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Hybrid Systems with Memory: Modelling and Stability Analysis via Generalized Solutions

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Abstract: Hybrid systems with memory are dynamical systems exhibiting both hybrid and delay phenomena. We present a general modelling framework for such systems using hybrid functional inclusions, whose generalized solutions are defined on hybrid time domains and evolve in the phase space of hybrid memory arcs equipped with the graphical convergence topology. We prove a general existence result based on some basic conditions on the data of hybrid systems. We further establish sufficient conditions for the stability analysis of hybrid systems with memory using Lyapunov-Razumikhin (L-R) functions. We demonstrate the stability results on a general nonlinear system with switching dynamics and state jumps.

1. INTRODUCTION

Hybrid systems with memory refer to dynamical systems exhibiting both hybrid and delay phenomena. Control systems with delayed hybrid feedback and interconnected hybrid systems with network delays are particular examples of such systems. In fact, delays are often inevitable in many control applications (Sipahi et al., 2011) and often cause instability and/or loss of robustness (Cloosterman et al., 2009).

Motivated by considerations of robust stability in hybrid control systems, generalized solutions of hybrid inclusions defined on hybrid time domains have been proposed to study hybrid systems (Goebel et al., 2004, 2012; Sanfelice et al., 2008). These generalized solutions have led to the successful extensions of many stability analysis results and tools known for classical nonlinear systems, including converse Lyapunov theorems, to a hybrid setting (see Goebel et al. (2012); Sanfelice et al. (2007) and references therein).

As a first step to studying hybrid systems with delays via generalized solutions, Liu and Teel (2012) defined such solutions using hybrid functional inclusions and established some basic existence and nominal well-posedness results. The case considered in Liu and Teel (2012), however, assumes that the flow and jump sets are subsets of the Euclidean space. This leaves open the general (and more challenging) case, where the flow and jump sets are subsets of the space of hybrid memory arcs. The main purpose of this paper is to establish this general case.

Another contribution of this paper is a set of sufficient conditions for the stability analysis of generalized solutions for hybrid systems. These conditions are formulated using Lyapunov-Razumikhin (L-R) functions rather than Lyapunov-Krasovskii (L-K) functionals, partially because L-R functions are easier to construct than L-K functionals, which is even more so in the hybrid setting. We demonstrate the stability results on a general nonlinear system with switching dynamics and state jumps.

We note that asymptotic stability for hybrid systems with delays have been studied extensively in the past in various settings (see, e.g., Liu and Ballinger (2001); Liu and Shen (2006); Liu et al. (2011a,b); Yan and Özbay (2008); Yuan et al. (2003)). General results on robust asymptotic stability along the lines of Goebel et al. (2012), however, are still not available for hybrid systems with delays. Most current results and tools for such systems rely on standard concepts such as uniform convergence and distance (for piecewise continuous functions), whereas we believe that graphical convergence/topology should play a more prominent role in handling the discontinuities in hybrid systems with delays, especially where robustness properties of the solutions are concerned (such as robust stability against perturbations in the hybrid system data).

2. PRELIMINARIES

Notation: Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with its norm denoted by $|\cdot|$; $\mathbb{Z}$ the set of all integers; $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{R}_{\leq 0} = (-\infty, 0]$, $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \cdots\}$, and $\mathbb{Z}_{\leq 0} = \{0, -1, -2, \cdots\}$; $C([a, b], \mathbb{R}^n)$ the set of all continuous functions from $[a, b]$ to $\mathbb{R}^n$.

2.1 Hybrid systems with memory

The definitions of hybrid time domains and hybrid arcs (Goebel et al., 2012; Goebel and Teel, 2006) (extended to hybrid systems with memory in (Liu and Teel, 2012)) are introduced below.

Definition 1. A subset $E \subseteq \mathbb{R} \times \mathbb{Z}$ is called a *compact hybrid time domain with memory* if $E = E_{\geq 0} \cup E_{\leq 0}$, where

$$E_{\geq 0} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

and

$$E_{\leq 0} = \bigcup_{k=1}^{K} ([s_k, s_{k-1}], -k+1)$$

for some finite sequence of times $s_K \leq \cdots \leq s_1 \leq s_0 = 0 = t_0 \leq t_1 \leq \cdots \leq t_J$. The set E is called a *hybrid time domain with memory* if, for all $(T, J) \in E_{\geq 0}$ and all $(S, K) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, $(E_{\geq 0} \cap ([0, T] \times \{0, 1, \cdots, J\})) \cup (E_{\leq 0} \cap ([-S, 0] \times \{-K, -K+1, \cdots, 0\}))$ is a compact hybrid time domain with memory. The set $E_{\leq 0}$ is called a *hybrid memory domain*.

Remark 2. It is easy to see that, for a hybrid time domain $E = E_{\leq 0} \cup E_{\geq 0}$, $E_{\geq 0}$ is a union of a finite or infinite sequence of sets of the form $[t_j, t_{j+1}] \times \{j\} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, with the last interval (if existent) possibly of the form $[t_j, T)$ with T finite or $T = \infty$, while $E_{\leq 0}$ is a union of a finite or infinite sequence of sets of the form $[s_k, s_{k-1}] \times \{-k+1\} \subseteq \mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0}$, with the last interval (if existent) possibly of the form $(S, t_{j-1}]$ only if $S = -\infty$.

Definition 3. A *hybrid arc with memory* is a pair consisting of a domain *dom* x, which is a hybrid time domain with memory, and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(\cdot, j)$ is locally absolutely continuous on $I_j = \{t : (t, j) \in dom\, x\}$ for each $j \in \mathbb{Z}$ such that I_j has nonempty interior. In particular, a hybrid arc x with memory is called a *hybrid memory arc* if $\operatorname{dom} x \subseteq \mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0}$. We shall simply use the term *hybrid arc* if we do not have to distinguish between the above two hybrid arcs.

The collection of all hybrid memory arcs is denoted by $\mathcal{M}$.

Definition 4. Given a hybrid arc and any $(t, j) \in \operatorname{dom} x$, we define an operator $\mathcal{A}_{[t,j]}$ that maps x to $\mathcal{A}_{[t,j]} x \in \mathcal{M}$ given by

$$\mathcal{A}_{[t,j]} x(s, k) = x(t+s, j+k),$$

for all $(s, k) \in \operatorname{dom} \mathcal{A}_{[t,j]} x$, where $\operatorname{dom} \mathcal{A}_{[t,j]} x$ is defined by $(s, k) \in \operatorname{dom} \mathcal{A}_{[t,j]} x$ if and only if $(t+s, j+k) \in \operatorname{dom} x$.

Definition 5. Data of a *hybrid system with memory* in $\mathcal{M}$ consists of four elements:

- a set $\mathcal{C} \subseteq \mathcal{M}$, called the *flow set*;
- a set-valued functional $\mathcal{F} : \mathcal{M} \rightrightarrows \mathbb{R}^n$, called the *flow map*;
- a set $\mathcal{D} \subseteq \mathcal{M}$, called the *jump set*;
- a set-valued functional $\mathcal{G} : \mathcal{M} \rightrightarrows \mathbb{R}^n$, called the *jump map*.

The system is denoted by $\mathcal{H}_\mathcal{M} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$.

Definition 6. A hybrid arc x is a *solution to the hybrid system* $\mathcal{H}_\mathcal{M}$ if $\mathcal{A}_{[0,0]} x \in \mathcal{C} \cup \mathcal{D}$ and:

(S1) for all $j \in \mathbb{Z}_{\geq 0}$ and almost all $t \in I_j$,

$$\mathcal{A}_{[t,j]} x \in \mathcal{C}, \quad \dot{x}(t, j) \in \mathcal{F}(\mathcal{A}_{[t,j]} x), \tag{1}$$

(S2) for all $j \in \mathbb{Z}_{\geq 0}$ and $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$\mathcal{A}_{[t,j]} x \in \mathcal{D}, \quad x(t, j+1) \in \mathcal{G}(\mathcal{A}_{[t,j]} x). \tag{2}$$

The solution x is called *nontrivial* if its positive domain $\operatorname{dom}_{\geq 0}(x) = \operatorname{dom} x \cap (\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0})$ has at least two points. It is called *complete* if $\operatorname{dom}_{\geq 0}(x)$ is unbounded. It is called *maximal* if there does not exist another solution y to $\mathcal{H}_\mathcal{M}$ such that $\operatorname{dom} x$ is a proper subset of $\operatorname{dom} y$ and $x(t, j) = y(t, j)$ for all $(t, j) \in \operatorname{dom} x$. The set of all maximal solutions to $\mathcal{H}_\mathcal{M}$ is denoted by $\mathcal{S}_{\mathcal{H}_\mathcal{M}}$.

2.2 *Preliminaries on set-valued analysis*

We need a few regularity conditions on the hybrid data to establish certain results on basic existence and well-posedness for $\mathcal{H}_\mathcal{M}$. To formulate these regularity conditions, we need to recall a few definitions from set-valued analysis. The set-valued analysis concepts recalled here for mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$ can be found in Chapter 5 of Rockafellar and Wets (1998) (see also Chapter 5 of Goebel et al. (2012) for set-valued analysis in the hybrid systems setting).

Definition 7. (Set convergence). Consider a sequence of sets $\{H_i\}_{i=1}^\infty$ in $\mathbb{R}^n$. The *outer limit* of the sequence, denoted by $\limsup_{i\to\infty} H_i$ is the set of all $x \in \mathbb{R}^n$ for which there exists a subsequence $x_{i_k} \in S_{i_k}$, $k = 1, 2, \cdots$, such that $x_{i_k} \to x$. The *inner limit* of $\{H_i\}_{i=1}^\infty$, denoted by $\liminf_{i\to\infty} H_i$, is the set of all $x \in \mathbb{R}^n$ for which there exists some i_0 and a sequence $x_i \in S_i$ ($\forall i \geq i_0$) such that $x_i \to x$. The *limit* of $\{H_i\}_{i=1}^\infty$ exists if $\limsup_{i\to\infty} H_i = \liminf_{i\to\infty} H_i$ and it is then given by $\lim_{i\to\infty} H_i = \limsup_{i\to\infty} H_i = \liminf_{i\to\infty} H_i$.

Definition 8. (Set-valued mappings). Let $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$. Its domain, range, and graph are defined by

$$\operatorname{dom} S := \{x : S(x) \neq \emptyset\},$$
$$\operatorname{rge} S := \{y : \exists\, x \text{ s.t. } y \in S(x)\},$$
$$\operatorname{gph} S := \{(x, y) : y \in S(x)\},$$

respectively. The mapping S is called *outer semicontinuous* at $x \in \mathbb{R}^m$ if for every sequences of points $x_i \to x$ and $y_i \to y$ with $y_i \in S(x_i)$, we have $y \in S(x)$. It is *locally bounded* at $x \in \mathbb{R}^m$ if there exists a neighborhood U_x of x such that the set $S(U_x) := \bigcup_{x' \in U_x} S(x') \subseteq \mathbb{R}^n$ is bounded. It is said to be *outer semicontinuous* (respectively, *locally bounded*) *relative to a set* $H \subseteq \mathbb{R}^m$, if the mapping defined by $S(x)$ for $x \in H$ and by $\emptyset$ for $x \notin H$ is outer semicontinuous (respectively, locally bounded) at each $x \in H$.

By convention, a mapping S is said to be outer semicontinuous or locally bounded if it is so relative to its domain.

Definition 9. (Graphical convergence). A sequence $S_i : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of mappings is said to *converge graphically* to some $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ if $\lim_{i\to\infty} \operatorname{gph} S_i = \operatorname{gph} S$. In particular, a sequence of hybrid arcs $\varphi_i : \operatorname{dom} \varphi_i \to \mathbb{R}^n$ converges graphically to some $\varphi : \mathbb{R}^2 \rightrightarrows \mathbb{R}^n$ if $\lim_{i\to\infty} \operatorname{gph} \varphi_i = \operatorname{gph} \varphi$. We use $\xrightarrow{\text{gph}}$ to denote graphical convergence.

2.3 *The space* $(\mathcal{M}, \mathbf{d})$

The space of all hybrid memory arcs is not a vector space, since different hybrid arcs can have different domains. In this section, we recall from Rockafellar and Wets (1998) a quantity that characterizes the set convergence of closed nonempty sets (and hence graphical convergence of outer semicontinuous mappings) and use this distance to define a metric on $\mathcal{M}$. Let cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$ denote the collection of all nonempty, closed subsets of $\mathbb{R}^n$. Given $\rho \geq 0$, for each pair $A, B \in$ cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$, define

$$\mathbf{d}_\rho(A,B) := \max_{|z|\leq\rho} \left|d(z,A) - d(z,B)\right|.$$

where $d(z,H)$ for $z \in \mathbb{R}^n$ and $H \subseteq \mathbb{R}^n$ is defined by $\inf_{w\in H}|w-z|$. Furthermore, define

$$\mathbf{d}(A,B) := \int_0^\infty \mathbf{d}_\rho(A,B)e^{-\rho}d\rho,$$

which is called the *(integrated) set distance* between A and B. This distance indeed characterizes set convergence of sets in cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$ as recalled below.

Theorem 10. (Rockafellar and Wets (1998), Theorem 4.42). A sequence $S_i \in$ cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$ converges to S if and only $\mathbf{d}(S_i, S) \to 0$. Moreover, the space (cl-sets$_{\neq\emptyset}(\mathbb{R}^n), \mathbf{d}$) is a separable, locally compact, and complete metric space.

Since outer semicontinuity of a set-valued map is characterized by the closedness of its graph, the set distance defined above can be naturally applied to characterize the graphical convergence of set-valued maps (and hence hybrid arcs). Let osc-maps$_{\neq\emptyset}(\mathbb{R}^2;\mathbb{R}^n)$ denote the collection of all outer semicontinuous set-valued functions from $\mathbb{R}^2$ to $\mathbb{R}^n$ that have nonempty domains. Given $\rho \geq 0$, for each pair $S, T \in$ osc-maps$_{\neq\emptyset}(\mathbb{R}^2;\mathbb{R}^n)$, define

$$\mathbf{d}_\rho(S,T) := \mathbf{d}_\rho(\operatorname{gph} S, \operatorname{gph} T)$$

and

$$\mathbf{d}(S,T) := \mathbf{d}(\operatorname{gph} S, \operatorname{gph} T),$$

which is called the *graphical distance* for mappings in osc-maps$_{\neq\emptyset}(\mathbb{R}^2;\mathbb{R}^n)$.

While hybrid memory arcs are single-valued, they can be seen as set-valued maps from $\mathbb{R}^2$ to $\mathbb{R}^n$, which are outer semicontinuous, since their graphs are always closed by definition. Therefore, as a corollary of Theorem 10 above, the graphical distance, restricted to $\mathcal{M}\times\mathcal{M}$, defines a metric space $(\mathcal{M}, \mathbf{d})$.

Corollary 11. The space $(\mathcal{M}, \mathbf{d})$ is a separable metric space.

Note that $(\mathcal{M}, \mathbf{d})$ is not complete, since the limit of a sequence of graphically convergent hybrid memory arcs may not be a hybrid memory arc. If such compactness is needed, the following subspace of $(\mathcal{M}, \mathbf{d})$ can be used.

Given $b, \lambda \in \mathbb{R}_{\geq 0}$, define

$$\mathcal{M}_b := \Big\{\varphi \in \mathcal{M} : \sup_{(s,k)\in\operatorname{dom}\varphi} |\varphi(s,k)| \leq b\Big\},$$

$$\mathcal{M}_{b,\lambda} := \Big\{\varphi \in \mathcal{M}_b : \varphi \text{ is } \lambda\text{-Lipschitz}\Big\},$$

where $\varphi \in \mathcal{M}$ is said to be *λ-Lipschitz* if

$$\left|\varphi(s',k) - \psi(s'',k)\right| \leq \lambda\left|s - s'\right|$$

holds for all (s,k), $(s',k) \in \operatorname{dom}\varphi$.

Theorem 12. The space $(\mathcal{M}_{b,\lambda}, \mathbf{d})$ is a separable, locally compact, and complete metric space.

Proof. It suffices to show that $\mathcal{M}_{b,\lambda}$ is a closed subspace of $(\mathcal{M}, \mathbf{d})$. Consider a sequence $\varphi_i \in \mathcal{M}_{b,\lambda}$ such that $\mathbf{d}(\varphi_i, \varphi) \to 0$ as $i \to \infty$ for some $\varphi \in \mathcal{M}$. We need to prove that $\varphi \in \mathcal{M}_{b,\lambda}$. Note that the sequence $\{\varphi_i\}_{i=1}^\infty$ is a bounded sequence and hence by definition locally eventually bounded. It follows from the argument in (Goebel et al., 2012, Examples 5.3 and 5.19) that $\operatorname{dom}\varphi = \lim_{i\to\infty}\operatorname{dom}\varphi_i$ is a hybrid memory domain. Moreover, since for each $(s,k) \in \operatorname{dom}\varphi$, there exist $(s_i,k_i) \in \operatorname{dom}\varphi_i$ such that $(s_i,k_i) \to (s,k)$ as $i \to \infty$. It follows that $(s,k) \in [-r,0]\times[-m,0]$ since $(s_i,k_i) \in [-r,0]\times[-m,0]$ for all i. This shows $\operatorname{dom}\varphi \subseteq [-r,0]\times[-m,0]$.

For each $k \in \mathbb{Z}_{\leq 0}$, let $I^k = \{s \in \mathbb{R}_{\leq 0} : (s,k) \in \operatorname{dom}\varphi\}$. Let I_i^k be similarly defined for φ_i. It follows from the very definition of set convergence that $\varphi_i(\cdot,k)$ converges graphically to $\varphi(\cdot,k)$. Now note that the sequence $\{\varphi_i(\cdot,k)\}_{i=1}^\infty$ is λ-Lipschitz. Suppose I^k is a nonempty set. Following the same argument as in the proof of (Goebel et al., 2012, Lemma 5.28), one can show that $\varphi(\cdot,k)$ is single-valued and λ-Lipschitz on I^k. In addition, $\varphi_i(\cdot,k)$ converges uniformly to $\varphi(\cdot,k)$ on every compact subset of $\operatorname{int}(I_i^k)$. This concludes that $\varphi \in \mathcal{M}_{b,\lambda}$. ∎

The following lemma shows that given a hybrid arc $x \in \mathcal{X}$, its memory $\mathcal{A}_{[t,j]}x$ at (t,j) can be regarded as a continuous function from I^j to $(\mathcal{M}, \mathbf{d})$, for each $j \in \mathbb{Z}$ such that I^j has nonempty interior.

Lemma 13. Let $x \in \mathcal{X}$ be a hybrid arc with memory. For each $j \in \mathbb{Z}$ such that I^j has nonempty interior, the function $a : I^j \to \mathcal{M}$ defined by $a(t) := \mathcal{A}_{[t,j]}x$ is uniformly continuous on each compact subinterval U of I^j. Moreover, if $x(\cdot,j)$ is λ-Lipschitz on U, then a is $\max(\lambda,1)$-Lipschitz.

This lemma shows that by considering the graphical convergence topology on $\mathcal{M}$, we can establish the above continuity property of $\mathcal{A}_{[t,j]}x$, which is a fundamental property that is needed for studying functional differential equations (Hale and Lunel, 1993).

2.4 *Regularity assumptions on hybrid data of $\mathcal{H}_\mathcal{M}$*

We now introduce a few regularity conditions on the hybrid data, especially on $\mathcal{F}$ and $\mathcal{G}$, which are regarded as set-valued mappings from the space $(\mathcal{M}, \mathbf{d})$ to $\mathbb{R}^n$. These regularity conditions will allow us to establish certain basic existence results in the next section. Given a subset $\mathcal{M}' \subseteq \mathcal{M}$ and a functional $\mathcal{F} : \mathcal{M} \rightrightarrows \mathbb{R}^n$, we use the notation $\mathcal{F}|_{\mathcal{M}'}$ to denote the mapping defined by $\mathcal{F}(\varphi)$ for $\varphi \in \mathcal{M}'$ and by $\emptyset$ for $\varphi \notin \mathcal{M}$.

Definition 14. A set-valued functional $\mathcal{F} : \mathcal{M} \rightrightarrows \mathbb{R}^n$ is said to be *outer semicontinuous* at $\varphi \in \mathcal{M}$, if for every sequences of hybrid memory arcs $\varphi_i \xrightarrow{\text{gph}} \varphi$ and $y_i \to y$ with $y_i \in \mathcal{F}(\varphi_i)$, we have $y \in \mathcal{F}(\varphi)$.

Definition 15. A set-valued functional $\mathcal{F} : \mathcal{M} \rightrightarrows \mathbb{R}^n$ is said to be *locally bounded* at $\varphi \in \mathcal{M}$ if there exists a neighborhood $\mathcal{U}_\varphi$ of φ such that the set $\mathcal{F}(\mathcal{U}_\varphi) := \bigcup_{\psi\in\mathcal{U}_\varphi}\mathcal{F}(\psi) \subseteq \mathbb{R}^n$ is bounded.

In the above definitions, $\mathcal{F}$ is said to be outer semicontinuous (respectively, locally bounded) *relative* to a set $\mathcal{M}' \subseteq \mathcal{M}$, if the mapping $\mathcal{F}|_{\mathcal{M}'}$ is outer semicontinuous (respectively, locally bounded) at each $\varphi \in \mathcal{M}'$. Finally, the mapping $\mathcal{F}$ is said to be outer semicontinuous (respectively, locally bounded) if it is so relative to its domain.

Assumption 16. The following is a list of basic conditions on the data of the hybrid system $\mathcal{H}_{\mathcal{M}} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$.

(A1) $\mathcal{C}$ and $\mathcal{D}$ are closed subsets of $\mathcal{M}$;
(A2) $\mathcal{F}$ is outer semicontinuous and locally bounded relative to the set $\mathcal{C}$ and $\mathcal{F}(\varphi)$ is nonempty and convex for each $\varphi \in \mathcal{C}$;
(A3) $\mathcal{G}$ is outer semicontinuous and locally bounded relative to $\mathcal{D}$, and $\mathcal{G}(\varphi)$ is nonempty for each $\varphi \in \mathcal{D}$.
(A3′) $\mathcal{G}$ is nonempty for each $\varphi \in \mathcal{D}$.

Definition 17. For any $\varphi \in \mathcal{K} \subseteq \mathcal{M}$, we define $\mathcal{T}_{\mathcal{K}}(\varphi) \subseteq \mathbb{R}^n$ by $v \in \mathcal{T}_{\mathcal{K}}(\varphi)$ if and only if, for any $\varepsilon > 0$, there exist $h \in (0, \varepsilon]$ and $x_h \in C([0, h], \mathbb{R}^n)$ such that
(1) $x_h(0) = \varphi(0, 0)$ and

$$\frac{x_h(h) - x_h(0)}{h} \in v + \varepsilon\mathbb{B};$$

(2) the hybrid memory arc ψ_{x_h} defined by

$$\psi_{x_h}(s, k) = \begin{cases} x_h(h+s), & \forall s \in [-h, 0],\ k = 0, \\ \varphi(h+s, k), & \forall (h+s, k) \in \operatorname{dom}\varphi, \end{cases} \quad (3)$$

lies in $\mathcal{K}$.

3. EXISTENCE OF SOLUTIONS

Theorem 18. Let $\mathcal{H}_{\mathcal{M}} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ satisfy the conditions (A1), (A2), and (A3′) in Assumption 16. Let $\varphi \in \mathcal{C} \cup \mathcal{D}$. If, for every $\xi \in \mathcal{C} \backslash \mathcal{D}$,

$$\mathcal{F}(\xi) \cap \mathcal{T}_{\mathcal{C}}(\xi) \neq \emptyset, \quad (4)$$

then there exists a nontrivial solution to $\mathcal{H}_{\mathcal{M}}$ from every initial condition $\varphi \in \mathcal{C} \cup \mathcal{D}$ such that $\varphi \in \mathcal{M}_{b,\lambda}$ for some b, λ. Moreover, every such maximal solution x satisfies exactly one of the following conditions:

(a) x is complete;
(b) $\operatorname{dom}_{\geq 0}(x)$ is bounded, the interval I_J has nonempty interior, and $\limsup_{t \to T^-} |x(t, J)| = \infty$, where $J = \sup_j \operatorname{dom} x$ and $T = \sup_t \operatorname{dom} x$;
(c) $\varphi(T, J) \notin \mathcal{C} \cup \mathcal{D}$, where $(T, J) = \sup \operatorname{dom} x$.

Furthermore, if $\mathcal{G}(\varphi) \subseteq \mathcal{C} \cup \mathcal{D}$ for all $\varphi \in \mathcal{D}$, then (c) above does not occur.

Due to space limit, we will only present a sketch of the proof for Theorem 18 here and leave the full detailed proof to forthcoming publications.

Sketch of Proof. ***Local existence:*** If $\varphi \in \mathcal{D}$, then the hybrid arc x with $\mathcal{A}_{[0,0]}x = \varphi$ and $x(0, 1) = z$ with any $z \in \mathcal{G}(\varphi)$ provides a desired solution. Otherwise, $\varphi \in \mathcal{C} \backslash \mathcal{D}$ and the viability condition (4) is satisfied at a. Given any $a > 0$, define

$$\mathcal{M}_S := \{\psi \in \mathcal{C} \cap \mathcal{M}_{b,\lambda} : |\psi(0, 0) - \varphi(0, 0)| \leq a\},$$

where $b := \|\varphi\| + a$ and $\lambda > 1$ is such that $\mathcal{F}(\psi) \subseteq (\lambda - 1)\mathcal{B}$ for all $\|\psi\| \leq b$ and φ is λ-Lipschitz. Clearly, $\mathcal{M}_S$ is a closed set in $(\mathcal{M}, \mathbf{d})$.

The idea main is to construct a series of approximate solutions that converge, within $\mathcal{M}_S$, a true flow solution to the hybrid system $\mathcal{H}_{\mathcal{M}}$. We rely on the following claim to construct a series of approximate solutions. The viability condition (4) is essential for its proof.

Claim: For each $\varepsilon \in (0, 1)$, there exists positive numbers $\{h_k\}_{k=1}^p$, real vectors $\{v_k\}_{k=1}^p$, and hybrid arcs $\{y_k\}_{k=1}^p$ such that $\sum_{k=1}^{p-1} h_k \leq \frac{a}{\lambda + (1+\lambda)\varepsilon} < \sum_{k=1}^{p} h_k$ and

$$\begin{cases} \mathcal{A}_{[0,0]}y_k \in \mathcal{M}_S,\ \mathcal{A}_{[h_k,0]}y_k \in \mathcal{C},\ v_k \in \mathcal{F}(\mathcal{A}_{[0,0]}y_k), \\ \dfrac{y_k(h_k, 0) - y_k(0, 0)}{h_k} \in v_k + \varepsilon\mathbb{B}, \\ \mathcal{A}_{[h_{k-1},0]}y_{k-1} \text{ and } \mathcal{A}_{[0,0]}y_k \text{ are } (1/\varepsilon, h_k\varepsilon)\text{-close}, \end{cases} \quad (5)$$

holds for all $k = 1, \cdots, p$, where $y_0 = \varphi$, $h_0 = 0$, and the domain of each y_k, $k = 1, \cdots, p$, contains $[0, h_k] \times \{0\}$.

Construction of Approximated Solutions: Define a hybrid arc y_ε by $\mathcal{A}_{[0,0]}y_\varepsilon = y_0 = \varphi$ and

$$y_\varepsilon(s, 0) = y_{i+1}(s - \sum_{k=0}^{i} h_k, 0) + \sum_{k=0}^{i} [y_k(h_k, 0) - y_{k+1}(0, 0)],$$

if

$$s \in [\sum_{k=0}^{i} h_k, \sum_{k=0}^{i+1} h_k], \quad i \in \{0, \cdots, p-1\}.$$

We further define a hybrid arc x_ε by $\mathcal{A}_{[0,0]}x_\varepsilon = \mathcal{A}_{[0,0]}y_\varepsilon = y_0 = \varphi$ and

$$x_\varepsilon(s, 0) = \frac{s - \sum_{k=0}^{i} h_k}{h_{i+1}} \Big[y_\varepsilon(\sum_{k=0}^{i+1} h_k, 0) - y_\varepsilon(\sum_{k=0}^{i} h_k, 0) \Big] + y_\varepsilon(\sum_{k=0}^{i} h_k, 0),$$

if

$$s \in [\sum_{k=0}^{i} h_k, \sum_{k=0}^{i+1} h_k], \quad i \in \{0, \cdots, p-1\}.$$

Convergence to a True Solution: Given any $T_0 < \frac{a}{\lambda}$, choose a strictly decreasing sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $T_0 < \frac{a}{\lambda + (1+\lambda)\varepsilon_1}$ and $\varepsilon_n \to 0$ as $n \to \infty$. The sequence of hybrid arcs $X_n := x_{\varepsilon_n}$, $n = 1, 2, \cdots$, are defined on $\operatorname{dom}\varphi \cup [0, T_0] \times \{0\}$ and satisfy $\mathcal{A}_{[0,0]}X_n = \varphi$ for all n. Moreover, each $X_n(\cdot, 0)$ is λ-Lipschitz on $[0, T_0]$. By Ascoli's theorem, there exists a subsequence of $X_n(\cdot, 0)$ (still denoted by X_n) converges uniformly to a function Y on $[0, T_0]$. We can define a hybrid arc X with domain $\operatorname{dom}\varphi \cup [0, T_0] \times \{0\}$ and $\mathcal{A}_{[0,0]}X = \varphi$. Moreover, $X(\cdot, 0)$ is also λ-Lipschitz on $[0, T_0]$ and hence $\dot{X}(\cdot, 0)$ exists almost everywhere on $[0, T_0]$ and $\dot{X}(\cdot, 0) \in L^\infty([0, T_0], \mathbb{R}^n)$.

The goal is to prove that

$$\dot{X}(t, 0) \in \mathcal{F}(\mathcal{A}_{[t,0]}X), \quad \text{for almost all} \quad t \in (0, T_0). \quad (6)$$

and $\mathcal{A}_{[t,0]}X \in \mathcal{C}$ for all $[0, T_0]$.

Verifying (a)–(c): These can be verified by standard argument on continuation of solutions, adapted for hybrid systems: by flowing based on local boundedness of $\mathcal{F}$ and by jumps on conditions of $\mathcal{G}$. ■

4. STABILITY ANALYSIS USING LYAPUNOV-RAZUMIKHIN FUNCTIONS

In this section, we establish Lyapunov sufficient conditions for the asymptotic stability analysis of hybrid systems with delays.

Definition 19. Let $\mathcal{H}_\mathcal{M}$ be a hybrid system in $\mathcal{M}$ and $\mathcal{W} \subseteq \mathbb{R}^n$ be a closed set. The set $\mathcal{W}$ is said to be *uniformly globally $\mathcal{KL}$ pre-asymptotically stable* for $\mathcal{H}_\mathcal{M}$ if there exists $\mathcal{KL}$ function β such that any solution φ to $\mathcal{H}_\mathcal{M}$ satisfies

$$|\varphi(t,j)|_\mathcal{W} \leq \beta(\|\mathcal{A}_{[0,0]}\varphi\|^\Delta_\mathcal{W}, t+j), \tag{7}$$

where $\Delta > 0$ is a given constant and $|x|_\mathcal{W} := \inf_{y\in\mathcal{W}} |y-x|$ for $x \in \mathbb{R}^n$ and

$$\|\varphi\|^\Delta_\mathcal{W} = \sup_{\substack{(t,j)\in \mathrm{dom}\,\varphi \\ -\Delta\leq t+j\leq 0}} \inf_{y\in\mathcal{W}} |y-\varphi(t,j)|$$

for $\varphi \in \mathcal{M}$.

Theorem 20. Let $\mathcal{H}_\mathcal{M} = (\mathcal{C},\mathcal{F},\mathcal{D},\mathcal{G})$ be a hybrid system with memory and let $\mathcal{W} \subseteq \mathbb{R}^n$ be a closed set. If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}^+$, $\mathcal{K}_\infty$ functions α_i ($i = 1,2$), and positive constants $\mu > q$ and $\rho < 1$ such that the following hold:

(i) $\alpha_1(|\varphi(0,0)|_\mathcal{W}) \leq V(\varphi(0,0)) \leq \alpha_2(|\varphi(0,0)|_W)$ for all $\varphi \in \mathcal{C}\cup\mathcal{D}\cup\mathcal{G}(\mathcal{D})$;
(ii) $\nabla V(\varphi(0,0))\cdot f \leq -\mu V(\varphi(0,0)) + q\overline{V}_{[0,0]}(\varphi)$ for all $\varphi \in \mathcal{C}$ and $f \in \mathcal{F}(\varphi)$;
(iii) $V(g) \leq \rho\overline{V}_{[0,0]}(\varphi)$ for all $\varphi \in \mathcal{D}$ and $g \in \mathcal{G}(\varphi)$,

where $\overline{V}_{[0,0]}(\varphi) = \max_{-\Delta\leq s+k\leq 0} V(\varphi(s,k))$, then $\mathcal{W}$ is uniformly globally pre-asymptotically stable for $\mathcal{H}_\mathcal{M}$.

Proof. Let $\lambda \in \left(0, -\frac{\ln\rho}{\Delta}\right]$ be such that $-\mu + qe^{\Delta\lambda} + \lambda \leq 0$. This is always possible since $0 < \rho < 1$ and $\mu > q$. Consider $V(x(t,j))$ for $x \in \mathcal{S}_{\mathcal{H}_\mathcal{M}}$ and $(t,j) \in \mathrm{dom}\,x$. Fix any $\varepsilon > 0$. We claim that

$$V(x(t,j)) \leq \overline{V}_{[0,0]}e^{-\lambda(t+j)} + \varepsilon, \quad \forall (t,j)\in \mathrm{dom}\,x, \tag{8}$$

where $\overline{V}_{[0,0]} = \max_{-\Delta\leq s+k\leq 0} V(x(s,k))$. Define

$$(t,j) = \inf\left\{(s,k)\in \mathrm{dom}\,x : V(x(s,k)) > \overline{V}_{[0,0]}e^{-\lambda(s+k)} + \varepsilon\right\},$$

where inf is defined based on the lexicographical order on pairs $(t,j) \in \mathrm{dom}\,x$. Clearly, $(t,j) \in \mathrm{dom}\,x$. Moreover, since $V(x(s,k)) < \overline{V}_{[0,0]}e^{-\lambda(s+k)} + \varepsilon$ for all $s+k \leq 0$, we have $t+j > 0$. We consider two cases. If $(t,j-1) \in \mathrm{dom}\,x$, then we have

$$\begin{aligned} V(x(t,j)) &\leq \rho\overline{V}_{[t,j-1]} \leq \rho(e^{\Delta\lambda}\overline{V}_{[0,0]}e^{-\lambda(t+j)} + \varepsilon) \\ &< \overline{V}_{[0,0]}e^{-\lambda(t+j)} + \varepsilon, \end{aligned}$$

where we have used $\rho e^{\Delta\lambda} < 1$. This contradicts the definition of (t,j). Since $(t,j-1) \notin \mathrm{dom}\,x$, it follows from the continuity of $x(\cdot,j)$ on I_j that $V(x(t,j)) \leq \overline{V}_{[0,0]}e^{-\lambda(t+j)} + \varepsilon$. If $(t,j+1) \in \mathrm{dom}\,x$, we can similarly show that $V(x(t,j+1)) \leq \overline{V}_{[0,0]}e^{-\lambda(t+j+1)} + \varepsilon$, which again contradicts the definition of (t,j). If $(t,j+1) \notin \mathrm{dom}\,x$, we have $(t,j) \in \mathrm{int}(I_j)$ and $V(x(t,j)) = \overline{V}_{[0,0]}e^{-\lambda(t+j)} + \varepsilon$. Now consider

$$\begin{aligned} &\frac{d[V(x(t,j)) - \overline{V}_{[0,0]}e^{-\lambda(t+j)}]}{dt} \\ &= -\mu V(x(t,j)) + q\overline{V}_{[t,j]} + \lambda\overline{V}_{[0,0]}e^{-\lambda(t+j)} \\ &\leq -\mu(\overline{V}_{[0,0]}e^{-\lambda(t+j)} + \varepsilon) + (qe^{\Delta\lambda} + \lambda + q\varepsilon)\overline{V}_{[0,0]}e^{-\lambda(t+j)} \\ &= (-\mu + qe^{\Delta\lambda} + \lambda)\overline{V}_{[0,0]}e^{-\lambda(t+j)} - (\mu - q)\varepsilon \\ &< (-\mu + qe^{\Delta\lambda} + \lambda)\overline{V}_{[0,0]}e^{-\lambda(t+j)} \leq 0, \end{aligned}$$

which implies that $V(x(t,j)) - \overline{V}_{[0,0]}e^{-\lambda(t+j)}$ is strictly decreasing on $[t, t+h]$ for sufficiently small h. This contradicts the definition of (t,j). Therefore, (8) holds. Since $\varepsilon > 0$ is arbitrary chosen, we have actually proved

$$V(x(t,j)) \leq \overline{V}_{[0,0]}e^{-\lambda(t+j)}, \quad \forall (t,j)\in \mathrm{dom}\,x. \tag{9}$$

This, together with condition (i), implies that $\mathcal{W}$ is uniformly globally pre-asymptotically stable for $\mathcal{H}_\mathcal{M}$. ■

5. AN EXAMPLE

Example 1. Consider a hybrid system:

$$\left.\begin{cases} \dot{z} = f_p(z, z(t-r_f)) \\ \dot{p} = 0 \\ \dot{\tau} = 1 \end{cases}\right\} \quad z\in\mathbb{R}^n,\ p\in\mathcal{P},\ \tau\in[0,\delta], \tag{10}$$

$$\left.\begin{cases} z^+ = g_p(z, z(t-r_g)) \\ p^+ \in \mathcal{P} \\ \tau^+ = 0 \end{cases}\right\} \quad z\in\mathbb{R}^n,\ p\in\mathcal{P},\ \tau\in\{\delta\}. \tag{11}$$

Let $\psi = (\varphi, p, \tau) \in \mathcal{M}$. One way of interpreting the hybrid system above is to use the following data:

$$\mathcal{F} = \begin{bmatrix} \bigcup_{(-r_f,k)\in\mathrm{dom}\,\psi} f_p(\varphi(0,0), \varphi(-r_f,k)) \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{G} = \begin{bmatrix} \bigcup_{(-r_g,k)\in\mathrm{dom}\,\psi} g_p(\varphi(0,0), \varphi(-r_g,k)) \\ \mathcal{P} \\ 0 \end{bmatrix}$$

$$\mathcal{C} = \{\psi = (\varphi,p,\tau)\in\mathcal{M} : \tau(0,0)\subseteq[0,\delta]\}.$$

$$\mathcal{D} = \{\psi = (\varphi,p,\tau)\in\mathcal{M} : \tau(0,0) = \delta\}.$$

$$\mathcal{W} = \{0\in\mathbb{R}^n\}\times\mathcal{P}\times[0,\delta].$$

Clearly, $|\psi(0,0)|_\mathcal{W} = |\varphi(0,0)|$ and $\|\psi\|^\Delta_\mathcal{W} = \|\varphi\|^\Delta$ for all $\psi = (\varphi,p,\tau)\in\mathcal{M}$, where $\|\varphi\|^\Delta = \sup_{\substack{(t,j)\in\mathrm{dom}\,\varphi \\ -\Delta\leq t+j\leq 0}} |\varphi(t,j)|$.

Let $x = (z,p,\tau) \in \mathbb{R}^{n+2}$ and consider a Lyapunov function candidate of the form $V(x) := U_p(z)e^{-\sigma\tau}$, where σ is a constant to be determined and, for $p \in \mathcal{P}$, each $U_p : \mathbb{R}^n \to \mathbb{R}^+$ satisfies

(i) $\alpha_1(|z|) \leq U_p(z) \leq \alpha_2(|z|)$ for all $z\in\mathbb{R}^n$ and $p\in\mathcal{P}$;
(ii) there exist constants q and $\hat{q} > 0$ such that $\nabla U_p(z)\cdot f_p(z,\hat{z}) \leq qU_p(z) + \hat{q}U_p(\hat{z})$ for all $(z,\hat{z})\in\mathbb{R}^{2n}$ and $p\in\mathcal{P}$;
(iii) there exist a constant $\rho > 0$ such that $U_p(g_p(z,\hat{z})) \leq \rho U_p(z) + \hat{\rho}U_p(\hat{z})$ for all $(z,\hat{z})\in\mathbb{R}^{2n}$ and $p,q,q'\in\mathcal{P}$.

Now we verify that conditions of Theorem 20 hold. First, for $\psi = (\varphi, p, \tau) \in \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}(\mathcal{D})$,

$$\alpha_1(|\psi(0,0)|_W)e^{-\delta|\sigma|} \leq V(\psi(0,0)) = U_p(\varphi(0,0))e^{-\sigma\tau} \leq \alpha_2(|\psi(0,0)|_W)e^{\delta|\sigma|}.$$

Hence, condition (i) is verified. Second,

$$\begin{aligned}\nabla V(\psi(0,0)) \cdot f &= [\nabla U_p(\varphi(0,0))e^{-\sigma\tau}, 0, -\sigma U_p(\varphi(0,0))e^{-\sigma\tau}] \cdot \begin{bmatrix} f_c \\ 0 \\ 1 \end{bmatrix} \\ &= e^{-\sigma\tau}(\nabla U_p(\varphi(0,0)) \cdot f_c - \sigma U_p(\varphi(0,0))),\end{aligned}$$

where $f \in \mathcal{F}(\varphi)$ and hence $f_c = f_p(\varphi(0,0), \varphi(-r_f, k))$ for some $(-r_f, k) \in \operatorname{dom}\psi$. It follows from condition (ii) above on U_p that

$$\begin{aligned}&\nabla V(\psi(0,0)) \cdot f \\ &\leq e^{-\sigma\tau}(qU_p(\varphi(0,0)) + \hat{q}U_p(\varphi(-r_f,k)) - \sigma U_p(\varphi(0,0))) \\ &\leq -\sigma U_p(\varphi(0,0))e^{-\sigma\tau} + qU_p(\varphi(0,0))e^{-\sigma\tau} \\ &\quad + \hat{q}\mu U_{p(-r_f,k)}(\varphi(-r_f,k))e^{-\sigma\tau(-r_f,k)}e^{|\sigma|\delta} \\ &\leq -\sigma V(\psi(0,0)) + \hat{q}\mu e^{|\sigma|\delta}U_{p(-r_f,k))}(\varphi(-r_f,k))e^{-\sigma\tau(-r_f,k))} \\ &\leq (-\sigma + q)V(\psi(0,0)) + \bar{q}\overline{V}_{[0,0]}(\psi),\end{aligned}$$

where $\bar{q} = \hat{q}\mu e^{|\sigma|\delta}$, $\mu \geq 1$ is such that $U_p(z) \leq \mu U_q(z)$ for all $z \in \mathbb{R}^n$, and $\overline{V}_{[0,0]}(\psi) = \max_{-\Delta \leq s+k \leq 0} V(\psi(s,k))$, provided that $-\Delta \leq -r_f + k \leq 0$ whenever $(-r_f, k) \in \operatorname{dom}\psi$. Hence, condition (ii) of Theorem 20 is verified if $\sigma - q > \bar{q}$. Finally, for $g = (g_d, p, 0) \in \mathcal{G}(\psi)$, $V(g) = U_p(g_d)$, where $g_d \in \bigcup_{(-r_g,k)\in \operatorname{dom}\psi} g_p(\varphi(0,0), \varphi(-r_g,k))$. It follows from condition (iii) above on U_p that

$$\begin{aligned}V(g) = U_p(g_d) &\leq \rho U_p(\varphi(0,0)) + \hat{\rho}U_p(\varphi(-r_g,k)) \\ &\leq \rho e^{\sigma\delta}U_p(\varphi(0,0))e^{-\sigma\delta} \\ &\qquad + \hat{\rho}\mu e^{\max(\sigma,0)\delta}U_{p(-r_g,k)}(\varphi(-r_g,k)))e^{-\sigma\tau(-r_g,k)} \\ &\leq \bar{\rho}\overline{V}_{[0,0]}(\psi),\end{aligned}$$

for some $(-r_g, k) \in \operatorname{dom}\psi$, where $\bar{\rho} = \rho e^{\sigma\delta} + \hat{\rho}\mu e^{\max(\sigma,0)\delta}$. Hence, condition (iii) of Theorem 20 is verified if $\bar{\rho} < 1$. Thus, the set $\mathcal{W}$ is uniformly globally pre-asymptotically stable for the hybrid system defined above, if

$$\sigma - q > \bar{q} = \hat{q}\mu e^{|\sigma|\delta} \text{ and } \bar{\rho} = \rho e^{\sigma\delta} + \hat{\rho}\mu e^{\max(\sigma,0)\delta} < 1 \quad (12)$$

hold simultaneously.

Remark 21. In particular, the analysis above applies to following two cases:

(i) $q + \hat{q} < 0$, $\rho + \hat{\rho}\mu > 1$, and $\hat{\rho}\mu < 1$. This may correspond to the case where the dynamics during flow are stable, whereas the jump dynamics are not. The conditions in (12) can always be satisfied by choosing $\sigma \in (q + \hat{q}, 0)$ and δ sufficiently large;

(ii) $q + \hat{q} > 0$ and $\rho + \hat{\rho}\mu < 1$. This may correspond to the case where the dynamics during flow are unstable, whereas the jump dynamics are stable. The conditions in (12) can always be satisfied by choosing $\sigma > q + \hat{q}\mu > 0$ and δ sufficiently small.

6. CONCLUSIONS

We have proved in this paper a general existence result for the generalized solutions of hybrid systems with memory. Moreover, we have provided a set of sufficient conditions for the stability analysis of such systems via generalized solutions. The stability results are illustrated by a general nonlinear system with switching dynamics and state jumps. We believe that the proposed framework can lead to the development of a robust stability theory for hybrid systems with delays in this direction.

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