Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Robust Time-Varying Model Predictive Control with Application to Mobile Robot Unmanned Path Tracking

Mitra Bahadorian * **Ray Eaton** * **Tim Hesketh** * **Borislav Savkovic** *

* *The School of Electrical Engineering and Telecommunications, The University of New South Wales, Kensington, NSW, 2052, Australia (e-mail:(mitrab,r.eaton,t.hesketh)@unsw.edu.au, boris.savkovic@gmail.com)*

Abstract: This work presents a tube-based Robust Model Predictive Controller (RMPC) with an application to the control of a mobile robot performing unmanned path tracking. In addition to robustness against unknown but bounded uncertainties, the RMPC algorithm proposed in this work is implementable (i.e. it imposes low complexity and conservativeness), thus making it highly amenable to real-time applications. This is due to an improved formulation of the tube over previous works (here a tube is sequence of sets) which is constructed by predicting the evolution of the difference between the actual uncertain system under control and its respective nominal disturbance-free system. Moreover, a feedback corrective controller formulated as a time-varying finite-time Linear Quadratic Regulator (LQR) is proposed to regulate the uncertain system around its respective nominal uncertainty-free system and thereby suppress the effect of uncertainties acting on the actual system. The proposed RMPC algorithm is applied to a Pioneer $P3-DX$ mobile robot platform performing unmanned path tracking. Experimental results demonstrate robust and stable performance of the proposed RMPC algorithm.

Keywords: Model Predictive Control, Robust Control, Unmanned Path Tracking, Mobile Robotics.

1. INTRODUCTION

Mobile robot unmanned path tracking is a problem of practical importance in the field of robotics and autonomous vehicles. The aim is to have a mobile robot (or an unmanned vehicle) follow a given reference path autonomously. There are two key issues to be taken into account. One is the physical constraints imposed on robot's control inputs. The other one is the scenario of a mobile robot traveling in an unstructured environment (i.e. an environment with sources of uncertainties/disturbances). To develop a suitable control strategy, one needs to take these issues into account.

Previously, different control approaches such as feedback linearization (as in Shojaei et al. (2009)), sliding mode (as in Eaton et al. (2009) and Fang et al. (2006)) and Model Predictive Control (MPC)(as in Lenain et al. (2005) and Xie and Fierro (2008)) were applied to the mobile robot unmanned path tracking problem. Our group has previously also contributed considerably to the MPC literature relating to control of mobile robots (see Bahadorian et al. (2011), Bahadorian et al. (2012)). In this work, we are particularly interested in applying the MPC control scheme to the problem of unmanned path tracking. MPC is an optimal and predictive control methodology which is capable of handling constraints of a system (i.e. constraints on states and/or inputs) explicitly. To apply the MPC control approach to unmanned path tracking problem, one possibility is to form a linearized model of the mobile robot's dynamics to be used by MPC to predict the future behavior of the system under control (see Klancar and Skrjanc (2007), Xie and Fierro (2008)).

The key shortcoming of the conventional MPC paradigm pertains to its inability to cope with disturbances and uncertainties acting on a control system. Issues here relate to how to achieve robustness while satisfying all the constraints. A summary of these issues may be found in Bemporad and Morari (1999) and Mayne et al. (2000). One of the key methods to robustify MPC against uncertainties and disturbances is the tube-based robustifying approach, the aim of which is to properly restrict the constraints of a nominal disturbance-free system and employ a form of auxiliary corrective feedback controller to suppress the effects of uncertainty (see e.g. Mayne and Langson (2001), Mayne et al. (2005), Mayne et al. (2006), Mayne et al (2009), Richards (2005a), Richards (2005b) and Richards and How (2006)). The key idea behind the method is to obtain an optimized input for the nominal disturbance-free system which is subject to more restricted (i.e. tighter) constraints than the actual system. A linear feedback controller is then employed to ensure that the actual system follows this nominal and optimal trajectory, resulting in robustness despite uncertainty. The formulation of the overall control law guarantees that the actual system evolves around the disturbance-free system within the designated tubes, while satisfying constraints of the actual system.

In this work, the core idea is to extend the results from Mayne and Langson (2001), Mayne et al. (2005), Mayne et al. (2006), Mayne et al (2009) and Bahadorian et al. (2011) to develop an implementable robust MPC (i.e. with low complexity and conservativeness). We are interested in formulating a dynamic (i.e. time-varying) corrective feedback controller (i.e. an auxiliary controller which ensures robust evolution of actual system around MPC optimized trajectories as mentioned previously) as a part of our RMPC development.

Previously our group has developed the theoretical and sim-

⋆ This work was supported by the Australian Research Council (ARC).

ulation framework to address the problem of robust MPC for mobile robots in Bahadorian et al. (2011) and Bahadorian et al. (2012). In the present paper we extend our previous works and apply the method to a real robotic platform which is a Pioneer $P3-DX$ mobile robot platform from ActiveMedia.
The contribution of this work is to develop an implementable RMPC. The proposed RMPC control approach carries low conservativeness due to our particular formulation of tubes. Also, the proposed formulation of the auxiliary feedback controller as a time-varying finite-horizon Linear Quadratic Regulator (LQR) is very light in terms of complexity and may be easily computed online. These two improved features make the proposed RMPC control paradigm a fast low complexity controller, therefore a highly promising control candidate for mobile robot control in real time. As stated previously, our group has previously developed a version of such an approach for mobile robot unmanned path tracking employing a different linearized model (see Bahadorian et al. (2011) and Bahadorian et al. (2012)).
The rest of this work is organized as follows: in section II, the motion dynamics of wheeled mobile robots as well as the trajectory tracking problem are introduced. Section III presents the main result of this work, the new Robust MPC (RMPC) algorithm to be applied to mobile robot unmanned path tracking problem. In section IV, the performance of the proposed control approach is verified by presenting experiential results obtained by applying our RMPC algorithm to a Pioneer $P3-DX$ mobile robot platform. Finally in section V concluding remarks are presented.
Notations used in this paper:
Let $\mathbb{S}_1$ and $\mathbb{S}_2$ represent two sets where $\mathbb{S}_1, \mathbb{S}_2 \subseteq \mathbb{R}^n$. Then the Minkowski sum of $\mathbb{S}_1$ and $\mathbb{S}_2$ is denoted by $\mathbb{S}_1 \oplus \mathbb{S}_2 = \{a+b : a \in \mathbb{S}_1, b \in \mathbb{S}_2\}$ and Minkowski difference between $\mathbb{S}_1$ and $\mathbb{S}_2$ is denoted by $\mathbb{S}_1 \ominus \mathbb{S}_2 = \{z : \forall b \in \mathbb{S}_2, (z+b) \in \mathbb{S}_1\}$.
Vectors are written in **bold** font in this paper, e.g. $\mathbf{x}$ is a vector, but x is a scalar.

2. APPLICATION

2.1 Dynamics of wheeled mobile robot

The system under control is a wheeled mobile robot (WMR) performing trajectory tracking. Under the nonholomonic constraints and with no slip assumption, the kinematic model of the robot can be described as:

$$\begin{aligned} \dot{x}_c &= v_c \cos\theta_c \\ \dot{y}_c &= v_c \sin\theta_c \\ \dot{\theta}_c &= \omega_c \end{aligned} \tag{1}$$

where the pair (x_c, y_c) denotes the position of the robot in a Carthesian frame and θ_c denotes the orientation of the robot. Also v_c and ω_c denote the linear and angular velocity of the robot respectively. The compact form of the system (1) can be presented as follows:

$$\dot{\mathbf{x}}_c = f(\mathbf{x}_c, \mathbf{u}_c) \tag{2}$$

where $\mathbf{x}_c = [x_c, y_c, \theta_c]' \in \mathbb{R}^3$ denotes the state of the mobile robot and $\mathbf{u}_c = [v_c, \omega_c]' \in \mathbb{R}^2$ represents the input to the robot. It is notable that both state and input of the robot are time-dependant, i.e. for example by x_c we mean $x_c(k)$ (where k denotes a time step), however for the sake of brevity we suppress the time-dependency notation.

2.2 Path Tracking Problem

To describe the path tracking problem, let us introduce a virtual mobile robot moving on the desired reference track. This latter virtual robot is referred to as the reference robot. The dynamics of the reference robot are the same as the actual robot, i.e.:

$$\dot{\mathbf{x}}_r = f(\mathbf{x}_r, \mathbf{u}_r) \tag{3}$$

where $\mathbf{x}_r = [x_r, y_r, \theta_r]'$ denotes the reference state and $\mathbf{u}_r = [v_r, \omega_r]'$ represents the corresponding reference input. With the assumption of the reference robot moving on the reference track, the path tracking problem consists of finding a control law $\mathbf{u}_c = [v_c, \omega_c]'$ to enforce the difference between the reference and actual robots (i.e. $\mathbf{x}_e = \mathbf{x}_r - \mathbf{x}_c$) to zero, i.e.:

$$\lim_{t\to\infty} \mathbf{x}_e = \lim_{t\to\infty} (\mathbf{x}_r - \mathbf{x}_c) = 0 \tag{4}$$

To this aim, by attaching the center of coordinate frame to the tracking robot (i.e. using a local coordinate frame), the difference between the reference robot and actual robot's state can be described as Kanayama et al. (1990):

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta_c & \sin\theta_c & 0 \\ -\sin\theta_c & \cos\theta_c & 0 \\ 0 & 0 & 1 \end{bmatrix} (\mathbf{x}_r - \mathbf{x}_c) \tag{5}$$

The error dynamics can be obtained by differentiating (5) along both (2) and (3) and rearranging the terms, therefore:

$$\begin{aligned} \dot{x}_e &= \omega_e y_e - v_e + v_r \cos\theta_e \\ \dot{y}_e &= -\omega_e x_e + v_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r - \omega_e \end{aligned} \tag{6}$$

Let us define the input term $\mathbf{u}_c$ as following:

$$\mathbf{u}_c := \mathbf{u}_r - \mathbf{u}_e = \begin{bmatrix} v_r \cos\theta_e - v_e \\ \omega_r - \omega_e \end{bmatrix} \tag{7}$$

Note that in expression (7), the terms $v_r \cos\theta_e$ and ω_r relate to the feed-forward linear and angular velocities respectively. Also, $\mathbf{u}_e$ is defined as:

$$\mathbf{u}_e := [v_e, \omega_e]' \tag{8}$$

By linearizing the system from expression (6) around the equilibrium point (i.e. $\mathbf{x}_e = 0$ and $\mathbf{u}_e = 0$), the following error-based dynamics will be obtained:

$$\dot{\mathbf{x}}_e = \begin{bmatrix} 0 & \omega_r & 0 \\ -\omega_r & 0 & v_r \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}_e + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}_e \tag{9}$$

Using a sufficiently short sampling time T_s to discretize the system (9), the subsequent discrete time system can be obtained:

$$\mathbf{x}_e(k+1) = \begin{bmatrix} 1 & \omega_r T_s & 0 \\ -\omega_r T_s & 1 & v_r T_s \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_e(k) + \begin{bmatrix} T_s & 0 \\ 0 & 0 \\ 0 & T_s \end{bmatrix} \mathbf{u}_e(k) \tag{10}$$

A compact form of the discrete time system from expression (10) can be defined as:

$$\mathbf{x}_e(k+1) = A(k)\mathbf{x}_e(k) + B\mathbf{u}_e(k) \tag{11}$$

with $A(k)$ denoting the error-based state matrix and B denoting the error-based input matrix. Note that without loss of generality, by substituting $B(k) = B$ in expression (11), the following formulation for the discretized error dynamics of mobile robot may be achieved:

$$\mathbf{x}_e(k+1) = A(k)\mathbf{x}_e(k) + B(k)\mathbf{u}_e(k) \tag{12}$$

By making the tracking error $\mathbf{x}_e$ bounded with sufficiently small bounds, it is ensured that the mobile robot always stays close to the reference track and does not deviate beyond the designated bounds. Therefore, the following constraints are imposed on the tracking error $\mathbf{x}_e$:

$$\mathbf{x}_e(k) \in \mathbb{X}_e \subseteq \mathbb{R}^3 \tag{13}$$

where $\mathbb{X}_e$ denotes a compact convex set. Due to physical constraints, the input to the mobile robot is also limited, i.e. $\mathbf{u}_c(k) \in \mathbb{U}_c \subseteq \mathbb{R}^2$ where $\mathbb{U}_c$ denotes a compact convex set. Employing expression (7), the bounds on the error-based input $\mathbf{u}_e(k)$ can be derived as:

$$\mathbf{u}_e(k) \in \mathbb{U}_e \subseteq -\mathbb{U}_c \oplus \mathbf{u}_r(k) \tag{14}$$

To establish a more realistic model, the effect of linearization and also uncertainties (such as noisy sensory readings or any occurring delays in the system) are modelled by an unknown but bounded uncertainty term $\mathbf{w}(k) \in \mathbb{W} \subseteq \mathbb{R}^3$. Note that the set $\mathbb{W}$ denotes a a-priori defined convex set which contains the origin. Therefore, the error-based model from expression (12) can be rewritten as:

$$\mathbf{x}_e(k+1) = A(k)\mathbf{x}_e(k) + B(k)\mathbf{u}_e(k) + \mathbf{w}(k) \tag{15}$$

The system from expression (15) is now defined in the framework of a constrained time-varying system subject to uncertainty. In the next section of this paper, we are going to present a robust MPC algorithm to control system (15).

3. CONTROLLER

The aim is to develop a RMPC control scheme for the error-based discretized linear time-varying system, with n states and m inputs, of the form:

$$\mathbf{x}_e(k+1) = A(k)\mathbf{x}_e(k) + B(k)\mathbf{u}_e(k) + \mathbf{w}(k) \tag{16}$$

where $\mathbf{x}_e(k) \in \mathbb{R}^n$ and $\mathbf{u}_e(k) \in \mathbb{R}^m$ denote the error-based state and the error-based input of the system (16) and $\mathbf{w}(k) \in \mathbb{R}^n$ denotes an unknown but bounded uncertainty acting on the system. The state $\mathbf{x}_e(k)$ and input $\mathbf{u}_e(k)$ of the system (16) are subject to the following constraints:

$$\mathbf{x}_e(k) \in \mathbb{X}_e \subseteq \mathbb{R}^m \tag{17}$$

$$\mathbf{u}_e(k) \in \mathbb{U}_e(k) \subseteq \mathbb{R}^n \tag{18}$$

with description of $\mathbb{X}_e$ and $\mathbb{U}_e(k)$ from section before. The unknown uncertainty $\mathbf{w}(k)$ modelling model-plant mismatch and/or noisy sensory readings is also assumed to be bounded, i.e.:

$$\mathbf{w}(k) \in \mathbb{W} \subseteq \mathbb{R}^n \tag{19}$$

The system from (16) may be converted to a nominal disturbance-free system, to be used in the subsequent section, of the form:

$$\bar{\mathbf{x}}_e(k+1) = A(k)\bar{\mathbf{x}}_e(k) + B(k)\bar{\mathbf{u}}_e(k) \tag{20}$$

where

$$\bar{\mathbf{x}}_e(k) \in \overline{\mathbb{X}}_e \subseteq \mathbb{R}^m \tag{21}$$

$$\bar{\mathbf{u}}_e(k) \in \overline{\mathbb{U}}_e(k) \subseteq \mathbb{R}^n \tag{22}$$

with $\bar{\mathbf{x}}_e(k)$ and $\bar{\mathbf{u}}_e(k)$ denoting the nominal error-based state and the nominal error-based input respectively. The sets $\overline{\mathbb{X}}_e$ and $\overline{\mathbb{U}}_e(k)$ represent suitably-defined nominal state and input constraints sets respectively. The formulation of these sets will be presented later on.

3.1 Formulation of tube-based RMPC

To control the system from equation (16), we propose the following RMPC control law:

RMPC Controller:

$$\mathbf{u}_e(k) = \bar{\mathbf{u}}_e(k) + \mathbf{u}_{cor}(k) \tag{23}$$

$$\bar{\mathbf{x}}_e(0) = \mathbf{x}_e(0) \tag{24}$$

$$\bar{\mathbf{u}}_e(k) = \bar{\mathbf{u}}_e^*(0|k) \tag{25}$$

$$\{\bar{\mathbf{u}}_e^*(i|k)\}_{i=0}^{N-1} = arg\ min_{\{\bar{\mathbf{u}}_e(i|k)\}_{i=0}^{N-1}} \bar{J}_k(\bar{\mathbf{x}}_e(k)) \tag{26}$$

where $\mathbf{u}_{cor}(k)$ is a localized corrective feedback control term to suppress and correct for the effect of uncertainties acting on the system. The term $\bar{J}_k(\bar{\mathbf{x}}_e(k))$ denotes the cost function of the following finite horizon optimization problem:

Finite Horizon Optimization Problem $P_k(\bar{\mathbf{x}}_e(k))$

$$\bar{J}_k(\bar{\mathbf{x}}_e(k)) = \sum_{i=0}^{N-1} ||\bar{\mathbf{x}}_e(i|k)||_Q^2 + ||\bar{\mathbf{u}}_e(i|k)||_R^2 + (1/2)||\bar{\mathbf{x}}_e(N|k)||_{Q_f}^2 \tag{27}$$

subject to the following constraints on $\bar{\mathbf{x}}_e(i|k)$ and $\bar{\mathbf{u}}_e(i|k)$:

$$\bar{\mathbf{x}}_e(i|k) \in \overline{\mathbb{X}}_e(i|k) \tag{28}$$

$$\bar{\mathbf{u}}_e(i|k) \in \overline{\mathbb{U}}_e(i|k) \tag{29}$$

$$\bar{\mathbf{x}}_e(N|k) \in \overline{\mathbb{X}}_f \tag{30}$$

$$\bar{\mathbf{x}}_e(0|k) = \bar{\mathbf{x}}_e(k) \tag{31}$$

where N denotes the prediction horizon of the MPC optimization problem and with $0 \leq i \leq N-1$. The notation $(i|k)$ denotes the prediction at $i \geq 0$ steps ahead from current time $k \geq 1$. Furthermore we have:

- Matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are both symmetric positive definite matrices denoting the weight matrices on nominal error-based state and nominal error-based input respectively.
- The notation $||.||_R$ denotes the weighted Euclidian norm defined by $||\mathbf{x}||_R = \sqrt{\mathbf{x}'R\mathbf{x}}$.
- The set $\overline{\mathbb{X}}_f \in \mathbb{R}^n$ and matrix $Q_f \in \mathbb{R}^{n\times n}$ represent the terminal set of optimization problem $P_k(\bar{\mathbf{x}}_e(k))$ and the terminal weight matrix respectively.

3.2 Formulation of the localized feedback controller

In this section, formulation of the corrective feedback input $\mathbf{u}_{cor}(k)$ from equation (23) is presented. Let us define $\delta(k)$ as the difference between the actual error-based state $\mathbf{x}_e(k)$ and the nominal disturbance-free error-based state $\bar{\mathbf{x}}_e(k)$, i.e.:

$$\delta(k) \triangleq \mathbf{x}_e(k) - \bar{\mathbf{x}}_e(k) \tag{32}$$

To achieve corrective action for this difference, by employing a time-varying finite-horizon LQR with state weighting matrix Q_{LQR}, input weighting matrix R_{LQR} and terminal weighting matrix Q_{ff}, we propose a corrective input term $\mathbf{u}_{cor}(k)$ of the form:

$$\mathbf{u}_{cor}(k) = G(k)\delta(k) \tag{33}$$

where:

$$G(k) = -(R_{LQR} + B^T(k)P_1B(k))^{-1}B^T(k)P_1A(k) \tag{34}$$

with notation T denoting matrix transposition. The term P_1 is determined by the following recursive "Riccatti" equation:

$$P_N = Q_{ff}$$

$$P_{z-1} = Q_{LQR} + A^T(z-1|k)P_zA(z-1|k) - A^T(z-1|k)P_zB(z-1|k)(R_{LQR} + B^T(z-1|k)P_zB(z-1|k))^{-1} .B^T(z-1|k)P_zA(z-1|k) \tag{35}$$

where $2 \leq z \leq N$. Also later on in the subsequent section we will need the formulation of $G(i|k)$ as follows:

$$G(i|k) = (-R_{LQR} + B^T(i|k)P_1B(i|k))^{-1}B^T(i|k)P_1A(i|k) \quad (36)$$

Using expressions (16), (20) and (23) by employing analogous arguments to the ones used in Mayne and Langson (2001), evolution of $\delta(k)$ is governed by a linear difference equation of the form:

$$\delta(k+1) = A_G(k)\delta(k) + \mathbf{w}(k) \quad (37)$$

where $A_G(k) = A(k) + B(k)G(k)$. Note that a controller $G(k)$ of the form (34) keeps $\delta(k)$ bounded.

3.3 Formulation of the nominal constraint sets

While the corrective input term $\mathbf{u}_{cor}(k)$ from expressions (23) and (33) relates to keeping the actual error-based state $\mathbf{x}_e(k)$ close to the nominal state $\bar{\mathbf{x}}_e(k)$, it is also required to formulate restricted constraint sets $\overline{\mathbb{X}}_e(i|k)$ and $\overline{\mathbb{U}}_e(i|k)$ (from equations (28) and (29)), such that the error-based state $\mathbf{x}_e(k)$ and the error-based input $\mathbf{u}_e(k)$ satisfy the constraints $\mathbb{X}_e$ and $\mathbb{U}_e(k)$ from expressions (21) and (22) respectively. To this aim, we formulate nominal constraint sets $\overline{\mathbb{X}}_e(i|k)$ and $\overline{\mathbb{U}}_e(i|k)$ as follow:

$$\overline{\mathbb{X}}_e(i|k) = \mathbb{X}_e \ominus \mathbb{T}(i|k) \quad (38)$$

$$\overline{\mathbb{U}}_e(i|k) = \mathbb{U}_e(k) \ominus G(i|k)\mathbb{T}(i|k) \quad (39)$$

where $k \geq 1$ and $0 \leq i \leq N-1$ and where $G(i|k)$ is defined by expression (36). By unrolling expression (37) for N steps into the future time, we propose the following formulation for $\mathbb{T}(i|k)$:

$$\mathbb{T}(i|k) = \begin{cases} \{\delta(k)\} & , \text{if } i = 0 \\ (A(i-1|k) + B(i-1|k)G(i-1|k))\mathbb{T}(i-1|k) \oplus \mathbb{W}, & \text{if } i \geq 1 \end{cases} \quad (40)$$

The set $\mathbb{T}(i|k)$ is the set of all possible deviations of actual error-based state $\mathbf{x}_e(i+k)$ from nominal error-based state $\bar{\mathbf{x}}_e(i+k)$ (i.e. $\delta(i+k)$)) over the prediction horizon N. This is based on the known fact that the deviation of $\mathbf{x}_e(i+k)$ from $\bar{\mathbf{x}}_e(i+k)$ evolves according to the linear difference equation from expression (37).

3.4 Remarks on robustness, stability and feasibility properties of of the proposed RMPC

Remark 1. **Robustness of the proposed RMPC**
Robustness of the proposed RMPC is guaranteed, if there exists a scalar $\varepsilon > 0$, such that $\forall t, ||\delta(k)|| < \varepsilon$. This may be achieved by following the approach from Richards (2005b) by picking receding horizon LQR terminal weight Q_{ff} as a matrix with large eigenvalues to ensure that the terminal prediction $\delta(N|k)$ of the difference between actual error-based state $\mathbf{x}_e(k)$ and nominal error-based state $\bar{\mathbf{x}}_e(k)$ is practically zero. Then $\delta(k)$ is ensured to be bounded as proved in section 2.4 page 43 of Richards (2005a). This is ensured here, since $G(k)$ may be seen as an unconstrained MPC controller, driven by disturbance term $\mathbf{w}(k)$ from expression (37). Thus the corrective input $\mathbf{u}_{cor}(k)$ keeps the actual error-based state $\mathbf{x}_e(k)$ within a bounded set around the nominal error-based state $\bar{\mathbf{x}}_e(k)$, i.e. keeps $\delta(k)$ bounded. This is a tube-like controller as in Mayne and Langson (2001)- Mayne et al (2009). However in the present case the tubes and feedback matrices $G(i|k)$ are time-varying.

Remark 2. **Stability of the proposed RMPC**
The formulation of the constraint restrictions from equations (38) and (39) imposes suitable restrictions to account for the effect of uncertainties by leaving sufficient space for system perturbations so as to avoid constraint violations (see e.g. Mayne et al. (2005)- Mayne et al (2009)). However in order to achieve stability, further conditions must be imposed. While restrictions imposed in equations (38) and (39) are sufficient if the feedback matrices $G(i|k)$ are constant and time-invariant (see e.g. Mayne et al (2009)), further conditions must be imposed in the present case in order to ensure stability. This may be achieved by imposing the condition that the constraint restrictions for $\overline{\mathbb{X}}_e(i|k)$ and $\overline{\mathbb{U}}_e(i|k)$ from equations (38) and (39) are only imposed if $\{\bar{\mathbf{u}}_e^*(1|k-1), \bar{\mathbf{u}}_e^*(2|k-1), ..., \bar{\mathbf{u}}_e^*(N-1|k-1), 0\}$ is a feasible solution for the optimization problem $P_k(\bar{\mathbf{x}}_e(k))$ at time k, i.e. if the optimal prediction trajectory from time $k-1$ starting with $\bar{\mathbf{u}}_e^*(1|k-1)$ is a feasible trajectory at time k. With this condition, then by standard MPC stability arguments, it immediately follows that $\overline{V}_k(\bar{\mathbf{x}}_e(k)) = \min \overline{J}_k(\bar{\mathbf{x}}_e(k))$ is a Lyapunov function satisfying $\overline{V}_k(\bar{\mathbf{x}}_e(k)) - \overline{V}_{k-1}(\bar{\mathbf{x}}_e(k-1)) < -||\bar{\mathbf{x}}_e(k-1)||_Q$ (see e.g. Goodwin et al. (2005) or Mayne et al. (2000)). If however $\{\bar{\mathbf{u}}_e^*(1|k-1), \bar{\mathbf{u}}_e^*(2|k-1), ..., \bar{\mathbf{u}}_e^*(i-1|k-1), 0\}$ is not a feasible solution to optimization problem $P_k(\bar{\mathbf{x}}_e(k))$ at time k, then we replace the constraints from equations (38) and (39) at time k by the following :

$$\overline{\mathbb{X}}_e(i|k) = \overline{\mathbb{X}}_e(i+1|k-1) \quad (41)$$

$$\overline{\mathbb{U}}_e(i|k) = \overline{\mathbb{U}}_e(i+1|k-1) \quad (42)$$

i.e. we employ the constrains from time $k-1$ with a prediction $i+1$ (from time $k-1$) as the constrains at time k for predictions i times ahead (from time k). This choice, by construction, ensures that $\{\bar{\mathbf{u}}_e^*(1|k-1), \bar{\mathbf{u}}_e^*(2|k-1), ..., \bar{\mathbf{u}}_e^*(N-1|k-1), 0\}$ is a feasible trajectory for optimization problem $P_k(\bar{\mathbf{x}}_e(k))$ at time k (this follows since $\{\bar{\mathbf{u}}_e^*(1|k-1), \bar{\mathbf{u}}_e^*(2|k-1), ..., \bar{\mathbf{u}}_e^*(N-1|k-1), 0\}$ is the optimal solution to optimization problem $P_{k-1}(\bar{\mathbf{x}}_e(k-1))$ from previous time step $k-1$, however under the constrains given by equations (41) and (42) (as opposed to equations (38) and (39)). Thus because $\{\bar{\mathbf{u}}_e^*(1|k-1), \bar{\mathbf{u}}_e^*(2|k-1), ..., \bar{\mathbf{u}}_e^*(N-1|k-1), 0\}$ is feasible for optimization problem $P_k(\bar{\mathbf{x}}_e(k))$ at time k, it immediately follows that $\overline{V}_k(\bar{\mathbf{x}}_e(k)) - \overline{V}_{k-1}(\bar{\mathbf{x}}_e(k-1)) < -||\bar{\mathbf{x}}_e(k-1)||_Q$ again. This way stability is achieved in any case. An intuitive explanation of the robustifications employed here is that we first check if an optimal solution (subject to the constrains in equations (38) and (39)) may be achieved. If this solution may not be guaranteed to give stability, then we revert to the "safe" constrains from equations (41) and (42). Thus our controller first attempts optimality under constraints from equations (38) and (39), and if the resulting optimization does not guarantee stability, then a "safe" control (from an extension from the previous time step $k-1$) is employed, employing the constrains from equations (41) and (42). This way stability is ensured. Note that if the second case from above occurs, then the corrective control input $\mathbf{u}_{cor}(k) = G(1|k-1)\delta(k)$ is applied instead of the control input $\mathbf{u}_{cor}(k) = G(k)\delta(k) = G(0|k)\delta(k)$ from equation (34). This way the one-stead ahead prediction from time $k-1$ (i.e. $G(1|k-1)$) is used at time k due to reversion to the "safe" case that is required for ensuring stability of the nominal dynamics.

Remark 3. **Feasibility of the proposed RMPC**
Employing the robustification, the argument in the previous paragraph also ensures recursive feasibility, i.e. if the optimization problem $P_k(\bar{\mathbf{x}}_e(k))$ is feasible at time $k = 1$, then it will be

feasible for all $k > 1$. This follows by standard MPC arguments, see e.g. Goodwin et al. (2005) and Mayne et al. (2000).

4. EXPERIMENTAL RESULTS

A series of experiments has been carried out on a Pioneer $P3-DX$ mobile robot platform from ActiveMedia to validate the robust performance of the proposed RMPC algorithm. The position and orientation of the robot were obtained from on-board odometry encoders of the Pioneer robot. Prior to running the main experiments, the odometry readings of the robot were calibrated by performing a series of well-known UMB-Mark calibrating tests (see Borenstein and Feng (1994)). The reference trajectory was defined as an inverted S-shape track, as shown in a dark solid line in Fig. (1), starting at initial position and orientation $[0,0,0]'$.
To keep the robot close to the reference track, the bounds on $\mathbb{X}_e$ from equation (13) were chosen as:

$$\mathbb{X}_e = \{[x_e, y_e, \theta_e]' : |x_e| \leq 0.3(m), |y_e| \leq 0.3(m), |\theta_e| \leq \pi/6(rad)\} \quad (43)$$

The input constraints were set to be:

$$\mathbb{U}_c = \{[v_c, \omega_c]' : |v_c| \leq 0.5(m/s), |\omega_c| \leq 0.9(rad/s)\} \quad (44)$$

The bounds on the uncertainty set $\mathbb{W}$ were identified by running a series of experiments and taking the worse-case scenario into account. This experimental method of determining $\mathbb{W}$ is in line with previous works by Richards (2005a) and Richards (2005b). In short, we recorded the deviation from the undisturbed (assuming zero disturbance) and actual dynamics, thereby giving us a range of values of $\mathbf{w}(k)$. The set $\mathbb{W}$ was then selected to be sufficiently large so as to encompass all such observed values $\mathbf{w}(k)$. The main sources of uncertainty were identified as limited acceleration of the robot as well as a delay of around $0.4s$ in the system. We found the following choice of the set $\mathbb{W}$ to capture the size of the uncertainty/disturbances:

$$\mathbb{W} = \{[w_1, w_2, w_3]' : |w_1| \leq 0.05(m), |w_2| \leq 0.05(m), |w_3| \leq 0.05(rad)\} \quad (45)$$

The prediction horizon N was chosen to be $N = 5$. The state and input weighting matrices were chosen as:

$$Q = Q_{LQR} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$R = R_{LQR} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.01 \end{bmatrix}$$

with LQR terminal weighting matrix Q_{ff} chosen as $Q_{ff} = 10Q$.

Three experiments were carried out by setting the initial error of the robot with respect to the reference track (solid black line from Fig. (1)) to three different values $\mathbf{x}_{01}, \mathbf{x}_{02}$ and $\mathbf{x}_{03}$ such that $\mathbf{x}_{01} = [-0.15, 0.05, \pi/12]'$, $\mathbf{x}_{02} = [0.10, -0.15, 0]'$ and $\mathbf{x}_{03} = [0, 0.2, 0]'$. The nominal (red/green/blue dashed lines) and actual trajectory (red/green/blue solid lines) of the robot were obtained from the planar plot of the states $\bar{x}_c, \bar{y}_c$ and x_c, y_c respectively. It can be seen that in Fig. (1), regardless of the initial position of the robot, the nominal trajectories (red/green/blue dashed lines) which are derived from the ideal model of the system, always converge to the reference track (solid black line). Due to the effect of uncertainties acting on the actual system (e.g. delays), the actual trajectories (ref/green/blue solid lines) wobble around the respective nominal trajectories (red/green/blue dashed lines) without getting too far away from them, thereby validating robust performance of the proposed algorithm.

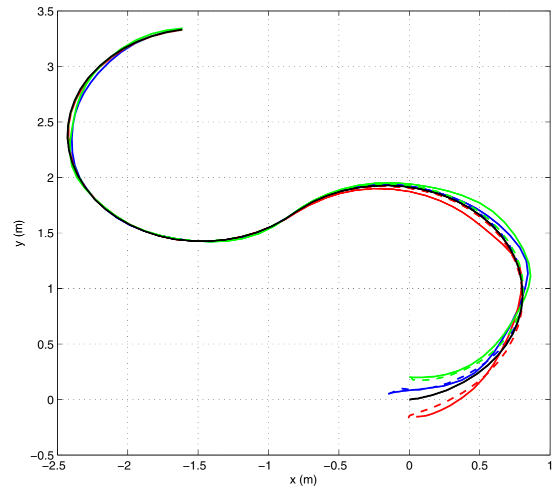

Fig. 1. Plot of the reference (solid black line), the actual (solid red/green/blue line) and the nominal (dashed red/green/blue line) trajectories of the Pioneer $P3-DX$ mobile robot, for all three experiments.

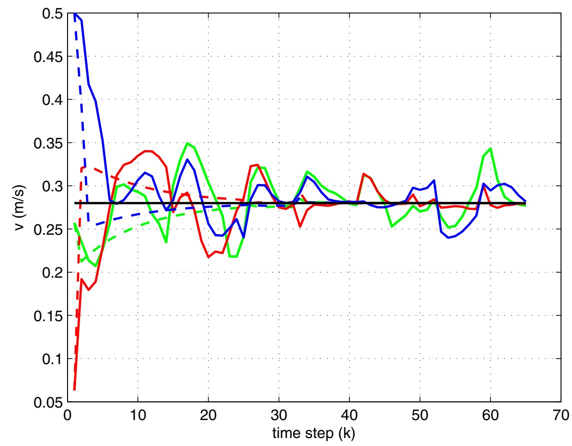

Fig. 2. Plot of the reference (solid black line), the actual (solid red/green/blue line) and the nominal (dashed red/green/blue line) input v_c to the Pioneer $P3-DX$ mobile robot, for all three experiments

Figures (2) and (3) depict the reference (solid black line), the nominal (dashed red/green/blue line) and the actual (solid red/green/blue line) linear velocity v_c and angular velocity ω_c, which are the control inputs to the mobile robot. When the nominal input is applied to the disturbance-free system, the actual input which is formulated as $\bar{\mathbf{u}}_c + \mathbf{u}_{cor}$ ensures that despite the uncertainty $\mathbf{w}$, the robot performs robust reference tracking. Also, for all three experiments, the nominal inputs (dashed red/green/blue line) $\bar{v}_c$ and $\bar{\omega}_c$ are active over an interval at the beginning of the experiment thereby steering the disturbance-free trajectories (dashed red/green/blue line) from Fig. (1) to the reference track. The actual inputs v_c and ω_c wobble around $\bar{v}_c$ and $\bar{\omega}_c$ respectively, due to the action of $\mathbf{u}_{cor}(k)$

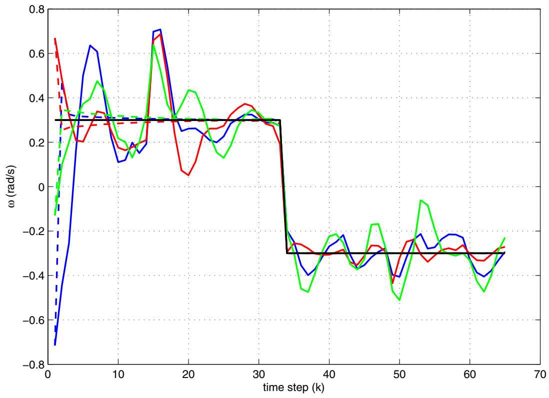

Fig. 3. Plot of reference (solid black line), actual (solid red/green/blue line) and nominal (dashed red/green/blue line) input ω_c to the Pioneer $P3-DX$ mobile robot, for all three experiments.

from expression (33), which corrects/suppresses the effect of uncertainty $\mathbf{w}(k)$ on the system, therefore ensuring system robustness. Moreover, it can be seen that the actual inputs (i.e. $[v_c, \omega_c]'$) to the robot stay within the assigned constrain sets $\mathbb{U}_c$ from expression (44), illustrating robust constraint satisfaction.

5. CONCLUSION

This paper presented a tube-based robust MPC (RMPC) control algorithm with an application to the problem of mobile robot unmanned path tracking. The improved formulation of the tube in this work over previous robust MPC methods employing tubes, is based on the prediction of the difference between the actual uncertain system and the nominal disturbance-free system. Also, a corrective feedback controller (formulated as a time-varying finite-horizon LQR) ensures regulation of the actual system around the nominal system within the designated tubes, guaranteeing robust performance. The proposed formulation of the corrective feedback controller is very light in terms of computational complexity, thereby making the proposed algorithm applicable within a real-time setting. To verify robust performance of the proposed RMPC algorithm, a series of experiments were conducted employing a Pioneer $P3-DX$ mobile robot. The proposed RMPC algorithm was applied to a Pioneer $P3-DX$ robot to perform unmanned path tracking. The experimental results illustrate robust and stable performance of the proposed RMPC control algorithm.

REFERENCES

M. Bahadorian, B. Savkovic, R. Eaton, T. Hesketh, Toward a Robust Model Predictive Controller applied to Mobile Vehicle Trajectory Tracking Control, *Proceedings of 18th IFAC World Congress*, Milan, Italy, Sep 2011

M. Bahadorian, B. Savkovic, R. Eaton, T. Hesketh, Robust Model Predictive Control For Automated Trajectory Tracking Of An Unmanned Ground Vehicle, *Proceedings of the* 2012 *American Control Conference*, Montreal, Canada, June 2012

A. Bemporad and M. Morari, Robust model predictive control, a survey, *Robustness in identification and control*, Vol. 245, 1999

J. Borenstein and L. Feng UMBmark - A Method for Measuring, Comparing, and Correcting Dead-reckoning Errors in Mobile Robots, *Number:UM-MEAM-94-22,* 1994

R. Eaton, J. Katupitiya, H. Pota, and K. W. Siew, Robust sliding mode control of an agricultural tractor under the influence of slip, *Proceedings of IEEE/ASME International Conference on Advanced Intelligent Mechatronics*, 2009

H. Fang, R. Fan, B. Thuilot, and P. Martinet, Trajectory tracking control of farm vehicles in presence of sliding, *Robotics and Autonomous Systems*, vol. 54, pp. 828 839, 2006

G.C. Goodwin, M.M. Seron, and J.A. De Dona, Constrained Control and Estimation, *Springer-Verlag, Heidelberg* 2005

Y. Kanayama, Y. Kimura, F. Miyazaki, and T. Noguchi, A stable tracking control method for an autonomous mobile robot, *Proceedings of IEEE Int. Conf. on Robotics and Automation*, pp. 384-389, 1990

G. Klancar and I. Skrjanc, Tracking-error model-based predictive control for mobile robots in real time, *Robotics and Autonomous Systems*, 55(6), pp. 460-469, June 2007

F. Kuhne, W. F. Lages and J. M. G. Da Silva Jr, Model predictive control of mobile robot using linearization, *Proceedings of IEEE International Conference on Mechatronics and Robotics*, Aachen, Germany, Sep 2004

R. Lenain, B. Thuilot, C. Cariou and P. Martinet, Model Predictive Control for Vehicle Guidance in Presence of Sliding: Application to Farm Vehicles Path Tracking, *Proceedings of the* 2005 *IEEE International Conference on Robotics and Automation* Barcelona, Spain, April 2005

D.Q. Mayne, J.B. Rawlings, C.V. Rao and P.O.M. Scokaert, Constrained model predicitive control : Stability and optimality, *Automatica*, 36 , pp. 789-814, 2000

D.Q. Mayne and W. Langson, Robustyfying model predictive control of constrained linear systems, *Electronics Letters*, 37(23), pp. 1422-1423, 2001

D.Q. Mayne, M.M. Seron, and S.V. Rakovic, Robust model predictive control of constrained linear systems with bounded disturbances, *Automatica*, 41(2), pp. 219-224, 2005

D.Q. Mayne, S.V. Rakovic, R. Findeisen, and F. Allgower, Robust output feedback model predictive control of constrained linear systems, *Automatica*, 42(7), pp. 1217-1222, 2006

D.Q. Mayne, S.V. Rakovic, R. Findeisen, and F. Allgower, Robust output feedback model predictive control of constrained linear systems: Time-varying case, *Automatica*, 45(9), pp. 2082-2087, 2009

A.G. Richards, Robust Constrained Model Predictive Control, *Doctoral Dissertation, MIT, Available online at http://dspace.mit.edu/handle/1721.1/28914*, 2005

A.G. Richards, Robust Model Predictive Control for Time-Varying Systems, *Proceedings of the* 44*th IEEE Conference on Decision and Control, and the European Control Conference*2005, Seville, Spain, Dec 2005

A.G. Richards and J. How, Robust Stable Model Predictive Control with Constraint Tightening, *Proceedings of the* 2006 *American Control Conference*, Minnesota, USA, June 2006.

K. Shojaei, A. Tarakameh and A.M. Shahri, Adaptive Trajectory Tracking of WMRs based on Feedback Linearization Technique, *Proceedings of* 2009 *International Conference on Mechatronics and Automation*, Changchun, China, Aug 2009

F. Xie and R. Fierro, First-state contractive model predictive control of nonholonomic robots, *Proceedings of* 2008 *American Control Conference*, Washington, USA, June 2008