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Perturbation analysis for optimal production planning of a manufacturing system with influence machine degradation

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Abstract:

In this paper, we considered a failure-prone manufacturing system composed by a single-product machine, a stock and a customer who demands a stochastic quantity of product. To describe the proposed manufacturing system, a discrete flow model is adopted and which takes into account machine failure, lost demands and machine degradation. The goal of this paper is to determine the optimal production planning taken into account service level by minimizing the sum of production, inventory, lost sales and degradation costs. Perturbation analysis method is applied to the discrete flow model for optimizing the proposed system. Then the trajectories of production rate, stock level, degradation rate, and lost demands are studied and the perturbation analysis estimators are determined. These estimators are shown to be unbiased and then they are implanted in an optimization algorithm which determines the optimal production planning in the presence of service level.

Keywords: Manufacturing system, discrete flow model, degradation machine, production planning, perturbation analysis.

1. INTRODUCTION

In recent years, the determination of an optimal production plan has been paying attention to manufacturing management. We find in the literature many works on optimal production planning. Dobos (2003) considered a reverse logistics system with constant demand and delay, the author determined the optimal inventory stores levels and production rate, the objective is to minimize the sum of the holding costs in the stores and costs of the manufacturing, remanufacturing and disposal. Sethi et al. (1997) considered a stochastic failure prone manufacturing system with a single machine and constant demand. The authors determined the optimal production planning and then they found the rate of production over time in order to minimize the average cost of production and surplus. Dahane et al. (2012) studied a single randomly repairable and failing manufacturing system producing two types of products. The authors used a genetic algorithm for determining simultaneously the optimal production rate of the first product during each period *k* (*k* periods over a finite horizon) and the optimal duration of the production interval of the second product in order to maximize the total expected profit. Turki et al. (2013) taken into account service level, the authors determined the optimal production planning which minimizes the sum of production, inventory, and lost sales costs. Besides, there is a few works in the literature that consider the machine degradation cost which has an influence on the optimal production planning. Ayed et al. (2012) considered a randomly failing manufacturing system which has to satisfy a random demand during a finite horizon given a required service level. In order to satisfy this demand, another subcontracting production system is considered. The authors integrated the effect of the machine degradation by introducing a unitary degradation cost, and then they determined the optimal production plan by minimising the sum of the production, the inventory and the degradation costs. In this paper, an optimal production planning will be determined under hypotheses of service level (Hajej et al. (2009)). This optimal production planning minimizes the sum of production, inventory, lost sales and degradation costs.

Discrete flow model is used (Vasic and Ruskin (2011)) to describe the system and to consider the service level. The use of the discrete flow model is explained by the fact that this model is more realistic for discrete manufacturing systems than stochastic flow model (Markou and Panayiotou (2007)). Indeed, discrete flow model allows tracking individual parts part by part either in performance evaluation or real-time flow control and is generally easier to simulate. Under this model, an interesting optimization method will be developed which seeks to minimize the total expected cost via a simulation.

Perturbation Analysis (PA) (Ho et al. (1979)) (Yu and Cassandras (2004)) is a technique which allows obtaining sample path derivatives of a random variable with respect to some parameters of interest (e.g., stock level, production rate …). The relevant advantage of PA method is that the simulation based on PA allows reducing the simulation time comparing to a classical simulation method. Indeed, the optimization algorithm based on PA computes at every step the gradient estimators which correspond to the new value of a parameter of interest. Therefore, the value of gradient estimator allows orienting quickly the algorithm to the optimal value. Yu and Cassandras (2004) applied perturbation analysis method to a stochastic flow model and then derived gradient estimators of throughput and buffer overflow metrics with respect to production control parameters, then they used them as gradient estimators in simple iterative schemes for adjusting thresholds in order to optimize an objective function that trades off throughput and buffer overflow costs. The authors showed that these gradient estimators are unbiased before using them in the optimization algorithm. Indeed, the unbiasedness is the principal condition for making the application of PA useful in practice, since it enables the use of the sample PA derivative in control and optimization methods that employ stochastic gradient-based techniques. Then, these estimators could be used in stochastic approximation algorithm. Unfortunately to show the unbiasedness of the estimators in the discrete setting (discrete flow model) is more complicate than in the stochastic fluid model setting. Despite this disadvantage, we will use the discrete flow model to modeling our system by the fact this model is very realistic and more precise than stochastic fluid model which sometime does not maintain the identity of some important parameters of manufacturing systems (e.g., service level, delivery time).

The main contribution of this paper is to apply the PA method on the discrete flow model and to study the trajectories of production rate, stock level, lost demands and failure rate in order to derive gradient estimators. These gradient estimators will be showed that are unbiased and they will be used in optimization algorithm for determining the optimal production planning.

The paper is organized as follows. The manufacturing system with service level is presented in section 2. In section 3 the PA approach is applied on the discrete flow model. The optimisation algorithm and numerical results are presented in section 4. Finally, the last section concludes the paper and gives some perspectives to our work.

2. PROBLEM FORMULATION AND EXPLANATION

We consider a manufacturing system which produces one type of product and composed by a single machine M, a store S and a customer. We denote by $d(k)$ the customer demand which is random and given by a Normal distribution. The demand $d(k)$ is satisfied from the stock S with inventory service level α. The stock S is filled up by the machine M (see Fig. 1). The goal of this paper is to determine the optimal production planning taken into account service level.

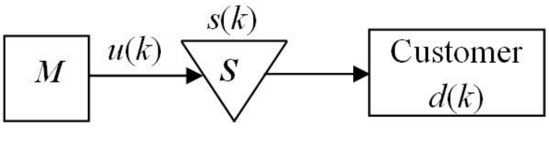

Fig. 1. Manufacturing system.

The following parameters are used in the model formulation:
Δt: length of a production period
H : number of production periods in the planning horizon
$H.\Delta t$: length of the finite planning horizon
$u(k)$: production rate of machine M during period k (k=0,1,…, H-1)
$U=\{u(0), u(1), ..., u(H-1)\}$
$\hat{d}(k)$: average demand during period k (k=0, 1,…, H)
$V_d(k)$: variance of demand during period k (k=0, 1,…, H)
$s(k)$: stock level at the end of period k (k=0, 1,…,H)
$\hat{s}(k)$: average stock level during period k (k=0, 1,…,H)
$s_0(k)$: initial stock level
$L(k)$: number of unsatisfied demands at the end of period k
cp: unit production cost of machine M
cs: inventory holding cost of one product unit during one period at the stock S
cs^- : unit lost sales cost
c_λ: unit degradation cost
$\lambda(k,t)$: failure rate over the horizon $H\Delta t$
U_{max}: maximal production rate of machine M
U_{min} : minimal production rate of machine M
α : probability index related to customer satisfaction and expressing the service level
$u_\alpha(k)$: minimum cumulative production quantity during the period k.

The stock level at the period $k+1$ equals to the stock level at the period k plus the production rate of machine M during period k, minus the customer demand during period k. Therefore, the stock level at the period $k+1$is given by the following equation:

$$s(k+1) = s(k) + u(k) - d(k) \tag{1}$$

The service level requirement constraint for each period is expressed as follow:

$$PROB\ (s(k+1) \geq 0) \geq \alpha \tag{2}$$

In what follows, we present a constraint which defines an upper and lower bounds on the production level for each period k:

$$U_{min} \leq u(k) \leq U_{max} \tag{3}$$

However, for this service level constrain we have another important transformation which changes the service level constraint into equivalent, but deterministic inequalities by specifying through the following lemma a minimum cumulative production quantity depending on the service level requirements.

Lemma 1:

$$PROB\ (s(k+1) \geq 0) \geq \alpha \Rightarrow \left(u(k) \geq u_\alpha(k)\right) \quad k = 0,1,...,H$$

Where $u_\alpha(k)$ represents a minimum cumulative production quantity expressed as follows:

$u_\alpha(k) = V_{d,k} \cdot \varphi^{-1}_{d,k}(\alpha) + \hat{d}(k) - s(k);\ k = 0,1,...,H-1$, with:

$V_{d,k}$: Variance of demand $d(k)$ at period k

$\varphi_{d,k}(\alpha)$: Cumulative Gaussian distribution function with mean $\hat{d}_k$ and finite variance $V_{d_k} \geq 0$

$\varphi^{-1}_{d,k}(\alpha)$: Inverse distribution function.

Proof of lemma1: $s(k+1) = s(k) + u(k) - d(k)$

$$\Rightarrow \text{Prob}\left(s(k+1)\geq 0\right)\geq \alpha$$

$$\Rightarrow \text{Prob}\left(s(k)+u(k)-d(k)\geq 0\right)\geq \alpha$$

$$\Rightarrow \text{Prob}\left(s(k)+u(k)\geq d(k)\right)\geq \alpha$$

$$\Rightarrow \text{Prob}\left(s(k)+u(k)-\hat{d}(k)\geq d(k)-\hat{d}(k)\right)\geq \alpha$$

$$\Rightarrow \text{Prob}\left(\frac{s(k)+u(k)-\hat{d}(k)}{V_{d,k}}\geq \frac{d(k)-\hat{d}(k)}{V_{d,k}}\right)\geq \alpha \qquad \text{(A)}$$

With $\left(\frac{d(k)-\hat{d}(k)}{V_{d,k}}\right)$ is a Gaussian random variable with an identical distribution as $d(k)$.

It is possible from (A) to determine a lower bound for the control variable, assuming that φ is a probability distribution function and f a probability density function. Hence,

$$(A)\Rightarrow \varphi_{d,k}\left(\frac{s(k)+u(k)-\hat{d}(k)}{V_{d,k}}\right)\geq \alpha \qquad \text{(B)}$$

Since $\lim_{-\infty}\varphi_{d,k}\to 0$ and $\lim_{+\infty}\varphi_{d,k}\to 1$ we conclude that $\varphi_{d,k}$ is strictly increasing. We note that $\varphi_{d,k}$ is indefinitely differentiable, so we conclude that $\varphi_{d,k}$ is invertible.

$$\text{Thus } (B)\Rightarrow \frac{s(k)+u(k)-\hat{d}(k)}{V_{d,k}}\geq \varphi^{-1}_{d,k}(\alpha)$$

$$\Leftrightarrow s(k)+u(k)-\hat{d}(k)\geq V_{d,k}\cdot\varphi^{-1}_{d,k}(\alpha)$$

$$\Leftrightarrow u(k)\geq V_{d,k}\cdot\varphi^{-1}_{d,k}(\alpha)+\hat{d}(k)-s(k)$$

Thus

$$\text{Prob}\left(s(k+1)\geq 0\right)\geq \alpha \quad\Rightarrow\quad \left(u(k)\geq V_{d,k}\cdot\varphi^{-1}_{d,k}(\alpha)+\hat{d}(k)-s(k)\right)$$

Q.E.D.

The number of unsatisfied demands during the period k depends on minimum cumulative production quantity and production rate of machine M during the period k. The number of unsatisfied demands is defined as follows:

$$L(k)=\begin{cases} u_\alpha(k)-U_{\max} & \textit{If } u_\alpha(k)>U_{\max} \\ u_\alpha(k)-u(k) & \textit{If } u(k)<u_\alpha(k) \\ 0 & \text{otherwise} \end{cases} \qquad (4)$$

As we said in the introduction, we consider in this paper the machine degradation which has impact on the production plan. We note that we suppose that the failure rate $\lambda(k,t)$ is increasing in both time and production rate $u(k)$. As the machine production rate is variable over the horizon $H\Delta t$, the degradation will be variable too. We consider that the failure rate is continuous and cumulative; then the failure rate is expressed as follow:

$$\lambda(k,t)=\lambda(k,\Delta t)+\frac{u(k)}{U_{\max}}\lambda_n(t) \qquad (5)$$

With $\lambda(0,0)=\lambda(0)$ and $\lambda_n(t)$ is the nominal failure rate corresponding to the maximal production rate.
The failure rate function can be expressed in the following way by:

$$\lambda(k,t)=\lambda(0)+\sum_{l=1}^{k-1}\frac{u(l)}{U_{\max}}\lambda_n(k,\Delta t)+\frac{u(k)}{U_{\max}}\lambda_n(t)$$

$$\text{with } \quad t\in[0,\Delta t] \qquad (6)$$

To study the influence of the degradation on the production plan, we consider a unitary degradation cost c_λ.

Remark1: to simplify the notation of the failure rate we replace $\lambda(k,t)$ by $\lambda(k)$.

The cost function at the period k, which is composed by the inventory cost, lost sales cost and degradation cost is given by:

$$C(k)=cp.u(k)+cs.s(k)+cs^-.L(k)+c_\lambda.\lambda(k) \qquad (7)$$

The total cost over the horizon H is given by:

$$CT=\sum_{k=0}^{k=H-1}C(k) \qquad (8)$$

We will study in the following section the sample path trajectories of $u(k)$, $s(k)$, $L(k)$ and $\lambda(k)$. This study will allow us to determine the PA estimators and prove that these estimators are unbiased.

3. PA APPROACH

In this section we turn our attention, for applying the PA method to the discrete flow model. Indeed, The PA is an approach intended to estimate gradients of performances metric with respect to some parameters of interest. The principle of this method consists in observing and analyzing two sample paths, one is the nominal sample path ($u(k)$), and the other is the perturbed sample path ($u^\delta(k)$) (see Fig. 2). We assumed that the production rate during period k is increased by a perturbation, denoted by δ. In this paper, we consider $\delta>0$ (similar results could be easily obtained for $\delta<0$) and we evaluate the resulting changes in stock level $s(k)$ and number of unsatisfied demands $L(k)$ using geometric arguments.

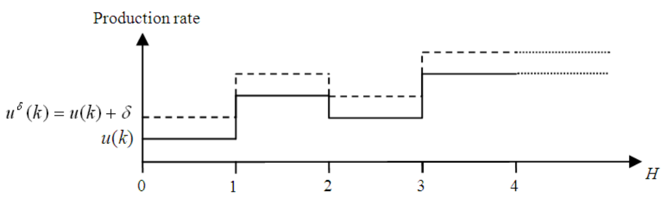

Fig. 2. Perturbed and nominal trajectories of the Production rate.

We consider the following assumptions for PA study:

- To comparing the both sample trajectories, the same distribution of random variables (demand customer, degradation law) is used.

- The maximal production rate, minimal production rate and unit costs are the same for both sample trajectories.
- The initial stock levels for the nominal and perturbed trajectories are equals.
- The initial failure rates for the nominal and perturbed trajectories are equals (i.e. $\lambda^{\delta}(0) = \lambda(0)$).

The following notations are used:

- $s^{\delta}(k)$: stock level during the period k for the perturbed trajectory;
- $u^{\delta}(k)$: production rate during the period k for the perturbed trajectory;
- $L^{\delta}(k)$: number of unsatisfied demands during the period k for the perturbed trajectory.
- $u^{\delta}_{\alpha}(k)$: minimum cumulative production quantity during the period k for the perturbed trajectory.
- $\lambda^{\delta}(k)$: failure rate during the period k for the perturbed trajectory.

3.1 Trajectories study

Stock level trajectories

The stock level during the period k for the perturbed trajectory is defined as follows:

$$s^{\delta}(k+1) = s^{\delta}(k) + u^{\delta}(k) - d(k) \tag{9}$$

Lemm2: $s^{\delta}(k) = s(k) + k.\delta$ for all $k = 0,1,\ldots, H$

Proof: $s(k) = s(k-1) + u(k-1) - d(k-1) =$

$$s(0) + \sum_{i=0}^{k-1} u(i) - \sum_{i=0}^{k-1} d(i)$$

$$s^{\delta}(k) = s^{\delta}(k-1) + u^{\delta}(k-1) - d(k-1) =$$

$$s^{\delta}(0) + \sum_{i=0}^{k-1} u^{\delta}(i) - \sum_{i=0}^{k-1} d(i)$$

We have $s^{\delta}(0) = s(0)$ (assumption) and

$\sum_{i=0}^{k-1} u^{\delta}(i) = \sum_{i=0}^{k-1} (u(i) + k) = \sum_{i=0}^{k-1} u(i) + k.\delta$, then

$$s^{\delta}(k) = s(0) + \sum_{i=0}^{k-1} u(i) + k.\delta - \sum_{i=0}^{k-1} d(i) = s(k) + k.\delta$$

Q.E.D.

Number of unsatisfied demands trajectories

The number of unsatisfied demands during the period k for the perturbed trajectory is defined as follows:

$$L^{\delta}(k) = \begin{cases} u^{\delta}_{\alpha}(k) - U_{\max} & \text{If } u^{\delta}_{\alpha}(k) > U_{\max} \\ u^{\delta}_{\alpha}(k) - u^{\delta}(k) & \text{If } u^{\delta}(k) < u^{\delta}_{\alpha}(k) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

with $u^{\delta}_{\alpha}(k) = V_{d,k} \cdot \varphi^{-1}_{d,k}(\alpha) + \hat{d}(k) - s^{\delta}(k)$; $k = 0,1,\ldots,H-1$

Lemma3: for all $k = 0,1,\ldots, H$ we have

$$L^{\delta}(k) - L(k) = \begin{cases} -k.\delta & \text{If } u^{\delta}_{\alpha}(k) > U_{\max} \text{ and } u_{\alpha}(k) > U_{\max} \\ u(k) - (k.\delta + U_{\max}) & \text{If } u^{\delta}_{\alpha}(k) > U_{\max} \text{ and } u(k) < u_{\alpha}(k) \\ u^{\delta}_{\alpha}(k) - U_{\max} & \text{If } u^{\delta}(k) > U_{\max} \text{ and } u_{\alpha}(k) \le u(k) \le U_{\max} \\ -\delta \cdot (1+k) & \text{If } u^{\delta}(k) < u^{\delta}_{\alpha}(k) \text{ and } u(k) < u_{\alpha}(k) \\ u^{\delta}_{\alpha}(k) - u^{\delta}(k) & \text{If } u^{\delta}(k) < u^{\delta}_{\alpha}(k) \text{ and } u_{\alpha}(k) \le u(k) \le U_{\max} \\ u(k) - u_{\alpha}(k) & \text{If } u^{\delta}_{\alpha}(k) \le u^{\delta}(k) \le u_{\max} \text{ and } u(k) < u_{\alpha}(k) \\ 0 & \text{Otherwise} \end{cases}$$

Proof:

Based on equations (4) and (10) we have 7 cases

- Case 1: if $u^{\delta}_{\alpha}(k) > U_{\max}$ and $u_{\alpha}(k) > U_{\max}$, then $L^{\delta}(k) = u^{\delta}_{\alpha}(k) - U_{\max}$ and $L(k) = u_{\alpha}(k) - U_{\max}$, thus we have $L^{\delta}(k) - L(k) = u^{\delta}_{\alpha}(k) - U_{\max} - (u_{\alpha}(k) - U_{\max})$
  $= u^{\delta}_{\alpha}(k) - u_{\alpha}(k) = s(k) - s^{\delta}(k) = -k.\delta$.
- Case 2 : if $u^{\delta}_{\alpha}(k) > U_{\max}$ and $u(k) < u_{\alpha}(k)$, we have $L^{\delta}(k) = u^{\delta}_{\alpha}(k) - U_{\max}$ and $L(k) = u_{\alpha}(k) - u(k)$ then $L^{\delta}(k) - L(k) = u^{\delta}_{\alpha}(k) - U_{\max} - u_{\alpha}(k) + u(k) = u(k) - (k.\delta + U_{\max})$
- Case 3 : if $u^{\delta}_{\alpha}(k) > U_{\max}$ and $u_{\alpha}(k) \le u(k) \le U_{\max}$, we have $L^{\delta}(k) = u^{\delta}_{\alpha}(k) - U_{\max}$ and $L(k) = 0$ then $L^{\delta}(k) - L(k) = u^{\delta}_{\alpha}(k) - U_{\max}$
- Case 4 : if $u^{\delta}(k) < u^{\delta}_{\alpha}(k)$ and if $u(k) < u_{\alpha}(k)$, we have $L^{\delta}(k) = u^{\delta}_{\alpha}(k) - u^{\delta}(k)$ and $L(k) = u_{\alpha}(k) - u(k)$ then $L^{\delta}(k) - L(k) = u^{\delta}_{\alpha}(k) - u^{\delta}(k) - (u_{\alpha}(k) - u(k))$
  $= -k.\delta - \delta = -\delta.(k+1)$
- Case 5 : if $u^{\delta}(k) < u^{\delta}_{\alpha}(k)$ and if $u_{\alpha}(k) \le u(k) \le U_{\max}$, then $L^{\delta}(k) = u^{\delta}_{\alpha}(k) - u^{\delta}(k)$ and $L(k) = 0$ then $L^{\delta}(k) - L(k) = u^{\delta}_{\alpha}(k) - u^{\delta}(k)$
- Case 6 : if $u^{\delta}_{\alpha}(k) \le u^{\delta}(k) \le U_{\max}$ and $u(k) < u_{\alpha}(k)$, we have $L^{\delta}(k) = 0$ and $L(k) = u_{\alpha}(k) - u(k)$ then $L^{\delta}(k) - L(k) = u(k) - u_{\alpha}(k)$
- Case 7 : if $u^{\delta}_{\alpha}(k) \le u^{\delta}(k) \le U_{\max}$ and $u_{\alpha}(k) \le u(k) \le U_{\max}$, we have $L^{\delta}(k) = 0$ and $L(k) = 0$ then $L^{\delta}(k) - L(k) = 0$

Q.E.D.

Failure rate trajectories

The failure rate during the period k for the perturbed trajectory is defined as follows:

$$\lambda^{\delta}(k,t) = \lambda(0) + \sum_{l=1}^{k-1} \frac{u^{\delta}(l)}{U_{\max}} \lambda_n(k, \Delta t) + \frac{u^{\delta}(k)}{U_{\max}} \lambda_n(t) \quad \text{with } t \in [0, \Delta t] \tag{11}$$

Lemma4:

for all $k = 0,1,\ldots, H$

$$\lambda^{\delta}(k,t)=\lambda(k,t)+\delta.\left[\left(\sum_{l=1}^{k-1}\frac{\lambda_n(k,\Delta t)}{U_{\max}}.(k-1)\right)+\left(\frac{\lambda_n(t)}{U_{\max}}\right)\right]$$

Proof:

$$\lambda^{\delta}(k,t)-\lambda(k,t)=\lambda(0)-\lambda(0)+\sum_{l=1}^{k-1}\frac{\lambda_n(k,\Delta t)}{U_{\max}}.\left[u^{\delta}(l)-u(l)\right]$$

$$+\frac{\lambda_n(t)}{U_{\max}}.\left[u^{\delta}(k)-u(k)\right]$$

$$=\delta.\left(\sum_{l=1}^{k-1}\frac{\lambda_n(k,\Delta t)}{U_{\max}}.(k-1)\right)+\delta.\left(\frac{\lambda_n(t)}{U_{\max}}\right)$$

$$=\delta.\left[\left(\sum_{l=1}^{k-1}\frac{\lambda_n(k,\Delta t)}{U_{\max}}.(k-1)\right)+\left(\frac{\lambda_n(t)}{U_{\max}}\right)\right]$$

Q.E.D.

In what follows, the PA estimators will be determined and their unbiasedness will be proved.

3.2 PA estimators

In this section we determine the estimators of the difference of each part (production, inventory, lost sales and degradation costs) of the expected cost.

The expected cost for the nominal trajectory is given by:

$$CT_{ex}=E\left[\sum_{k=0}^{k=H}C(k)\right].$$

Then $CT_{ex}=\left(cp.E\left[\sum_{k=0}^{k=H-1}u(k)\right]+cs.E\left[\sum_{k=0}^{k=H}s(k)\right]+cs^{-}.E\left[\sum_{k=0}^{k=H-1}L(k)\right]+c_{\lambda}.E\left[\sum_{k=0}^{k=H}\lambda(k)\right]\right)$

The expected cost for the perturbed trajectory is given by:

$CT_{ex}^{\delta}=E\left[\sum_{k=0}^{k=H}C^{\delta}(k)\right]$. With

$$C^{\delta}(k)=cp.u^{\delta}(k)+cs.\,s^{\delta}(k)+cs^{-}.L^{\delta}(k)+c_{\lambda}.\lambda^{\delta}(k)$$

Then $CT_{ex}=\left(cp.E\left[\sum_{k=0}^{k=H-1}u^{\delta}(k)\right]+cs.E\left[\sum_{k=0}^{k=H}s^{\delta}(k)\right]+cs^{-}.E\left[\sum_{k=0}^{k=H-1}L^{\delta}(k)\right]+c_{\lambda}.E\left[\sum_{k=0}^{k=H}\lambda^{\delta}(k)\right]\right)$

The difference between the expected cost for the perturbed and nominal trajectories is given by:

$$\Delta CT_{ex}=cp.E\left[\sum_{k=0}^{k=H-1}(u^{\delta}(k)-u(k))\right]+cs^{-}.E\left[\sum_{k=0}^{k=H-1}(L^{\delta}(k)-L(k))\right]$$

$$+cs.E\left[\sum_{k=0}^{k=H}(s^{\delta}(k)-s(k))\right]+c_{\lambda}.E\left[\sum_{k=0}^{k=H}(\lambda^{\delta}(k)-\lambda(k))\right]$$

According to lemmas 2, 3 and 4, we have:

$$\Delta CT_{ex}=cp.E\left[\sum_{k=0}^{k=H-1}(\delta)\right]+cs.E\left[\sum_{k=0}^{k=H}(k.\delta)\right]$$

$$+cs^{-}.E\left[\sum_{k=0}^{k=H-1}(L^{\delta}(k)-L(k))\right]+c_{\lambda}.E\left[\sum_{k=0}^{k=H}(\lambda^{\delta}(k)-\lambda(k))\right]$$

$$\Delta CT_{ex}=cp.E\left[H.\delta\right]+cs.E\left[\sum_{k=0}^{k=H}(k.\delta)\right]$$

$$+cs^{-}.E\left[\sum_{k=0}^{k=H-1}(L^{\delta}(k)-L(k))\right]+c_{\lambda}.E\left[\sum_{k=0}^{k=H}(\lambda^{\delta}(k)-\lambda(k))\right]$$

$$=cp.E\left[H.\delta\right]+cs.E\left[(H.(H+1)/2).\delta\right]$$

$$+cs^{-}.E\left[\sum_{k=0}^{k=H-1}(L^{\delta}(k)-L(k))\right]+c_{\lambda}.E\left[\sum_{k=0}^{k=H}(\lambda^{\delta}(k)-\lambda(k))\right]$$

$$=cp.H.\delta+cs.(H.(H+1)/2).\delta+$$

$$cs^{-}.E\left[\sum_{k=0}^{k=H-1}(L^{\delta}(k)-L(k))\right]+c_{\lambda}.E\left[\sum_{k=0}^{k=H}(\lambda^{\delta}(k)-\lambda(k))\right]$$

Then

$$\frac{\Delta CT_{ex}}{\Delta u}=cp.H+cs.(H.(H+1)/2)$$

$$+cs^{-}.E\left[\sum_{k=0}^{k=H-1}(L^{\delta}(k)-L(k))\right]/(\Delta u)$$

$$+c_{\lambda}.E\left[\sum_{k=0}^{k=H}(\lambda^{\delta}(k)-\lambda(k))\right]/(\Delta u)$$

For making these estimators useful in practice, the unbiasedness should be proved.

***Theorem*:** The estimators of the difference of each part of the expected average cost are unbiased, i.e.:

$$\frac{E\left[H\right]}{\Delta u}=E\left[\frac{H}{\Delta u}\right],\quad \frac{E\left[H.(H+1)/2\right]}{\Delta u}=E\left[\frac{H.(H+1)/2}{\Delta u}\right]$$

$$\frac{E\left[\sum_{k=0}^{k=H-1}(L^{\delta}(k)-L(k))\right]}{\Delta u}=E\left[\frac{\sum_{k=0}^{k=H-1}(L^{\delta}(k)-L(k))}{\Delta u}\right]\text{and}$$

$$\frac{E\left[\sum_{k=0}^{k=H}(\lambda^{\delta}(k)-\lambda(k))\right]}{\Delta u}=E\left[\frac{\sum_{k=0}^{k=H}(\lambda^{\delta}(k)-\lambda(k))}{\Delta u}\right]$$

$$+c_{\lambda}.E\left[\sum_{k=0}^{k=H}\delta.\left[\left(\sum_{l=1}^{k-1}\frac{\lambda_n(k,\Delta t)}{U_{\max}}.(k-1)\right)+\left(\frac{\lambda_n(t)}{U_{\max}}\right)\right]\right]$$

The proof of this theorem is similar to the the proof of theorem in Turki et al. (2013).

4. PA BASED OPTIMIZATION

In this section we will present how the PA estimators are determined by simulation and then used for determining the optimal planning. However, we will present in this section a

PA estimation algorithm which determines the PA estimators, an optimization algorithm and then the numerical results.

4.1 PA estimation algorithm

Let X, Y, Z and Q are the PA estimators (parameters) which will be used in the following algorithm. Indeed, for the first and second estimators are known and which are $X=H$ and $Y=H.(H+1)/2$ (see the previous section). For Z and Q we have this algorithm.

Beginning

$Q=0$, $Z=0$, $k=0$ //*Initialization*

Do

- $Z= Z+ (L^{\delta}(k) - L(k))$
- $Q= Q+ (\lambda^{\delta}(k) - \lambda(k))$
- Advance k.

While $k<H$
(Third estimator=Z)
(Fourth estimator=Q)

End

In what follows, an optimisation algorithm is presented and which determines the optimal production rate in a period k. Indeed, this algorithm is combined with Nelder-Mead method under MATHEMATICA software for determining the optimal production plan over the horizon H.

4.2 Optimization algorithm

Beginning

$u_i= U_{min}$, $u_s= U_{max}$ and $u_m= u_a(k)$ //*Initialisation*

Do

Step 1: $u(k)= u_s$

Step 2: Determine the estimators X, Y, Z and Q which correspond to $u(k)= u_s$ by using the PA estimation algorithm.

Step 3: Determine the difference estimation of the cost function $V(u(k))$ by using the estimators X, Y, Z and Q with $V(u(k)) = ((cp.\ X) + (cs.\ Y) + (cs^{-}.\ Z) + (c_{\lambda}.Q))$.

Step 4: If $V(u(k)) < 0$ then $u_i = u_m$ and $u_s= U_{max}$ return to step 1, else go to step 5.

Step 5: $u_m=$ **int** $[(u_s+ u_i)/2]$ then determine $V(u_m)$.

Step 6: If $V(u_m) < 0$ then $u_i= u_m$ else $u_s= u_m$.

Step 7: If $u_s <> u_i$ return to step 1, else go to step 8.

Step 8: the optimal production rate for the period k is equal to u_m.

End.

4.3 Numerical results

In this part, an example of optimal production planning is presented. The PA estimators are used in an optimization algorithm, which allows determining the optimal production planning. The following arbitrarily chosen input data are considered as an example to illustrate our approach:

H=20 months.

(i) Lower and upper boundaries of production capacities: $U_{min}=5$ and $U_{max}=14$.

cp =2 mu (*monetary unit*) $/k$, cs =1 mu/k cs^{-} =400 mu/k and c_{λ} =200 with λ_0=0 (degradation law characterized by a Weibull distribution).

(ii) The customer satisfaction degree, associated with the stock constraint, is equal to 95% (θ= 0.95).

In this part the PA estimators is used in an optimization algorithm, which allows us to determine the optimal production plan ($u^*(k)$) in Table 1.

Table1. Optimal production plan

$d(k)$	9	7	11	12	7	8	6	8	6	10
$u^*(k)$	13	6	10	7	5	3	11	6	9	9
$d(k)$	9	11	11	7	12	5	6	8	10	8
$u^*(k)$	8	14	8	7	4	5	14	13	5	6

5. CONCLUSIONS

In this paper, a manufacturing system composed by a single-product machine, a stock and a customer who demands a stochastic quantity of product is considered. A discrete fluid model is adopted to describe the system and take into account machine failure, lost demands and machine degradation. The PA method is applied to the discrete fluid model. The stock level, lost demands and failure rate trajectories is studied and analyzed. The perturbation analysis estimators are determined and shown to be unbiased. These estimators are then implemented in an optimization algorithm for determining the optimal production planning.

For future research, the production will be combined with maintenance and the PA method will be applied for determining the optimal production and maintenance plan.

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