principle. A preliminary discussion, concerning the initialization, is due.
In order to use the most recent information available on the state, at any time instant it would be natural to set the current value of the nominal state $\bar{x}_{t|t}$ to $\hat{x}_t$ (i.e., letting $\bar{x}_{t|t}$ be the *a posteriori* optimal conditional expectation value, in view of the estimated data $\hat{x}_t$). This corresponds also to setting $\Sigma_{t|t}$ to $\mathrm{diag}(\Sigma_{11,t|t-1}, 0)$, where $\Sigma_{11,t|t-1}$ is the prediction error covariance obtained using (4).
However, since we do not exclude the possibility of unbounded disturbances, this choice could in some cases lead to infeasible optimization problems, and the fundamental property of recursive feasibility would be lost. On the other hand, and in view of the terminal constraints (23), (24), it is quite simple to see that recursive feasibility is guaranteed provided that the considered mean is updated according to the prediction equation (6), which corresponds to a variance update given by (10). These considerations justify the choice of accounting for the initial conditions $(\bar{x}_t, \Sigma_t)$ as free variables. Accordingly, we have identified two alternative strategies.
Strategy 1 - Reset of the initial state: in the MPC optimization problem set $\bar{x}_{t|t} = \hat{x}_t$, $\Sigma_{t|t} = \mathrm{diag}(\Sigma_{11,t|t-1}, 0)$
Strategy 2 - Prediction: in the MPC optimization problem set $\bar{x}_{t|t} = \bar{x}_{t|t-1}$, $\Sigma_{t|t} = \Sigma_{t|t-1}$.
The S-MPC problem can now be stated as follows.

S-MPC problem: at any time instant t solve

$$\min_{\bar{x}_{t|t}, \Sigma_{t|t}, \bar{u}_{t\ldots t+N-1}, K_{t\ldots t+N-1}, L_{t\ldots t+N-1}} J$$

where J is defined in (15), (16), (18), subject to the initialization constraint, corresponding to Strategies 1 and 2, i.e.,

$$(\bar{x}_{t|t}, \Sigma_{t|t}) \in \{(\hat{x}_t, \mathrm{diag}(\Sigma_{11,t|t-1}, 0)), (\bar{x}_{t|t-1}, \Sigma_{t|t-1})\} \tag{29}$$

to the dynamics (14) and (10), to the linear constraints (21), (22) for all $k = 0, \ldots, N-1$, and to the terminal constraints (23), (24). □

Denoting with $\bar{u}_{t\ldots t+N-1|t} = \{\bar{u}_{t|t}, \ldots, \bar{u}_{t+N-1|t}\}$, $K_{t\ldots t+N-1|t} = \{K_{t|t}, \ldots, K_{t+N-1|t}\}$, $L_{t\ldots t+N-1|t} = \{L_{t|t}, \ldots, L_{t+N-1|t}\}$, and $(\bar{x}_{t|t}, \Sigma_{t|t})$ the optimal solution of the S-MPC problem, and according to the receding horizon principle, the feedback control law actually used is then given by (5) and the state observation evolves as (4). We define the S-MPC problem feasibility set as $\Xi := \{(\bar{x}_0, \Sigma_0) : \exists \bar{u}_{0\ldots N-1}, K_{0\ldots N-1}, L_{0\ldots N-1}$ such that (21), (22) hold for all $k = 0, \ldots, N-1$ and (23), (24) are verified$\}$. Note that, in view of the compactness of $\mathbb{X}$, see (2), the set Ξ results to be compact. The recursive feasibility and convergence properties of the resulting control system are discussed below. However, some preliminary comments are in order.

- At the initial time $t = 0$, the algorithm must be initialized by setting $\bar{x}_{0|0} = \hat{x}_0$ and $\Sigma_{0|0}$ =diag$(\Sigma_{11,0}, 0)$. In view of this, feasibility at time $t = 0$ amounts to $(\hat{x}_0, \Sigma_{0|0}) \in \Xi$.
- According to the problem statement, feasibility of S-MPC at time $t > 0$ amounts to $\{(\hat{x}_t,\text{diag}(\Sigma_{11,t}, 0)), (\bar{x}_{t|t-1}, \Sigma_{t|t-1})\} \bigcap \Xi \neq \emptyset$.
- The binary choice between Strategies 1 and 2 for the initialization of $\bar{x}_{t|t}$, $\Sigma_{t|t}$, see constraint (29), requires to solve at any time instant two optimization problems.
- A sequential procedure can be adopted to reduce the average overall computational burden. The optimization problem corresponding to Strategy 1 is first solved and, if it is infeasible, Strategy 2 must be used. On the contrary, if it is feasible, it is possible to compare the resulting value of the optimal cost function with the value of the cost using the sequences $\{\bar{u}_{t|t-1}, \ldots, \bar{u}_{t+N-2|t-1}, -\bar{K}\bar{x}_{t+N-1|t}\}$, $\{K_{t|t-1}, \ldots, K_{t+N-2|t-1}, \bar{K}\}$, $\{L_{t|t-1}, \ldots, L_{t+N-2|t-1}, \bar{L}\}$. If the optimal cost with Strategy 1 is lower, Strategy 1 can be used without solving the MPC problem for Strategy 2. This does not guarantee optimality, but the convergence properties of the method stated in the result below are recovered and the computational effort is reduced.

Now we are in the position to state the main result (the proof is given in the Appendix).

Theorem 1. If, at time $t = 0$, the S-MPC problem admits a solution, the optimization problem is recursively feasible and the state and input probabilistic constraints (2) and (3) are verified for all $t \geq 0$. Furthermore, if there exists $\rho \in (0, 1)$ such that the noise variance Ω verifies

$$\frac{(N + \frac{\beta}{\alpha})}{\alpha} tr(S_T \Psi \Omega \Psi^T) < \min(\rho \bar{\sigma}^2, \rho \lambda_{min}(\bar{\Sigma})) \quad (30)$$

where $\bar{\sigma}$ is the maximum radius of a ball, centered at the origin, included in $\bar{\mathbb{X}}_F$, and

$$\alpha = \min\{\lambda_{min}(Q), \frac{1}{2} tr\{Q^{-1}\}^{-1}\} \quad (31a)$$

$$\beta = \max\{\lambda_{max}(S), tr\{S_T\}\} \quad (31b)$$

then

$$\text{dist}(\|\bar{x}_t\|^2 + tr\{\Sigma_{t|t}\}, [0, \frac{1}{\rho\alpha}(N + \frac{\beta}{\alpha}) tr(S_T \Psi \Omega \Psi^T)]) \to 0$$

as $t \to +\infty$, where $\text{dist}(\zeta, \mathcal{Z}) := \inf\{\|\zeta - \eta\|, \eta \in \mathcal{Z}\}$ is the point-to-set distance from ζ to $\mathcal{Z}$. □

4. EXAMPLE

In this section an application of the proposed technique is presented. The system under control represents a point-mass moving in a two dimensional space, subject to a random disturbance and to a noise on the measure of each state variable. In particular the system is of type (1) with

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C = I_4, \quad F = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The control objective is to drive the state of that system to the origin while constraining it within a rectangle of vertices $(\pm 6, \pm 6)$, with bounded input $\|u\|_\infty \leq 2$. The disturbances w_t and v_t are zero mean white noises with covariance matrices respectively $W = 5I_2$ and $V = 0.1I_4$. For the MPC control problem we choose a prediction horizon of length $N = 8$ and weights $Q = 10^{-2}I_4$, $R = I_2$, and, for the computation of the terminal constraints, we set $\bar{W} = 2W$ and $\bar{V} = 2V$. The gains $\bar{K}$ and $\bar{L}$ are in turn obtained as the solution of an LQG control problem. The initial state of the system is set to $x_0 = [3.5, 0, 2, 4]^T$ and the probabilistic constraints are defined by $p_r^x = 0.2$, $p_s^u = 0.1$, and $\alpha_r^x = \alpha_s^u = 0.2$ $r = 1 \ldots n_r, s = 1 \ldots n_s$. Figure 1 shows the results of application of the proposed strategy.

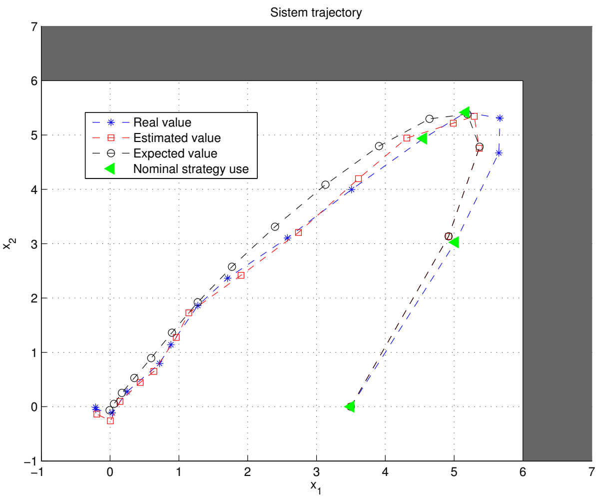

Fig. 1. Simulation results with $p^x = 0.2$ and $p^u = 0.1$.

5. CONCLUSIONS

In this paper a novel approach for output-feedback model predictive control of discrete-time linear stochastic systems is proposed. The main advantages of the presented algorithm are that it allows for the presence of possibly unbounded additive disturbances acting on the state and on the measurements, and that recursive feasibility and convergence are guaranteed under mild conditions.
The main concern related to the proposed method lies in the implementation issues: while corresponding state-feedback the algorithm (see Farina et al. (2013)) requires the solution to a computationally lightweight quadratic program including suitable linear matrix inequalities (LMIs), the problem to be solved online in this paper involves nonlinear matrix inequalities. This problem can be overcome by introducing simple matrix approximations allowing for a reformulation of the constraints as LMIs and for a dramatic reduction of the computational load. This issue will be subject of future work.

Appendix A

A.1 Proof of recursive feasibility

Assume that, at time instant t, a feasible solution of S-MPC is available, i.e., $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Xi$ with optimal sequences $\bar{u}_{t\ldots t+N-1|t}$, $K_{t\ldots t+N-1|t}$, and $L_{t\ldots t+N-1|t}$. We prove that, at time $t+1$, at least a feasible solution to S-MPC exists, i.e., $(\bar{x}_{t+1|t}, X_{t+1|t}) \in \Xi$ with admissible sequences $\bar{u}^f_{t+1\ldots t+N|t} = \{\bar{u}_{t+1|t}, \ldots, \bar{u}_{t+N-1|t}, -\bar{K}\bar{x}_{t+N|t}\}$, $K^f_{t+1\ldots t+N|t} = \{K_{t+1|t}, \ldots, K_{t+N-1|t}, \bar{K}\}$, and $L^f_{t+1\ldots t+N|t} = \{L_{t+1|t}, \ldots, L_{t+N-1|t}, \bar{L}\}$.

Constraint (21) is verified for all pairs $(\bar{x}_{t+1+k|t}, X_{t+1+k|t})$, $k = 0, \ldots, N-2$, in view of the feasibility of S-MPC at time t. Furthermore, in view of (23), (24), (28a), and the condition (27a), we have that

$$\begin{aligned} b^T\bar{x}_{t+N|t} &\leq (1-0.5\varepsilon)x^{max} - \tfrac{1-p^x}{2\varepsilon x^{max}p^x} b^T\bar{X}b \\ &\leq (1-0.5\varepsilon)x^{max} - \tfrac{1-p^x}{2\varepsilon x^{max}p^x} b^T X_{t+N|t} b \end{aligned}$$

i.e., constraint (21) is verified.

Analogously, constraint (22) is verified for all pairs $(\bar{u}_{t+1+k|t}, U_{t+1+k|t})$, $k = 0, \ldots, N-2$, in view of the feasibility of S-MPC at time t. Furthermore, in view of (23), (24), (28b), and the condition (27b), we have that

$$\begin{aligned} -c^T\bar{K}\bar{x}_{t+N|t} &\leq (1-0.5\varepsilon)u^{max} - \tfrac{1-p^u}{2\varepsilon u^{max}p^u} c^T\bar{U}c \\ &\leq (1-0.5\varepsilon)u^{max} - \tfrac{1-p^u}{2\varepsilon u^{max}p^u} c^T U_{t+N|t} c \end{aligned}$$

i.e., constraint (22) is verified.

Let $\Omega = \mathrm{diag}(W, V)$. In view of (23) and of the invariance property (25) it follows that $\bar{x}_{t+N+1|t} = (A - B\bar{K})\bar{x}_{t+N|t} \in \bar{\mathbb{X}}_f$ and, in view of (24)

$$\begin{aligned} \Sigma_{t+N+1|t} &= \Phi\Sigma_{t+N|t}\Phi^T + \Psi\Omega\Psi^T \\ &\preceq \Phi\bar{\Sigma}\Phi^T + \Psi\bar{\Omega}\Psi^T = \bar{\Sigma} \end{aligned}$$

hence verifying both (23) and (24) at time $t+1$.

A.2 Proof of convergence

In view of the feasibility, at time $t+1$ of the possibly suboptimal solution $\bar{u}^f_{t+1\ldots t+N|t}$, $K^f_{t+1\ldots t+N|t}$, $L^f_{t+1\ldots t+N|t}$, and $(\bar{x}_{t+1|t}, \Sigma_{t+1|t})$, we have that the optimal cost function computed at time $t+1$ is $J^*(t+1) = J^*_m(t+1) + J^*_v(t+1)$ [1]. In view of the optimality of $J^*(t+1)$

$$\begin{aligned} J^*(t+1) \leq\; & J_m(\bar{x}_{t+1|t}, \bar{u}^f_{t+1\ldots t+N|t}) \\ & + J_v(\Sigma_{t+1|t}, K^f_{t+1\ldots t+N|t}, L^f_{t+1\ldots t+N|t}) \end{aligned} \tag{A.1}$$

Note that

$$\begin{aligned} & J_m(\bar{x}_{t+1|t}, \bar{u}^f_{t+1\ldots t+N|t}) = \\ & J_m(\bar{x}_{t|t}, \bar{u}_{t\ldots t+N-1|t}) - \|\bar{x}_{t|t}\|^2_Q - \|\bar{u}_{t|t}\|^2_R + \\ & \|\bar{x}_{t+N|t}\|^2_Q + \|\bar{K}\bar{x}_{t+N|t}\|^2_R - \|\bar{x}_{t+N|t}\|^2_S + \\ & \|(A - B\bar{K})\bar{x}_{t+N|t}\|^2_S \end{aligned} \tag{A.2}$$

In view of (19)

$$\begin{aligned} & \|\bar{x}_{t+N|t}\|^2_Q + \|\bar{K}\bar{x}_{t+N|t}\|^2_R - \|\bar{x}_{t+N|t}\|^2_S + \\ & \|(A - B\bar{K})\bar{x}_{t+N|t}\|^2_S \leq 0 \end{aligned} \tag{A.3}$$

[1] For brevity, we denote $J^*(x_t, \bar{x}_{t|t-1}, \Sigma_{t|t-1})$ with $J^*(t)$, $J^*_m(x_t, \bar{x}_{t|t-1}, \Sigma_{t|t-1})$ with J^*_m(t), and $J^*_v(x_t, \bar{x}_{t|t-1}, \Sigma_{t|t-1})$ with $J^*_v(t)$

Furthermore, note that

$$J_m(\bar{x}_{t|t}, \bar{u}_{t\ldots t+N-1|t}) = J^*_m(t) \tag{A.4}$$

Now consider J_v. Denote, for better clarity, $S_T = \mathrm{diag}(S_L, S)$, $Q_T = \mathrm{diag}(Q, Q + \bar{K}^T R\bar{K})$. We compute that

$$\begin{aligned} & J_v(X_{t+1|t}, K^f_{t+1\ldots t+N|t}, L^f_{t+1\ldots t+N|t}) \\ & = J_v(X_{t|t}, K_{t\ldots t+N-1|t}, L_{t\ldots t+N-1|t}) \\ & - tr\{\begin{bmatrix} Q & 0 \\ 0 & Q + K^T_{t|t}RK_{t|t} \end{bmatrix} \Sigma_{t|t}\} + tr\{Q_T\Sigma_{t+N|t} \\ & + S_T\Phi\Sigma_{t+N|t}\Phi^T + S_T\Psi\Omega\Psi^T - S_T\Sigma_{t+N|t}\} \end{aligned} \tag{A.5}$$

Recall the following properties of the trace: $tr(A+B) = tr(A) + tr(B)$, $tr(AB) = tr(BA)$, being A and B matrices of suitable dimensions. In view of this, and recalling (19):

$$\begin{aligned} & tr\{Q_T\Sigma_{t+N|t} + S_T\Phi\Sigma_{t+N|t}\Phi^T - S_T\Sigma_{t+N|t}\} = \\ & tr\{(Q_T + \Phi^T S_T\Phi - S_T)\Sigma_{t+N|t}\} \leq 0 \end{aligned} \tag{A.6}$$

From (A.1)-(A.6) we obtain

$$\begin{aligned} J^*(t+1) \leq\; & J^*(t) - (\|\bar{x}_{t|t}\|^2_Q + \|\bar{u}_{t|t}\|^2_R) \\ & - tr\{\begin{bmatrix} Q & 0 \\ 0 & Q + K^T_{t|t}RK_{t|t} \end{bmatrix} \Sigma_{t|t}\} + tr(S_T\Psi\Omega\Psi^T) \end{aligned} \tag{A.7}$$

Furthermore, from the definition of $J^*(t)$ we also have that

$$\begin{aligned} J^*(t) \geq\; & \|\bar{x}_{t|t}\|^2_Q + \|\bar{u}_{t|t}\|^2_R \\ & + tr\{\begin{bmatrix} Q & 0 \\ 0 & Q + K^T_{t|t}RK_{t|t} \end{bmatrix} \Sigma_{t|t}\} \end{aligned} \tag{A.8}$$

Now, denote $\Omega_F = \{(\bar{x}, \Sigma) : \bar{x} \in \bar{\mathbb{X}}_F, \Sigma \preceq \bar{\Sigma}\}$. Assuming that $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Omega_F$ we have that $J^*(t) \leq J^{aux}_m(t) + J^{aux}_v(t)$, where

$$\begin{aligned} J^{aux}_m(t) =\; & \textstyle\sum_{k=t}^{t+N-1} \|(A - B\bar{K})^{t-k}\bar{x}_{t|t}\|^2_Q \\ & + \|\bar{K}(A - B\bar{K})^{t-k}\bar{x}_{t|t}\|^2_R + \|(A - B\bar{K})^N\bar{x}_{t|t}\|^2_S \end{aligned}$$

since $\{-\bar{K}\bar{x}_{t|t}, \ldots, -\bar{K}(A - B\bar{K})^{N-1}\bar{x}_{t|t}\}$ is a feasible input sequence. Therefore, recalling (19),

$$J^{aux}_m(t) \leq \|\bar{x}_{t|t}\|^2_S \tag{A.9}$$

Similarly, and recalling the properties of the trace and (19), we obtain that $J^{aux}_v(t)$ is equal to

$$\begin{aligned} & \textstyle\sum_{k=0}^{N-1} tr\{Q_T[\Phi^k\Sigma_{t|t}\Phi^{T(k)} + \sum_{i=0}^{k-1} \Phi^i\Psi\Omega\Psi^T\Phi^{T(i)}]\} \\ & + tr\{S_T[\Phi^N\Sigma_{t|t}\Phi^{T(N)} + \textstyle\sum_{i=0}^{N-1} \Phi^i\Psi\Omega\Psi^T\Phi^{T(i)}]\} \\ & = tr\{[\textstyle\sum_{k=0}^{N-1} \Phi^{T(k)}Q_T\Phi^k + \Phi^{T(N)}S_T\Phi^N]\Sigma_{t|t}\} \\ & + tr\{[\textstyle\sum_{k=1}^{N-1}\sum_{i=0}^{k-1} \Phi^{T(i)}Q_T\Phi^i + \sum_{i=0}^{N-1} \Phi^{T(i)}S_T\Phi^i]\Psi\Omega\Psi^T\} \\ & \leq tr\{S_T\Sigma_{t|t}\} + tr\{[S_T + \textstyle\sum_{k=1}^{N-1}(\Phi^{T(k)}S_T\Phi^k) \\ & + \textstyle\sum_{i=1}^{N-1} \Phi^{T(i)}Q_T\Phi^i]\Psi\Omega\Psi^T\} \\ & \leq tr\{S_T\Sigma_{t|t}\} + tr\{[S_T + \textstyle\sum_{i=1}^{N-1} S_T]\Psi\Omega\Psi^T\} \end{aligned}$$

Therefore

$$J^{aux}_v(t) \leq tr\{S_T\Sigma_{t|t}\} + N tr\{S_T\Psi\Omega\Psi^T\} \tag{A.10}$$

Combining (A.9) and (A.10) we obtain that, for all $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Omega_F$

$$J^*(t) \leq \|\bar{x}_{t|t}\|^2_S + tr\{S_T\Sigma_{t|t}\} + N\, tr\{S_T\Psi\Omega\Psi^T\} \tag{A.11}$$

From (A.7), (A.8) and (A.11) it is possible to derive robust stability-related results.

Before to proceed, recall that $tr\{S_T\Sigma_{t|t}\} = tr\{S_T^{\frac{1}{2}T}\Sigma_{t|t}S_T^{\frac{1}{2}}\}$ where $S_T^{\frac{1}{2}}$ is a matrix that verifies $S_T^{\frac{1}{2}T}S_T^{\frac{1}{2}} = S_T$. Therefore $tr\{S_T\Sigma_{t|t}\} = tr\{S_T^{\frac{1}{2}T}\Sigma_{t|t}S_T^{\frac{1}{2}}\} = \|\Sigma_{t|t}^{\frac{1}{2}}S_T^{\frac{1}{2}}\|^2_F$. On the other hand, denoting $Q_{T|t} = \mathrm{diag}(Q, Q + K^T_{t|t}RK_{t|t})$, it follows that $tr\{Q_{T|t}\Sigma_{t|t}\} = \|\Sigma_{t|t}^{\frac{1}{2}}Q_{T|t}^{\frac{1}{2}}\|^2_F$. Recall that the sub-moltiplicativity property holds also for the Frobenius norm, implying that $\|AB\|_F \leq \|A\|_F\|B\|_F$ and that $\|AB\|_F \geq (\|A^{-1}\|_F)^{-1}\|B\|_F$. In view of this, it follows that $tr\{S_T\Sigma_{t|t}\} \leq \|\Sigma_{t|t}^{\frac{1}{2}}\|_F^2\|S_T^{\frac{1}{2}}\|_F^2 = tr\{S_T\}tr\{\Sigma_{t|t}\}$ and that, also considering the matrix inversion Lemma, $tr\{Q_{T|t}\Sigma_{t|t}\} \geq (\|Q_{T|t}^{-\frac{1}{2}}\|_F^2)^{-1}\|\Sigma_{t|t}^{\frac{1}{2}}\|_F^2 \geq tr\{(\text{diag}(Q,Q))^{-1}\}^{-1}tr\{\Sigma_{t|t}\} = \frac{1}{2}tr\{Q^{-1}\}^{-1}tr\{\Sigma_{t|t}\}$. Define $V(\bar{x}_{t|t}, \Sigma_{t|t}) = \|\bar{x}_{t|t}\|^2 + tr\{\Sigma_{t|t}\}$ and $\omega = tr\{S_T\Psi\Omega\Psi^T\}$. In view of this, we can reformulate (A.7), (A.8) and (A.11) as follows.

$$J^*(t+1) \leq J^*(t) - \alpha V(\bar{x}_{t|t}, \Sigma_{t|t}) + \omega \quad \text{(A.12a)}$$
$$J^*(t) \geq \alpha V(\bar{x}_{t|t}, \Sigma_{t|t}) \quad \text{(A.12b)}$$
$$J^*(t) \leq \beta V(\bar{x}_{t|t}, \Sigma_{t|t}) + N\omega \quad \text{(A.12c)}$$

If $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Omega_F$ then, in view of (A.12c), (A.12a)

$$J^*(t+1) \leq J^*(t)(1 - \frac{\alpha}{\beta}) + (\frac{\alpha}{\beta}N + 1)\omega \quad \text{(A.13)}$$

Denote $b = \frac{1}{\rho}(N + \frac{\beta}{\alpha})$ with $\rho \in (0,1)$. If $J^*(t) \leq b\,\omega$, and provided that inequality (30) is verified, then one can prove that $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \mathcal{I}\Omega_F$, where $\mathcal{I}\Omega_F$ denotes the interior of Ω_F. In fact, $J^*(t) \leq b\,\omega$ implies that, in view of (A.12b)

$$V(\bar{x}_{t|t}, \Sigma_{t|t}) \leq \frac{b}{\alpha}\omega$$

This, considering (30), implies that

$$\|\bar{x}_{t|t}\|^2 < \bar{\sigma}^2 \quad \text{(A.14a)}$$
$$tr(\Sigma_{t|t}) < \lambda_{min}(\bar{\Sigma}) \quad \text{(A.14b)}$$

In view of (A.14a), then $\bar{x}_{t|t} \in \bar{\mathbb{X}}_F$. Furthermore, (A.14b) implies that $\lambda_{max}(\Sigma_{t|t}) < \lambda_{min}(\bar{\Sigma})$, which in turn implies that $\Sigma_{t|t} \prec \bar{\Sigma}$. Therefore, recalling (A.13), if $J^*(t) \leq b\,\omega$, then $J^*(t+1) \leq b\,\omega$ and the positive invariance of the set

$$D = \{(\bar{x}, \Sigma) : J^*(t) \leq b\,\omega\} \quad \text{(A.15)}$$

is guaranteed. From this point on, the proof follows similarly to Magni et al. (2006); Raimondo et al. (2009).

For $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Omega_F \backslash D$, it holds that $J^*(t) > b\,\omega$ which, in view of (A.12c), implies that

$$V(\bar{x}_{t|t}, \Sigma_{t|t}) > \frac{1}{\alpha}\omega$$

Therefore, considering (A.12a) we infer that

$$J^*(t+1) - J^*(t) < 0 \quad \text{(A.16)}$$

On the other hand, there exists constant $\bar{c} > 0$ such that, for all x_t with $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Xi\backslash\Omega_F$, there exists x_Ω with $(\bar{x}_\Omega, \Sigma_\Omega) \in \Omega_F\backslash D$ such that $-\alpha V(\bar{x}_{t|t}, \Sigma_{t|t}) \leq -\alpha V(\bar{x}_\Omega, \Sigma_\Omega) - \bar{c}$. This, in view of (A.12a) and (A.16), implies that, for all x_t with $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Xi\backslash\Omega_F$

$$J^*(t+1) - J^*(t) < -\bar{c} \quad \text{(A.17)}$$

This implies that there exists $T_1 > 0$ such that x_{t+T_1} is $(\bar{x}_{t+T_1|t+T_1}, \Sigma_{t+T_1|t+T_1}) \in \Omega_F$.

If, on the one hand $(\bar{x}_{t+T_1|t+T_1}, \Sigma_{t+T_1|t+T_1}) \in D$, in view of the positive invariance of D, $(\bar{x}_{t+k|t+k}, \Sigma_{t+k|t+k}) \in D$ for all $k \geq T_1$. If, on the other hand, $(\bar{x}_{t+T_1|t+T_1}, \Sigma_{t+T_1|t+T_1}) \in \Omega_F\backslash D$, recalling (A.13), (1), and (A.12b)

$$\begin{aligned} J^*(t+T_1+1) - J^*(t+T_1) & \\ &\leq -(1-\rho)\frac{\alpha}{\beta}J^*(t+T_1) \quad \text{(A.18)} \\ &\leq -(1-\rho)\frac{\alpha^2}{\beta}V(\bar{x}_{t+T_1|t+T_1}, \Sigma_{t+T_1|t+T_1}) \end{aligned}$$

In view of (A.17)-(A.18) there exists $T_2 \geq T_1$ such that, for all $\varepsilon > 0$

$$J(t+k) \leq \varepsilon + b\,\omega$$

for all $k \geq T_2$ which, from (A.12b), proves that $\text{dist}(\alpha V(\bar{x}_{t|t}, \Sigma_{t|t}), [0, b\omega]) \to 0$ as $t \to +\infty$.

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