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Reachability and Observability Graphs for Linear Positive Systems on Time Scales ⋆

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Abstract: Positive reachability and positive observability of positive linear systems on time scales are studied. Differential and integral calculus on time scales allows for unified treatment of continuous- and discrete-time systems. Reachability and observability graphs for positive systems are introduced. They are used to characterize positive reachability and positive observability for systems on homogeneous time scales. It is also shown that positive observability is dual to positive reachability. Positive minimality is introduced and characterized with the aid of Gram matrices, Hankel matrices and graphs.

Keywords: Linear systems. Reachability. Observability. Positive systems. Graphs. Multi-input/multi-output systems.

1. INTRODUCTION

For positive linear systems it is customary to consider various graphs related to the systems. The graphs are usually built on the concept of influence graph (see e.g. Farina and Rinaldi (2000); Kaczorek (2002)). For example, in Commault (2004) the graphs are used to characterize positive reachability of discrete-time systems and in Commault and Alamir (2007) to derive criteria of positive reachability of continuous-time systems. It appears that the criteria for positive reachability in these two cases are significantly different. The goal of this paper is to unify these criteria using time-scale approach. Calculus on time scales allows for unified treatment of differential and difference equations. Continuous-time and discrete-time systems can be put into one framework of systems on time scales.

In Bartosiewicz (2012) and Bartosiewicz (2013) we proved necessary and sufficient conditions for positive observability and positive reachability of positive systems on arbitrary time-scales. The conditions were expressed with the aid of modified reachability and observability Gram matrices. A necessary and sufficient condition for positive reachability or observability is that the appropriate Gram matrix is monomial. We recall these results here and infer from them that positive reachability and positive observability are dual to each other. Also positive minimality is characterized with the aid of the modified Gram matrices.

The main contribution of this paper is graph characterization of positive reachability, observability, and minimality for systems on homogeneous time scales. This class of systems includes standard continuous- and discrete-time systems. The reachability and observability graphs considered here differ from influence graphs used in Commault (2004); Commault and Alamir (2007) — they have fewer edges. We admit edges that correspond to monomial columns or rows of appropriate matrices. In order to accommodate both continuous-time and discrete-time cases we use the time-scale exponential function of the matrix A which defines the dynamics of the system instead of A itself.

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2. PRELIMINARIES

We introduce here the main concepts, recall definitions and facts, and set notation. For more information on positive continuous-time and discrete-time systems, the reader is referred to e.g. Farina and Rinaldi (2000), and for information on time scales calculus, to e.g. Bohner and Peterson (2001).

2.1 Positive matrices

By $\mathbb{R}$ we shall denote the set of all real numbers, by $\mathbb{Z}$ the set of integers, and by $\mathbb{N}$ the set of natural numbers (without 0). We shall also need the set of nonnegative real numbers, denoted by $\mathbb{R}_+$ and the set of nonnegative integers $\mathbb{Z}_+$, i.e. $\mathbb{N}\cup\{0\}$. Similarly, $\mathbb{R}^k_+$ will mean the set of all column vectors in $\mathbb{R}^k$ with nonnegative components and $\mathbb{R}^{k\times p}_+$ will consist of $k \times p$ real matrices with nonnegative elements. If $A \in \mathbb{R}^{k\times p}_+$ we write $A \geq 0$ and say that A is *nonnegative*. A nonnegative matrix A will be called *positive* if at least one of its elements is greater than 0. Then we shall write $A > 0$.

A positive column or row vector is called *monomial* if one of its components is positive and all the other are zero. A monomial column in $\mathbb{R}^n_+$ has the form αe_k for some $\alpha > 0$ and $1 \leq k \leq n$, where e_k denotes the column with 1 at the kth position and other elements equal 0. Then we say that the column is *k-monomial.* An $n\times n$ matrix A is called *monomial* if all columns and rows of A are monomial. Then A is invertible and its inverse is also positive. Moreover, if a positive matrix A has a positive inverse, then A is monomial.

It will be convenient to extend the set of all real numbers adding one element. It will be denoted by ∞ and will mean the positive infinity. We set $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and $\bar{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$. If $a \in \mathbb{R}$ then we define $a + \infty = \infty$. Moreover, for $a \in \mathbb{R}$ and $a > 0$ we set $a/0 = \infty$ and $a/\infty = 0$. Of course $\infty > 0$. If a matrix A has elements from $\bar{\mathbb{R}}$, then the notions of nonnegativity and positivity have the same meanings as before and are denoted in the same way. Addition of such matrices is defined in the standard way.

2.2 Calculus on time scales

Calculus on time scales is a generalization of the standard differential calculus and the calculus of finite differences.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. In particular $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$ and $\mathbb{T} = q^{\mathbb{N}} := \{q^k, k \in \mathbb{N}\}$ for $q > 1$ are time scales. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. If $t_0, t_1 \in \mathbb{T}$, then $[t_0, t_1]_{\mathbb{T}}$ denotes the intersection of the ordinary closed interval with $\mathbb{T}$. Similar notation is used for open, half-open or infinite intervals.

For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ if $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ when $\sup \mathbb{T}$ is finite; the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ if $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ when $\inf \mathbb{T}$ is finite; the *forward graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$; the *backward graininess function* $\nu : \mathbb{T} \to [0, \infty)$ by $\nu(t) := t - \rho(t)$.

If $\sigma(t) > t$, then t is called *right-scattered*, while if $\rho(t) < t$, it is called *left-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then t is called *right-dense*. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*.

The time scale $\mathbb{T}$ is *homogeneous*, if μ and ν are constant functions. When $\mu \equiv 0$ and $\nu \equiv 0$, then $\mathbb{T} = \mathbb{R}$ or $\mathbb{T}$ is a closed interval (in particular a half-line). When μ is constant and greater than 0, then $\mathbb{T} = \mu\mathbb{Z} + c$ for some $c \in \mathbb{R}$.

If $M := \sup \mathbb{T}$ is finite and $\rho(M) < M$, then we set $\mathbb{T}^\kappa := \mathbb{T} \setminus \{M\}$. Otherwise $\mathbb{T}^\kappa := \mathbb{T}$. Thus $\mathbb{T}^\kappa$ is got from $\mathbb{T}$ by removing its maximal point if this point exists and is left-scattered.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. The *delta derivative of f at t*, denoted by $f^\Delta(t)$, is the real number with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. If $f^\Delta(t)$ exists, then we say that f *is delta differentiable at t*. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Example 1. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$. If $\mathbb{T} = h\mathbb{Z}$, then $f^\Delta(t) = \frac{f(t+h) - f(t)}{h}$. If $\mathbb{T} = q^{\mathbb{N}}$, then $f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Let $a, b \in \mathbb{T}$. Then the *delta integral* of f on the interval $[a, b)_{\mathbb{T}}$ is defined by

$$\int_a^b f(\tau)\Delta\tau := \int_{[a,b)_{\mathbb{T}}} f(\tau)\Delta\tau := F(b) - F(a).$$

If f is continuous, then it has an antiderivative.

It is more convenient to consider the half-open interval $[a, b)_{\mathbb{T}}$ than the closed interval $[a, b]_{\mathbb{T}}$ in the definition of the integral. If b is a left-dense point, then the value of f at b would not affect the integral. On the other hand, if b is left-scattered, the value of f at b is not essential for the integral (see Example 2). This is caused by the fact that we use delta integral, corresponding to the forward jump function.

Example 2. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(\tau)\Delta\tau = \int_a^b f(\tau)d\tau$, where the integral on the right is the usual Riemann integral. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int_a^b f(\tau)\Delta\tau = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

2.3 Linear systems on time scale

Let us consider the system of delta differential equations on a time scale $\mathbb{T}$:

$$x^\Delta(t) = Ax(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and A is a constant $n \times n$ matrix.

Remark 3. If $\mathbb{T} = \mathbb{R}$, then (1) is a system of ordinary differential equations $x' = Ax$. But for $\mathbb{T} = \mathbb{Z}$, (1) takes the difference form $x(t + 1) - x(t) = Ax(t)$, which can be transformed to the shift form $x(t+1) = (I + A)x(t)$. Thus to compare the definitions and the results stated for delta differential systems in the case $\mathbb{T} = \mathbb{Z}$ with those that were obtained for discrete-time systems in the shift form, one has to take this into account. One can easily transform the difference form to the shift form and vice versa.

Proposition 4. Equation (1) with initial condition $x(t_0) = x_0$ has a unique forward solution defined for all $t \in [t_0, +\infty)_{\mathbb{T}}$.

The *matrix exponential function* (at t_0) for A is defined as the unique forward solution of the matrix differential equation $X^\Delta = AX$, with the initial condition $X(t_0) = I$. Its value at t is denoted by $e_A(t, t_0)$.

Example 5. If $\mathbb{T} = \mathbb{R}$, then $e_A(t, t_0) = e^{A(t-t_0)}$. If $\mathbb{T} = h\mathbb{Z}$, then $e_A(t, t_0) = (I + A)^{(t-t_0)/h}$. If $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$, then $e_A(q^k t_0, t_0) = \prod_{i=0}^{k-1}(I + (q-1)q^i t_0 A)$ for $k \geq 1$ and $t_0 \in \mathbb{T}$.

Proposition 6. The following properties hold for every $t, s, r \in \mathbb{T}$ such that $r \leq s \leq t$:
i) $e_A(t, t) = I$;
ii) $e_A(t, s)e_A(s, r) = e_A(t, r)$;

Let us consider now a nonhomogeneous system

$$x^\Delta(t) = Ax(t) + f(t) \tag{2}$$

where f is rd-continuous.

Theorem 7. Let $t_0 \in \mathbb{T}$. System (2) for the initial condition $x(t_0) = x_0$ has a unique forward solution of the form

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^t e_A(t, \sigma(\tau))f(\tau)\Delta\tau. \tag{3}$$

3. POSITIVE CONTROL SYSTEMS

Let $n \in \mathbb{N}$ be fixed. From now on we shall assume that the time scale $\mathbb{T}$ consists of at least $n+1$ elements.

Let us consider a linear control system with output, denoted by Σ, and defined on the time scale $\mathbb{T}$:

$$x^{\Delta}(t) = Ax(t) + Bu(t) \quad (4a)$$
$$y(t) = Cx(t) \quad (4b)$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$.

We assume that the control u is a piecewise continuous function defined on some interval $[t_0, t_1)_{\mathbb{T}}$, depending on u, where $t_0, t_1 \in \mathbb{T}$. We shall assume that at each point $t \in [t_0, t_1)_{\mathbb{T}}$, at which u is not continuous, u is right-continuous and has a finite left-sided limit if t is left-dense. This allows to solve (4) step by step. Moreover, we can always evaluate $x(t_1)$. For t_1 being left-scattered we do not need the value of u at t_1, and for a left-dense t_1 we just take a limit of $x(t)$ at t_1.

Definition 8. We say that system Σ is *positive* if for any $t_0 \in \mathbb{T}$, any initial condition $x_0 \in \mathbb{R}^n_+$, any control $u : [t_0, t_1)_{\mathbb{T}} \to \mathbb{R}^m_+$ and any $t \in [t_0, t_1]_{\mathbb{T}}$, the solution x of (4a) satisfies $x(t) \in \mathbb{R}^n_+$ and the output $y(t) \in \mathbb{R}^p_+$.

By the separation principle we have the following characterization.

Proposition 9. The system Σ is positive if and only if $e_A(t, t_0) \in \mathbb{R}^{n\times n}_+$ for every $t, t_0 \in \mathbb{T}$ such that $t \geq t_0$, $B \in \mathbb{R}^{n\times m}_+$, and $C \in \mathbb{R}^{p\times n}_+$.

To state criteria of nonnegativity of the exponential matrix, let $\bar{\mu} = \sup\{\mu(t) : t \in \mathbb{T}\}$ and $A_{\mathbb{T}} := A + I/\bar{\mu}$, where I/∞ means the zero $n \times n$ matrix and $I/0$ is a diagonal matrix with ∞ on the diagonal. Thus for $\mathbb{T} = \mathbb{R}$, $A_{\mathbb{T}}$ is obtained from A by replacing the elements on the diagonal by ∞, for $\mathbb{T} = \mathbb{Z}$, $A_{\mathbb{T}} = A + I$, and for $\mathbb{T} = q^{\mathbb{N}}$, $A_{\mathbb{T}} = A$.

The following theorem unifies different criteria of nonnegativity of the exponential matrix for discrete- and continuous-time systems into one statement, in which, besides the matrix A, the graininess of the time scale is involved.

Theorem 10. (Bartosiewicz (2013)). The exponential matrix $e_A(t, t_0)$ is nonnegative for every $t, t_0 \in \mathbb{T}$ such that $t \geq t_0$ if and only if $A_{\mathbb{T}} \in \bar{\mathbb{R}}^{n\times n}_+$.

Corollary 11. The system Σ is positive if and only if $A_{\mathbb{T}} \in \bar{\mathbb{R}}^{n\times n}_+$, $B \in \mathbb{R}^{n\times m}_+$, and $C \in \mathbb{R}^{p\times n}_+$.

Remark 12. An $n \times n$ matrix with nonnegative elements outside the diagonal is called a *Metzler matrix*. Thus in the continuous-time case, the exponential matrix $e_A(t, t_0)$ is nonnegative for every $t > t_0$ if and only if A is a Metzler matrix. In that case the elements on the diagonal may be arbitrary. On the other hand, if the time scale $\mathbb{T}$ is the set $\mathbb{Z}$ of integer numbers, then $\mu \equiv 1$ and nonnegativity of the exponential matrix is equivalent to $A + I \geq 0$. In that case the delta differential equation $x^{\Delta}(k) = Ax(k)$ may be rewritten in the shift form as $x(k+1) = (A+I)x(k)$. Thus the condition $A + I \geq 0$ agrees with the necessary and sufficient condition of nonnegativity for discrete-time systems of the form $x(k+1) = Fx(k)$, where $k \in \mathbb{Z}$ (see Farina and Rinaldi (2000); Kaczorek (2002)).

4. POSITIVE REACHABILITY AND OBSERVABILITY

We recall here definitions and characterizations of positive reachability and positive observability obtained in Bartosiewicz (2012) and Bartosiewicz (2013). Then duality and minimality is investigated.

4.1 Positive reachability

If Σ is a positive system, then for a nonnegative initial condition x_0 and a nonnegative control u, the trajectory x stays in $\mathbb{R}^n_+$. For simplicity we assume that the initial condition is $x_0 = 0$. Let $x(t_1, t_0, 0, u)$ mean the trajectory of the system corresponding to the initial condition $x(t_0) = 0$ and the control u, and evaluated at time t_1.

Definition 13. (Bartosiewicz (2013)). Let $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. The *positive reachable set* (from 0) of the positive system Σ on the interval $[t_0, t_1]_{\mathbb{T}}$ is the set $\mathcal{R}^{[t_0,t_1]}_+$ consisting of all $x(t_1, t_0, 0, u)$, where u is a nonnegative control on $[t_0, t_1)_{\mathbb{T}}$.
The *positive reachable set* (from 0) *for the initial time* t_0 of Σ is

$$\mathcal{R}^{t_0}_+ = \bigcup_{t_1 \in \mathbb{T}, t_1 > t_0} \mathcal{R}^{[t_0,t_1]}_+$$

and the *positive reachable set* (from 0) of Σ is

$$\mathcal{R}_+ = \bigcup_{t_0 \in \mathbb{T}} \mathcal{R}^{t_0}_+.$$

The positive system Σ is *positively reachable on* $[t_0, t_1]_{\mathbb{T}}$ if $\mathcal{R}^{[t_0,t_1]}_+ = \mathbb{R}^n_+$, Σ is *positively reachable for the initial time* t_0 if $\mathcal{R}^{t_0}_+ = \mathbb{R}^n_+$ and Σ is *positively reachable* if $\mathcal{R}_+ = \mathbb{R}^n_+$.

The following propositions follow directly from the definitions:

Proposition 14. Let Σ be a positive system.
Σ is positively reachable on $[t_0, t_1]_{\mathbb{T}} \Rightarrow \Sigma$ is positively reachable for the initial time $t_0 \Rightarrow \Sigma$ is positively reachable.

Proposition 15. If $\tau_0 < t_0 < t_1$, then $\mathcal{R}^{[t_0,t_1]}_+ \subseteq \mathcal{R}^{[\tau_0,t_1]}_+$ and $\mathcal{R}^{t_0}_+ \subseteq \mathcal{R}^{\tau_0}_+$.

To study positive reachability let us introduce a modified Gram matrix related to the control system.

Definition 16. (Bartosiewicz (2013)). Let $M \subseteq \{1, \ldots, m\}$ and $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. For each $k \in M$ let S_k be a subset of $[t_0, t_1)_{\mathbb{T}}$ that is a union of finitely many disjoint intervals of $\mathbb{T}$ of the form $[\tau_0, \tau_1)_{\mathbb{T}}$, and let $\mathcal{S}_M = \{S_k : k \in M\}$. By *the Gram matrix of system* (4) *corresponding to* t_0, t_1, M *and* $\mathcal{S}_M$ we mean the matrix

$$W := W^{t_1}_{t_0}(M, \mathcal{S}_M)$$
$$:= \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau)) b_k b_k^T e_A(t_1, \sigma(\tau))^T \Delta\tau, \quad (5)$$

where b_k is the kth column of B.

The following characterization holds for arbitrary time scale.

Theorem 17. (Bartosiewicz (2013)). Let $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. Positive system (4) is positively reachable on $[t_0, t_1]_\mathbb{T}$ if and only if there are $M \subseteq \{1, \ldots, m\}$ and the family $\mathcal{S}_M = \{S_k : k \in M\}$ of subsets of $[t_0, t_1]_\mathbb{T}$ such that the matrix $W = W_{t_0}^{t_1}(M, \mathcal{S}_M)$ is monomial.

From the general characterization of positive reachability presented in Theorem 17 we can deduce more concrete results for particular time scales. For $\mathbb{T} = \mathbb{R}$ we get very restrictive conditions for positive reachability. The following result was first obtained in Commault and Alamir (2007).

Corollary 18. (Bartosiewicz (2013)). Let $\mathbb{T} = \mathbb{R}$ and $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$. Positive system (4) is positively reachable on $[t_0, t_1]$ if and only if A is diagonal and B contains an $n \times n$ monomial submatrix (so $m \geq n$).

For discrete homogeneous time scales the conditions for positive reachability are much less restrictive.

Corollary 19. (Bartosiewicz (2013)). Let $\mathbb{T} = \mu\mathbb{Z}$ for a constant $\mu > 0$. Let $t_0 \in \mathbb{T}$ and $t_1 = t_0 + k\mu$ for some $k \in \mathbb{N}$. System (4) is positively reachable on $[t_0, t_1]_\mathbb{T}$ if and only if the matrix $[B, (I + \mu A)B, \ldots, (I + \mu A)^{k-1}B]$ contains a monomial submatrix. If $k > n$ then in (7) k may be replaced by n.

4.2 Positive observability

Let $x(t, t_0, x_0, u)$ denote the solution of the dynamic part (4a) of system Σ corresponding to the initial time t_0, the initial state x_0, the control u, and evaluated at time $t \geq t_0$. Let $y(t, t_0, x_0, u) = Cx(t, t_0, x_0, u)$.

We want to recover the initial state in a linear and positive fashion. As the output is a sum of two terms, one depending only on the initial state and the other depending only on the control, we neglect the control part as no information can be got from it to recover the initial state. To simplify notation let us set: $y(t, t_0, x_0) := y(t, t_0, x_0, 0)$.

Definition 20. (Bartosiewicz (2012)). We say that system Σ is *positively observable on* $[t_0, t_1]_\mathbb{T}$ if there is a piecewise continuous positive map $\Phi : [t_0, t_1)_\mathbb{T} \to \mathbb{R}^{n \times p}$ such that for every $x_0 \in \mathbb{R}^n_+$:

$$\int_{t_0}^{t_1} \Phi(t) y(t, t_0, x_0) \Delta t = x_0.$$

System Σ is *positively observable for the initial time* t_0, if there is $t_1 > t_0$, $t_1 \in \mathbb{T}$, such that Σ is *positively observable on* $[t_0, t_1]_\mathbb{T}$.
System Σ is *positively observable* if there is $t_0 \in \mathbb{T}$ such that Σ is *positively observable for the initial time* t_0.

The following propositions follow directly from definitions:

Proposition 21. Let Σ be a positive system.
Σ is positively observable on $[t_0, t_1]_\mathbb{T} \Rightarrow \Sigma$ is positively observable for the initial time $t_0 \Rightarrow \Sigma$ is positively observable.

Proposition 22. Let $t_0 < t_1 < \tau_1$.
If Σ is positively observable on $[t_0, t_1]_\mathbb{T}$, then Σ is positively observable on $[t_0, \tau_1]_\mathbb{T}$.

To study positive observability let us introduce a modified observability Gram matrix related to the system.

Definition 23. (Bartosiewicz (2012)). Let $P \subseteq \{1, \ldots, p\}$ and $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. For each $k \in P$ let T_k be a subset of $[t_0, t_1)_\mathbb{T}$ that is a union of finitely many disjoint intervals of $\mathbb{T}$ of the form $[\tau_0, \tau_1)_\mathbb{T}$, and let $\mathcal{T}_P = \{T_k : k \in P\}$. By *the observability Gram matrix of system* (4) *corresponding to* t_0, t_1, P *and* $\mathcal{T}_P$ we mean the matrix

$$V := V_{t_0}^{t_1}(P, \mathcal{T}_P) := \sum_{k \in P} \int_{T_k} e_A(\tau, t_0)^T (c^k)^T c^k e_A(\tau, t_0) \Delta\tau, \tag{6}$$

where c^k is the kth row of C.

Then we have the following characterization:

Theorem 24. (Bartosiewicz (2012)). Let $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. System (4) is positively observable on $[t_0, t_1]_\mathbb{T}$ if and only if there are $P \subseteq \{1, \ldots, p\}$ and a family $\mathcal{T}_P = \{T_k : k \in P\}$ of subsets of $[t_0, t_1)_\mathbb{T}$ such that the matrix $V = V_{t_0}^{t_1}(P, \mathcal{T}_P)$ is monomial.

For $\mathbb{T} = \mathbb{R}$ we obtain very restrictive conditions for positive observability.

Corollary 25. (Bartosiewicz (2012)). Let $\mathbb{T} = \mathbb{R}$ and $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$. System (4) is positively observable on $[t_0, t_1]_\mathbb{T}$ if and only if A is diagonal and C contains an $n \times n$ monomial submatrix (so $p \geq n$).

For discrete homogeneous time scales the conditions for positive observability are much less restrictive.

Corollary 26. (Bartosiewicz (2012)). Let $\mathbb{T} = \mu\mathbb{Z}$ for a constant $\mu > 0$. Let $t_0 \in \mathbb{T}$ and $t_1 = t_0 + k\mu$ for some $k \in \mathbb{N}$. System (4) is positively observable on $[t_0, t_1]_\mathbb{T}$ iff the matrix

$$\mathcal{C} = \begin{pmatrix} C \\ C(I + \mu A) \\ \cdots \\ C(I + \mu A)^{k-1} \end{pmatrix} \tag{7}$$

contains a monomial $n \times n$ submatrix. If $k > n$ then in (7) k may be replaced by n.

4.3 Duality and minimality

Definition 27. Let Σ be the system

$$x^\Delta(t) = Ax(t) + Bu(t) \tag{8a}$$
$$y(t) = Cx(t) \tag{8b}$$

The *system dual to* Σ, denoted by Σ^T is given by

$$\tilde{x}^\Delta(t) = A^T \tilde{x}(t) + C^T \tilde{u}(t) \tag{9a}$$
$$\tilde{y}(t) = B^T \tilde{x}(t) \tag{9b}$$

Observe that $(\Sigma^T)^T = \Sigma$. It is easy to show the following:

Proposition 28. The system Σ is positive if and only if the system Σ^T is positive.

Now we can state the duality result concerning positive reachability and positive observability.

Theorem 29. Let Σ be a positive system on a homogeneous time scale $\mathbb{T}$.
The system Σ is positively reachable if and only if the system Σ^T is positively observable.
The system Σ is positively observable if and only if the system Σ^T is positively reachable.

Proof. Assume first that $\mu \equiv 0$. Then positive reachability of Σ is equivalent to the fact that A is diagonal and B contains a monomial $n \times n$ submatrix. This holds if and only if A^T is diagonal and B^T contains a monomial $n \times n$ submatrix, which is equivalent to positive observability of the dual system Σ^T.

Assume now that μ is constant and greater than 0. Then positive reachability of Σ is equivalent to the fact that the matrix $(B, (I + \mu A)B, \ldots, (I + \mu A)^{n-1}B)$ contains a monomial $n \times n$ submatrix. This holds if and only if its transpose

$$\begin{pmatrix} B^T \\ B^T(I + \mu A^T) \\ \vdots \\ B^T(I + \mu A^T)^{n-1} \end{pmatrix}$$

contains a monomial $n \times n$ submatrix, which is equivalent to positive observability of the dual system Σ^T.

To prove the second statement, it is enough to replace Σ with Σ^T in the first statement.

We do not know if Theorem 29 can be extended to nonhomogeneous time scales.

Definition 30. We say that a positive system Σ is *positively minimal* if it is positively reachable and positively observable. Positive minimality on $[t_0, t_1]_{\mathbb{T}}$ and positive minimality for the initial time t_0 are defined in an analogous way.

From Theorems 17 and 24 we can obtain the following:

Proposition 31. A positive system Σ is positively minimal if and only if there are sets $M \subseteq \{1, \ldots, m\}$ and $P \subseteq \{1, \ldots, p\}$ and families $\mathcal{S}_M = \{S_k : k \in M\}$ and $\mathcal{T}_P = \{T_k : k \in P\}$ such that the corresponding Gram matrices $W_{t_0}^{t_1}(M, \mathcal{S}_M)$ and $V_{t_0}^{t_1}(P, \mathcal{T}_P)$ are monomial.

Corollary 32. Let $\mu \equiv 0$. A positive system Σ is positively minimal if and only if B and C contain $n \times n$ monomial submatrices and A is diagonal.

Consider the finite Hankel matrix of the system Σ

$$H(\Sigma) = \begin{pmatrix} CB & C(I + \mu A)B & \ldots & C(I + \mu A)^{n-1}B \\ C(I + \mu A) & C(I + \mu A)^2 B & \ldots & C(I + \mu A)^n B \\ \vdots & \vdots & \ddots & \vdots \\ C(I + \mu A)^{n-1} & C(I + \mu A)^n B & \ldots & C(I + \mu A)^{2n-2}B \end{pmatrix}.$$

Corollary 33. Let $\mu > 0$ be constant. A positive system Σ is positively minimal if and only if $H(\Sigma)$ contains a monomial $n \times n$ submatrix.

Proof. From Corollaries 19 and 26 we obtain that Σ is positively minimal if and only if each of the matrices

$$\left(B, (I + \mu A)B, \ldots, (I + \mu A)^{n-1}B\right), \quad \begin{pmatrix} C \\ C(I + \mu A) \\ \vdots \\ C(I + \mu A)^{n-1} \end{pmatrix}$$

contains a monomial $n \times n$ submatrix. This is equivalent to that $H(\Sigma)$ contains a monomial $n \times n$ submatrix.

Corollary 34. Let $\mu > 0$ be constant and $m = p = 1$. A positive system Σ is positively minimal if and only if $H(\Sigma)$ is monomial.

5. REACHABILITY AND OBSERVABILITY GRAPHS

We shall describe now positive reachability and positive observability in the language of graphs. Let Σ be a positive control system with output on the time scale $\mathbb{T}$:

$$x^{\Delta}(t) = Ax(t) + Bu(t), \; y(t) = Cx(t), \qquad (10)$$

where $u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$. Let $b_1, \ldots, b_m$ denote columns of B, $e_A(t,\tau)_1, \ldots, e_A(t,\tau)_n$ columns of $e_A(t,\tau)$, and $c^1, \ldots, c^p$ rows of C. Let us fix $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. We consider two directed graphs $G_r(\Sigma)$ and $G_o(\Sigma)$ related to the system Σ and called *reachability and observability graphs*, respectively. In general they will depend also on t_0 and t_1. The first one corresponds to positive reachability and the second to positive observability. The set V_r of vertices of $G_r(\Sigma)$ is a disjoint union of the sets: $U = \{u_1, \ldots, u_m\}$ and $X = \{x_1, \ldots, x_n\}$, while V_o - the set of vertices of $G_o(\Sigma)$ - is a disjoint union of X and $Y = \{y_1, \ldots, y_m\}$. The set E_r of edges of $G_r(\Sigma)$ includes edges of two kinds: (u_k, x_j) and (x_j, x_i), $j \neq i$. The edge (u_k, x_j) belongs to E_r if and only if b_k is j-monomial. The edge (x_j, x_i) belongs to E_r if and only if there exists $\tau \in [t_0, t_1)_{\mathbb{T}}$ and $\epsilon > 0$ such that $\tau + \epsilon \leq t_1$, $(\tau, \tau + \epsilon)_{\mathbb{T}}$ is nonempty and for all $t \in (\tau, \tau + \epsilon)_{\mathbb{T}}$ the column $e_A(t,\tau)_j$ is i-monomial. Similarly, the set E_o of edges of $G_o(\Sigma)$ includes edges of two kinds: (x_j, x_i), $j \neq i$, and (x_i, y_k). The edge (x_j, x_i) belongs to E_o if and only if there exists $\tau \in [t_0, t_1)_{\mathbb{T}}$ and $\epsilon > 0$ such that $\tau + \epsilon \leq t_1$, $(\tau, \tau + \epsilon)_{\mathbb{T}}$ is nonempty and for all $t \in (\tau, \tau + \epsilon)_{\mathbb{T}}$ the column $(e_A(t,\tau)^T)_i$ is j-monomial. The edge (x_j, y_k) belongs to E_r if and only if c^k is j-monomial. As usual, a *path* in a graph is a sequence of vertices $(v_0, \ldots, v_k)$ such that for all $i = 1, \ldots, k$, (v_{i-1}, v_i) is an edge of the graph. Then v_0 is the *initial* vertex of the path and v_k is the *final* vertex. If $v_0 \in U$ then we say that the *path originates in* U and if $v_k \in Y$ then we say that the *path ends in* Y.

Remark 35. Reachability and observability graphs introduced here differ from the influence graphs studied in Farina and Rinaldi (2000); Kaczorek (2007); Commault (2004); Commault and Alamir (2007). First of all we employ monomial vectors in the definition, so the number of edges is significantly reduced. The reason for using only monomial vectors is that they are essential for characterizations of positive reachability and observability. Moreover instead of the matrix A we use the exponential function of A. This allows to express our conditions in the same form for continuous and discrete time.

Proposition 36. Let us assume that $\mu \equiv 0$. Then no edge of the form (x_j, x_i) belongs to E_r.

Proof. The condition $(x_j, x_i) \in E_r$ means that $\exp(A(t - \tau)_j$ is i-monomial for $t > \tau$. From continuity we get also that $\exp(A(t - t)_j$ is i-monomial. But this is possible only if $i = j$ and the edges (x_i, x_i) are not in E_r.

Proposition 37. Let us assume that μ is constant and greater than 0. If $(x_j, x_i) \in E_r$ then $(I + \mu A)_j$ is i-monomial.

Proof. The condition $(x_j, x_i) \in E_r$ means that $(I + \mu A)_j^{(t-\tau)/\mu}$ is i-monomial for $t > \tau$ and close to τ. Thus this holds in particular for $t = \sigma(\tau) = \tau + \mu$, which means that $(I + \mu A)_j$ is i-monomial.

Observe that for homogeneous time scales the property $(x_j, x_i) \in E_r$ does not depend on the interval $[t_0, t_1)_{\mathbb{T}}$.

Now we are able to give a characterization of positive reachability and positive observability using the language of graphs. We assume here that the time scale is homogeneous.

Theorem 38. Let $\mathbb{T}$ be a homogeneous time scale and $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. The system Σ is positively reachable on $[t_0, t_1]_{\mathbb{T}}$ if and only if for every $i = 1, \ldots, n$ there is a path in $G_r(\Sigma)$ that originates in U and whose final vertex is x_i.

Proof. By Theorem 17, Σ is positively reachable on $[t_0, t_1]_{\mathbb{T}}$ if and only if there are $M \subseteq \{1, \ldots, m\}$ and the family $\mathcal{S}_M = \{S_k : k \in M\}$ of subsets of $[t_0, t_1]_{\mathbb{T}}$ such that the matrix $W_{t_0}^{t_1}(M, \mathcal{S}_M) = \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau)) b_k b_k^T e_A(t_1, \sigma(\tau))^T \Delta\tau$ is monomial. This is equivalent to the following condition: for every $i = 1, \ldots, n$ there is $1 \leq k_i \leq m$ and an interval $[\alpha_i, \beta_i]_{\mathbb{T}}$ contained in $[t_0, t_1)_{\mathbb{T}}$ such that $e_A(t_1, \sigma(\tau)) b_{k_i} = \gamma(\tau) e_i$ for $\tau \in [\alpha_i, \beta_i]_{\mathbb{T}}$ and $\gamma(\tau) > 0$ (see Bartosiewicz (2012)). Let us now consider two cases. For $\mu \equiv 0$, $e_A(t_1, \sigma(\tau)) b_{k_i} = \exp(A(t_1 - \tau)) b_{k_i}$ is an analytic function of τ. Thus it is equal to $\gamma(\tau) e_i$ for $\tau \in \mathbb{R}$, in particular for $\tau = t_1$. This implies that b_{k_i} is i-monomial so the edge (u_{k_i}, x_i) belongs to E_r. This means that for every $i = 1, \ldots, n$ there is path in $G_r(\Sigma)$ that originates in U and whose final vertex is x_i. Now assume that for every i such a path exists. Since, by Proposition 36, there are no edges between x_i and x_j, then the path from u_{k_i} to x_i consists of the single edge (u_{k_i}, x_i), i.e. b_{k_i} is i-monomial. Moreover, Proposition 36 implies that $\exp(At)$ is diagonal for every $t \in \mathbb{R}$. Eventually $e_A(t_1, \sigma(\tau)) b_{k_i}$ is i-monomial for every $\tau \in \mathbb{R}$.

Let $\mathbb{T} = \mu\mathbb{Z}$ for a constant $\mu > 0$. Then $e_A(t_1, \sigma(\tau)) b_{k_i} = (I + \mu A)^{(t_1 - \tau - \mu)/\mu} b_{k_i}$. It has been shown in Kaczorek (2007) that to collect n linearly independent monomial columns in $\mathbb{R}^n$ of the form $(I + \mu A)^{s_i} b_{k_i}$ it is necessary and sufficient that for each $i = 1, \ldots, n$, there exist monomial b_{k_i} and $s_i \geq 0$ such that all the vectors $(I + \mu A)^s b_{k_i}$ for $s \leq s_i$ are monomial. This means that for each i there is a path in $G_r(\Sigma)$ starting at u_{k_i} and ending at x_j, possibly passing through other vertices of X. Now assume that for each i such a path exists. If $(u_{k_i}, x_{i_1}, \ldots, x_{i_s})$ is such a path, where $i_s = i$ then from Proposition 37 it follows that $(I + \mu A)_{i_j}$ is i_{j+1}-monomial for $j = 1, \ldots, s-1$. Thus $(I + \mu A)^s b_{k_i}$ is eventually i monomial. But it is equal to $(I + \mu A)^{(t_1 - \tau - \mu)/\mu} b_{k_i} = e_A(t_1, \sigma(\tau)) b_{k_i}$ for $\tau = t_1 - (s+1)\mu$. Now it is enough to take the interval $[\alpha_i, \beta_i]_{\mathbb{T}}$ that consists of the single element τ to get the required necessary and sufficient condition for positive reachability.

Theorem 39. Let $\mathbb{T}$ be a homogeneous time scale and $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. The system Σ is positively observable on $[t_0, t_1]_{\mathbb{T}}$ if and only if for every $i = 1, \ldots, n$, there is a path in $G_r(\Sigma)$ whose initial vertex is x_i and final vertex lies in Y.

Proof. Observe that positive observability is dual to positive reachability. Thus we can show the characterization of positive observability stated in the theorem constructing the dual system and using Theorem 38. However we need to reverse the order of edges so that we finish in Y and not start from there.

On homogeneous time scales positive reachability is equivalent to positive reachability on any sufficiently large $[t_0, t_1]$. Similar property holds for positive observability. Thus we have:

Corollary 40. Let $\mathbb{T}$ be a homogeneous time scale. The system Σ is positively reachable if and only if for every $i = 1, \ldots, n$, there is a path in $G_r(\Sigma)$ that originates in U and whose final vertex is x_i.
The system Σ is positively observable if and only if for every $i = 1, \ldots, n$, there is a path in $G_r(\Sigma)$ whose initial vertex is x_i and final vertex lies in Y.

We can also address positive minimality using graphs. Let us construct a new graph $G(\Sigma)$ whose set of vertices $V = U \cup X \cup Y$, and the set of edges $E = E_r \cup E_o$. However we want to remember the origin of each edge, so the edges will be colored: these belonging to E_r will be green and those belonging to E_o will be blue. Corollary 40 results in the following characterization of positive minimality.

Corollary 41. Let $\mathbb{T}$ be a homogeneous time scale. System Σ is positively minimal if and only if for every $i = 1, \ldots, n$, there is a path of $G(\Sigma)$ passing through x_i that originates in U, ends in Y, and consists of green edges up to x_i and of blue edges after x_i.

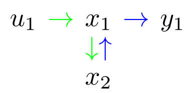
Example 42. Let $\mathbb{T} = \mathbb{Z}$ and

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; C = (1, 0).$$

Then

$$I + \mu A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } H(\Sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $H(\Sigma)$ is monomial, Σ is positively minimal, i.e. positively reachable and positively observable. Its graph $G(\Sigma)$ is given below.



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