Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Interval State and Unknown Inputs Estimation for Linear Time-Invariant Systems

David Gucik-Derigny *, **Tarek Raïssi** **, **Ali Zolghadri** *

* *University of Bordeaux, IMS-Lab, Automatic control group, 351 Cours de la liberation, 33405 Talence, France, (e-mail: david.gucik-derigny@u-bordeaux1.fr,ali.zolghadri@ims-bordeaux.fr).*
** *Conservatoire National des Arts et Metiers, Cedric-Lab, 292 Rue Saint-Martin, 75141, Paris, France, (e-mail: tarek.raissi@cnam.fr)*

Abstract: This paper deals with a set membership approach to design an Unknown Input Interval Observer for uncertain Linear Time-Invariant (LTI) continuous-time systems. The goal is to compute lower and upper bounds for unmeasured state as well as unknown inputs. The bounds are guaranteed under the assumption that external disturbances and noises are bounded with a priori known bounds. The proposed interval observer structure is based on decoupling the unknown input effect on the state dynamics by solving algebraic constraints on the estimation errors. Numerical simulations on a 5th-order lateral axis model of a fixed-wing aircraft are provided to demonstrate the efficiency of the proposed technique.

Keywords: Interval observers, bounded uncertainties, unknown inputs, change of coordinates, linear systems.

1. INTRODUCTION

Consider a system described by:

$$\begin{cases} \dot{x} \ = Ax + Bu + D\xi + \omega \\ y \ = Cx + \delta \\ \psi = Cx \end{cases}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state. The initial state vector x_0 belongs to an interval $[\underline{x}_0, \overline{x}_0]$. $u \in \mathbb{R}^m$ denotes the system input which belongs to an interval $[\underline{u}, \overline{u}]$. $\xi \in \mathcal{E} \subset \mathbb{R}^q$ is the unknown input vector of the model. $y \in \mathbb{R}^p$ and $\psi \in \mathbb{R}^p$ represent respectively the measured output affected by measurement noise and the free noise output. $\omega \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^p$ correspond respectively to the state and measurement noises. These disturbances are assumed to be bounded with a priori known bounds such that $|\omega| \leq \overline{\omega}$ and $|\delta| \leq \overline{\delta}$, where $\overline{\omega} \in \mathbb{R}^n$ and $\overline{\delta} \in \mathbb{R}^p$ are constant componentwise positive vectors. It is also assumed that $|\dot{\delta}| \leq \overline{\dot{\delta}}$ with $\overline{\dot{\delta}} \geq 0$. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}$ are constant matrices.

The problem of state estimation of systems described by (1) has many solutions and has been widely investigated in the literature. Popular and well-known observers are mainly based on, for instance, $\mathcal{H}_\infty$, Kalman filtering or Luenberger structures. In situations where external disturbances and noises are assumed bounded without any additional assumption, interval observers can be an appealing alternative approach. Basically, interval observers take a different approach from the conventional point observers since the latters converge asymptotically to the actual trajectory of the system model while interval observers provide guaranteed lower and upper bounds for the estimate at any time. Interval observers design requires stability and cooperative properties Mazenc and Bernard [2011], Raïssi et al. [2012]. One can distinguish two main set-membership approaches to perform state estimation for continuous-time systems. The first one (Alamo et al. [2005], Chisci et al. [1996], Jaulin [2002], Kieffer and Walter [2006], Kletting et al. [2006], Raïssi et al. [2004]) is based on the well-known prediction/correction mechanism as in the Kalman filter, where several geometrical forms for sets description such as parallelotopes, ellipsoids, zonotopes are used. The second approach Bernard and Gouzé [2004], Gouzé et al. [2000], Mazenc and Bernard [2011], Moisan et al. [2009], Raïssi et al. [2012] addresses closed loop interval observers where the measurements are taken as continuous-time data.

The state estimation of systems with unknown inputs within a set-membership framework has been addressed only in few works (Guerra et al. [2008], Rapaport and Gouzé [2002]). In Rapaport and Gouzé [2002], a parallelotopic unknown input interval observer is proposed. In Guerra et al. [2008], a robust fault detection using an unknown input interval observer with a zonotope representation is introduced based on an unknown input decoupling approach. In the above works, unknown inputs are not estimated. Following the main ideas reported in Hou and Müller [1992], the contribution of this paper is to design an unknown inputs interval estimator to derive guaranteed lower and upper bounds of the unmeasured state and the unknown inputs, consistent with the all a priori available knowledge. Furthermore, a similar approach for state estimation has been proposed in Efimov et al. [2012], but the unknown input estimation is not considered. As the unknown input estimation part needs the evaluation of the noisy measurements derivatives, High Order Sliding Modes (HOSM) differentiators will be used.

The paper is organized as follows. Section 2 recalls some useful preliminaries. Section 3 presents the problem statement and the assumptions used in the paper. Section 4 is devoted to the design of the unknown input interval observer. Section 5 presents some numerical simulations.

2. PRELIMINARIES

2.1 Notations

The set of real matrices with $n \times m$ elements is denoted by $\mathbb{M}_{n\times m}$, $I_n \in \mathbb{M}_{n\times n}$ depicts the identity matrix, $0_{n,m} \in \mathbb{M}_{n\times m}$ represents the null matrix which is denoted by 0 when there is no confusion. $\mathbb{M}^{+}_{n\times m}$ represents the set of positive real matrices.

Given a matrix $A \in \mathbb{M}_{n\times m}$, we denote $A^+ = max(0, A)$, $A^- = A - A^+$ and $\overline{\overline{A}} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. A^T and $A^\dagger$ denote respectively the transposition and pseudo-inverse matrices of A, with $A^\dagger = (A^T A)^{-1} A^T$.

In the following, the operators $\leq, <, >$ should be understood componentwise for vectors and matrices.

For $\underline{x}, \overline{x} \in \mathbb{R}^n$, $[\underline{x}, \overline{x}]$ denotes the set $\{x \in \mathbb{R}^n \mid \underline{x} \leq x \leq \overline{x}\}$.

The symbol $\|.\|$ denotes the euclidean norm $\mathcal{L}_2$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$ and for a measurable and locally essentially bounded input signal $u : \mathbb{R}_+ \to \mathbb{R}$, the symbol $|u|_{[t_0,t_1]}$ denotes its L_∞ norm:

$$|u|_{[t_0,t_1]} = ess\ sup\|u(t)\|, t \in [t_0, t_1],$$

if $t_1 = +\infty$, then we will simply write $|u|$.

Definition 1. A square matrix $A = (A_{ij}) \in \mathbb{M}_{n\times m}$ is said to be Metzler if $A_{ij} \geq 0$, $\forall i \neq j$.

Lemma 2. (cooperativity property) Smith [1995] Given a non-autonomous system described by $\dot{x}(t) = Ax(t) + B(t)$ where A is a Metzler matrix and $B(t) \geq 0$. Then, $x(t) \geq 0$, $\forall t > 0$ provided that $x(0) \geq 0$.

2.2 HOSM differentiator

Assume that the measurement noise is bounded with an a priori known bound $\overline{\delta}$. Then, the bounds of the output vector are given by:

$$\begin{aligned} \overline{y}(t) &= y(t) + \overline{\delta} \\ \underline{y}(t) &= y(t) - \overline{\delta} \end{aligned}, \quad \forall t \geq 0. \tag{2}$$

Given a s-th time differentiable signal $y_k(t)$, $k = 1, \ldots, p$. A HOSM differentiator Levant [2003] can be used to estimate the successive derivatives of $y_k(t)$. It is described by:

$$\begin{cases} \dot{q}^k_0 = \nu^k_0 \\ \nu^k_0 = -\lambda^k_0 |q^k_0 - y_k(t)|^{\frac{s}{s+1}} sign(q^k_0 - y_k(t)) + q^k_1 \\ \dot{q}^k_i = \nu^k_i \\ \nu^k_i = -\lambda^k_i |q^k_i - \nu^k_{i-1}|^{\frac{s-i}{s-i+1}} sign(q^k_i - \nu^k_{i-1}) + q^k_{i+1} \\ \dot{q}^k_s = -\lambda^k_s sign(q^k_s - \nu^k_{s-1}) \end{cases}$$

where λ^k_j, $k = 1, \ldots, p$, $j = 0, \ldots, s$ are positive constants which can be chosen based on a procedure proposed in Levant [2003].

Theorem 3. Levant [2003]: Let $y_k : \mathbb{R}_+ \to \mathbb{R}$, $k = 1, \ldots, p$ be a s-th time continuously differentiable signal and $|\delta_k| \leq \overline{\delta}_k$, then there exist $0 \leq T < +\infty$ and some constants $\mu^k_j > 0$, $j = 0, \ldots, s$ such that for all $t \geq T$:

$$|q^k_j(t) - \psi^{(j)}_k(t)| \leq \mu^k_j |\delta_k|^{\frac{s-j+1}{s+1}}, j = 0, \ldots, s. \tag{3}$$

□

The relation (3) represents the error bound of the j^{th} numerical output derivative estimate for $j = 0, \ldots, s$.

According to theorem 3, there exist parameters λ^k_j, $k = 1, \ldots, p$, $j = 0, \ldots, s$ and a time instant $T \in \mathbb{R}_+$ such that for all $t \geq T$ and for some constants $\mu^k_j > 0$

$$|q^k_j(t) - \psi^{(j)}_k(t)| \leq \mu^k_j \overline{\delta}_k^{\frac{s-j+1}{s+1}}, j = 0, \ldots, s. \tag{4}$$

Denote by $\epsilon(t)$:

$$\epsilon(t) = q_1(t) - \dot{\psi}(t) \tag{5}$$

In addition, from (1), we have:

$$\dot{\psi}(t) = \dot{y}(t) - \dot{\delta}(t) \tag{6}$$

Then, from (6) and (5), we get:

$$\dot{y}(t) = q_1(t) + \dot{\delta}(t) - \epsilon(t) \tag{7}$$

Finally, based on (7) and (4), we conclude that for all $t \geq T$, we have:

$$\begin{aligned} \overline{\dot{y}}(t) &= q_1(t) + \beta, \\ \underline{\dot{y}}(t) &= q_1(t) - \beta \end{aligned} \tag{8}$$

where $\beta = \overline{\dot{\delta}} + \mu_1 \overline{\delta}^{\frac{1}{2}} \geq 0$.

According to (6) and (5), we deduce also that

$$\begin{aligned} \overline{\dot{y}}(t) &= \dot{y}(t) + \zeta_1, \\ \underline{\dot{y}}(t) &= \dot{y}(t) - \zeta_2 \end{aligned} \tag{9}$$

where

$$\begin{aligned} \zeta_1 &= \overline{\dot{\delta}} - \dot{\delta} + \mu_1 \overline{\delta}^{\frac{1}{2}} + \epsilon \geq 0, \\ \zeta_2 &= \overline{\dot{\delta}} + \dot{\delta} + \mu_1 \overline{\delta}^{\frac{1}{2}} - \epsilon \geq 0. \end{aligned}$$

3. PROBLEM STATEMENT

The main goal of this work is to estimate the lower and upper bounds $\underline{\xi}$, $\overline{\xi} \in \mathbb{R}^q$ for the unknown input in (1) such that $\xi \in [\underline{\xi}, \overline{\xi}]$ and $\lim_{t\to\infty} |\overline{\xi} - \underline{\xi}| < \rho_\xi$ with ρ_ξ is a known positive constant. To this end, it is needed, first, to estimate the lower and upper bounds $\underline{x}$, $\overline{x} \in \mathbb{R}^n$ for the state $x \in \mathbb{R}^n$ which cannot be known exactly due to the presence of disturbances and noises, that is $x \in [\underline{x}, \overline{x}]$ and $\lim_{t\to\infty} |\overline{x} - \underline{x}| < \rho_x$ where ρ_x is a known positive constant.

In this work, an unknown input interval observer structure similar to the one proposed in Darouach et al. [1994] for certain LTI models is designed to deal with uncertain LTI models described by (1). The proposed interval observer is based on a decoupling of the unknown input effect on the state dynamics.

In the sequel, the following hypotheses are assumed to be verified.

Hypothesis 4.

$$\begin{aligned} &rank(CD) = rank(D) = q, \\ &q \leq p \end{aligned} \tag{10}$$

This hypothesis is known as "relative degree condition" and is a classical condition for existence of unknown input observers Fairman et al. [1984], Hou and Müller [1992], Levant [1980].

Hypothesis 5. There exists a matrix $W \in \mathbb{M}_{n\times p}$ such that $I_p + CW$ is nonsingular and the pair (PA, C) is detectable where $P = (I_n + EC)$ with

$$E = -D(CD)^{\dagger} + W(I_p - CD(CD)^{\dagger}) \tag{11}$$

This hypothesis is required to ensure the observer stability and the unknown input effect cancellation. It is clearly discussed in Darouach et al. [1994].

Hypothesis 6. There exists a gain matrix $K \in \mathbb{M}_{n\times p}$ and a state transformation matrix $T \in \mathbb{M}_{n\times n}$ of appropriate dimensions such that $T(PA - KC)T^{-1}$ is Metzler and Hurwitz.

Note that hypothesis 6 is usually verified and the computation of the transformation matrix T ensuring the cooperativity property is detailed in Combastel [2013], Mazenc and Bernard [2011], Raïssi et al. [2012].

Under hypothesis 6, there exists a non-singular state transformation $T \in \mathbb{M}_{n\times n}$:

$$z = Tx \tag{12}$$

for a given gain K such that $T(PA - KC)T^{-1}$ is Metzler. Hence, the system (1) is described in the coordinates (12) by:

$$\begin{cases} \dot{z} = TAT^{-1}z + TBu + TD\xi + T\omega \\ y = CT^{-1}z + \delta \end{cases} \tag{13}$$

4. MAIN RESULT

In this section, the interval observer design is introduced and detailed in two parts. A first part details the state estimation process and the second part is dedicated to the unknown input estimation.

4.1 State estimation

Given $L \in \mathbb{M}_{n\times p}$ and denote by:

$$\begin{aligned} K &= L + NE, \\ N &= PA - KC, \\ G &= PB \end{aligned} \tag{14}$$

Given the system (1) described by (13) in the coordinates (12) and consider the observer structure:

$$\begin{cases} \dot{\overline{z}} = TNT^{-1}\overline{z} + (TK)^{+}\overline{y} + (TK)^{-}\underline{y} \\ \quad +(TG)^{+}\overline{u} + (TG)^{-}\underline{u} + (-TE)^{+}\dot{\overline{y}} \\ \quad +(-TE)^{-}\dot{\underline{y}} + (TP)^{+}\overline{\varpi} - (TP)^{-}\overline{\varpi} \\ \dot{\underline{z}} = TNT^{-1}\underline{z} + (TK)^{+}\underline{y} + (TK)^{-}\overline{y} \\ \quad +(TG)^{+}\underline{u} + (TG)^{-}\overline{u} + (-TE)^{+}\dot{\underline{y}} \\ \quad +(-TE)^{-}\dot{\overline{y}} - (TP)^{+}\overline{\varpi} + (TP)^{-}\overline{\varpi} \end{cases} \tag{15}$$

where $(\dot{\overline{y}}, \dot{\underline{y}})$ and $(\overline{y}, \underline{y})$ are respectively defined in (8) and (2).

Theorem 7 states the first result of this paper by ensuring an interval state estimation in the coordinates z.

Theorem 7. Assume that $\underline{z}_0 \leq z(0) \leq \overline{z}_0$ and consider (15) with $\underline{z}(0) = \underline{z}_0$ and $\overline{z}(0) = \overline{z}_0$. Then, there exists a time instant $T \in \mathbb{R}^{+}$, $0 \leq T < +\infty$ such that the state $z(t)$, solution of (13), satisfies

$$z(t) \in [\underline{z}(t),\ \overline{z}(t)]\,,\ \forall t \geq T. \tag{16}$$

In addition, there exists $\rho_z > 0$ such that

$$\lim_{t\to\infty} |\overline{z}(t) - \underline{z}(t)| < \rho_z. \tag{17}$$

□

Proof. The proof is split into two steps. Firstly, upper and lower observation errors are shown to be positive for all $t \geq 0$ (i.e. $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$). In a second step, interval observer stability is proved.

Step 1: Observation errors positivity
The upper and lower observation errors are defined by

$$\begin{cases} \overline{e}_z = \overline{z} - z \\ \underline{e}_z = z - \underline{z} \end{cases} \tag{18}$$

Then, the dynamic observation errors are obtained by time derivating (18) and using equations (15), (9) and (2), the following equalities are deduced:

$$\begin{cases} \dot{\overline{e}}_z = TNT^{-1}\overline{z} + (TK)^{+}(CT^{-1}z + \delta) \\ \quad +(TK)^{-}(CT^{-1}z - \delta) + (TG)^{+}\overline{u} \\ \quad +(TG)^{-}\underline{u} + (-TE)^{+}(CT^{-1}\dot{z} + \dot{\delta}) \\ \quad +(-TE)^{-}(CT^{-1}\dot{z} - \dot{\delta}) + (TP)^{+}\overline{\varpi} \\ \quad -(TP)^{-}\overline{\varpi} - \dot{z} \\ \dot{\underline{e}}_z = \dot{z} - TNT^{-1}\underline{z} - (TK)^{+}(CT^{-1}z - \delta) \\ \quad -(TK)^{-}(CT^{-1}z + \delta) - (TG)^{+}\underline{u} \\ \quad -(TG)^{-}\overline{u} - (-TE)^{+}(CT^{-1}\dot{z} - \dot{\delta}) \\ \quad -(-TE)^{-}(CT^{-1}\dot{z} + \dot{\delta}) \\ \quad +(TP)^{+}\overline{\varpi} - (TP)^{-}\overline{\varpi} \end{cases} \tag{19}$$

By using the expression of $\dot{z}$ given in (13), the expression of K, N and G given in (14), the matrix P defined in hypothesis 5, and recalling that for any matrix $F \in \mathbb{M}_{n\times m}$, we have $F = F^{+} + F^{-}$, it can be easily shown that:

$$\begin{cases} \dot{\overline{e}}_z = TNT^{-1}\overline{e}_z + T(NP - PA + LC)T^{-1}z \\ \quad -TPD\xi + H_1(.) \\ \dot{\underline{e}}_z = TNT^{-1}\underline{e}_z - T(NP - PA + LC)T^{-1}z \\ \quad +TPD\xi + H_2(.) \end{cases} \tag{20}$$

where

$$\begin{aligned} H_1(.) = &(TG)^{+}(\overline{u} - u) - (TG)^{-}(u - \underline{u}) \\ &+(TP)^{+}(\overline{\varpi} - \omega) - (TP)^{-}(\overline{\varpi} + \omega) \\ &+(TK)^{+}(\overline{\delta} + \delta) - (TK)^{-}(\overline{\delta} - \delta) \\ &+(-TE)^{+}\zeta_1 - (-TE)^{-}\zeta_2 \end{aligned}$$

and

$$\begin{aligned} H_2(.) = &(TG)^{+}(u - \underline{u}) - (TG)^{-}(\overline{u} - u) \\ &+(TP)^{+}(\overline{\varpi} + \omega) - (TP)^{-}(\overline{\varpi} - \omega) \\ &+(TK)^{+}(\overline{\delta} - \delta) - (TK)^{-}(\overline{\delta} + \delta) \\ &+(-TE)^{+}\zeta_2 - (-TE)^{-}\zeta_1 \end{aligned}$$

Assume that the matrices N, K, E are solutions of the following algebric constraints:

$$\begin{cases} NP - PA + LC = N - PA + KC = 0 \\ PD = 0 \end{cases} \tag{21}$$

Then, according to (14), the equations (21) can be rewritten as:

$$\begin{cases} N + KC - ECA = A \\ ECD = -D \end{cases}. \tag{22}$$

Or in a compact form as:

$$[N\ K\ E]\,\Sigma = \Theta \tag{23}$$

where

$$\Sigma = \begin{bmatrix} I_n & 0 \\ C & 0 \\ -CA & CD \end{bmatrix} \text{ and } \Theta = [A\ -D]\,.$$

Under hypothesis 4, the solution set for (23) is parametrized by a matrix $Z = [Z_1\ K\ W]$ and it is given as follows:

$$[N\ K\ E] = \Theta\Sigma^{\dagger} + Z(I_{2n+p} - \Sigma\Sigma^{\dagger}) \tag{24}$$

where $\Sigma^{\dagger} = \begin{bmatrix} I_n & 0 & 0 \\ (CD)^{\dagger}CA & 0 & (CD)^{\dagger} \end{bmatrix}$.

Then, the expressions of E and N are given by:

$$\begin{cases} E &= -D(CD)^{\dagger} + W(I_p - CD(CD)^{\dagger}) \\ N &= (I + EC)A - KC \\ &= PA - KC \end{cases} \tag{25}$$

Under the algebraic constraints (21), where the condition $PD = 0$ allows one to cancel the unknown input ξ, the observation errors (20) have the following dynamics:

$$\begin{cases} \dot{\overline{e}}_z &= TNT^{-1}\overline{e}_z + H_1(.) \\ \dot{\underline{e}}_z &= TNT^{-1}\underline{e}_z + H_2(.) \end{cases} \tag{26}$$

Hypothesis 6 states that TNT^{-1} is Metzler and it is easy to verify that $H_1(.) \geq 0$ and $H_2(.) \geq 0$. In addition, the initial state $z(0)$ verifies $\underline{z}(0) \leq z(0) \leq \overline{z}(0)$. Thus, lemma 2 implies that property (16) holds.

Step 2: Interval observer stability
In the sequel, denote by E_z:

$$E_z = \begin{bmatrix} \overline{e}_z \\ \underline{e}_z \end{bmatrix}. \tag{27}$$

The differentiation of (27) leads to:

$$\dot{E}_z = \overline{\overline{J}}E_z + H(.) \tag{28}$$

where $\overline{\overline{J}}$ is defined in Section 2 with $J = TNT^{-1}$ and $H(.) = \begin{bmatrix} H_1(.) \\ H_2(.) \end{bmatrix}$.

Moreover, from the expressions of ζ_1 and ζ_2 in (9), there exists a constant $\overline{\zeta} \in \mathbb{R}^p$ such that $\zeta_1 \leq \overline{\zeta}$ and $\zeta_2 \leq \overline{\zeta}$, with:

$$\overline{\zeta} = 2(\overline{\dot{\delta}} + \mu_1\overline{\delta}^{\frac{1}{2}}).$$

Then, we can build a function $\overline{H}(.) : \mathbb{R}_+ \to \mathbb{R}^n$ such that

$$\begin{matrix} H_1(.) \leq \overline{H}(.) \\ H_2(.) \leq \overline{H}(.) \end{matrix} \tag{29}$$

with:

$$\begin{aligned} \overline{H}(.) &= ((TG)^+ - (TG)^-)(\overline{u} - \underline{u}) + 2((TP)^+ - (TP)^-)\overline{\omega} \\ &+ 2((TK)^+ - (TK)^-)\overline{\delta} + ((-TE)^+ - (-TE)^-)\overline{\zeta} \end{aligned}$$

Thus, since $\overline{\overline{J}}$ is Metzler and Hurwitz stable and $H(.)$ is bounded by a positive vector $\overline{H}^*(.) = \begin{bmatrix} \overline{H}(.) \\ \overline{H}(.) \end{bmatrix}$, then, based on theorem 2 in Gouzé et al. [2000], the total nonnegative error $\overline{e}_z - \underline{e}_z$ is asymptotically elementwise lower than the nonnegative vector:

$$\rho_z = -\overline{\overline{J}}^{-1}\overline{H}^* \tag{30}$$

Thus, property (17) holds with ρ_z defined in (30). □

Corollary 8. Under the conditions of theorem 7, we have $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ where x is the solution of (1) and

$$\begin{cases} \overline{x} = S^+\overline{z} + S^-\underline{z} \\ \underline{x} = S^+\underline{z} + S^-\overline{z} \end{cases} \tag{31}$$

with $S = T^{-1}$.

Proof. Since $Sz(t) = (S^+ + S^-)z(t)$, then, if $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$, we get:

$$S^+\underline{z}(t) + S^-\overline{z}(t) \leq Sz(t) \leq S^+\overline{z}(t) + S^-\underline{z}(t),$$

which means that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$.

Furthermore, the boundedness of $\underline{z}(t)$ and $\overline{z}(t)$ implies the boundedness of $\underline{x}(t)$ and $\overline{x}(t)$. □

In the following, we denote by:

$$\begin{cases} \overline{e}_x &= \overline{x} - x \\ \underline{e}_x &= x - \underline{x} \\ E_x &= \left[\overline{e}_x^T, \underline{e}_x^T\right] \end{cases}, \tag{32}$$

Therefore, from (31), (32) and (27), we have:

$$E_x = \mathcal{S}E_z \tag{33}$$

with $\mathcal{S} = \begin{bmatrix} S^+ & -S^- \\ -S^- & S^+ \end{bmatrix} \in \mathbb{M}^+_{2n\times 2n}$.

By using the upper bound in the observation error E_z described by (30) and the relation (33), we deduce that the total nonnegative error $\overline{e}_x - \underline{e}_x$ is asymptotically elementwise lower than the nonnegative vector:

$$\rho_x = -\mathcal{S}\overline{\overline{J}}^{-1}\overline{H}^* \tag{34}$$

4.2 Unknown input estimation

The goal of this subsection is to estimate upper and lower bounds of the unknown input ξ.

The differentiation of the output term of (13) leads to:

$$\dot{y} = CAT^{-1}z + CBu + CD\xi + C\omega + \dot{\delta} \tag{35}$$

Then, under hypothesis 4, we have:

$$\xi = M_0\dot{y} + M_1z + M_2u + M_3\omega + M_4\dot{\delta} \tag{36}$$

where $M_0 = (CD)^{\dagger}$, $M_1 = -(CD)^{\dagger}CAT^{-1}$, $M_2 = -(CD)^{\dagger}CB$, $M_3 = -(CD)^{\dagger}C$, $M_4 = -M_0$.
Based on (36), the upper and lower bounds of ξ are given by:

$$\begin{cases} \overline{\xi} = M_0^+\overline{\dot{y}} + M_0^-\underline{\dot{y}} + M_1^+\overline{y} + M_1^-\underline{y} + M_2^+\overline{u} \\ \quad +M_2^-\underline{u} + M_3^+\overline{\omega} - M_3^-\overline{\omega} + M_4^+\overline{\dot{\delta}} - M_4^-\overline{\dot{\delta}} \\ \underline{\xi} = M_0^+\underline{\dot{y}} + M_0^-\overline{\dot{y}} + M_1^+\underline{y} + M_1^-\overline{y} + M_2^+\underline{u} \\ \quad +M_2^-\overline{u} - M_3^+\overline{\omega} + M_3^-\overline{\omega} - M_4^+\overline{\dot{\delta}} + M_4^-\overline{\dot{\delta}} \end{cases} \tag{37}$$

where $\overline{\dot{y}}$ and $\underline{\dot{y}}$ are defined in (8).

With (37), we are in position to give a theorem ensuring the estimation of the unknown input domain.

Theorem 9. Assume that the hypotheses of theorem 7 are satisfied. Then, there exist a positive constant $\rho_\xi > 0$ and a time instant T, $0 \leq T < +\infty$ such that:

$$\xi(t) \in \left[\underline{\xi}(t),\ \overline{\xi}(t)\right],\ \forall t \geq T \tag{38}$$

$$\lim_{t\to\infty} |\overline{\xi}(t) - \underline{\xi}(t)| < \rho_\xi \tag{39}$$

where $\underline{\xi}(t)$ and $\overline{\xi}(t)$ are given by (37). □

Proof.

Define the upper and lower observation errors relative to the unknown input ξ as follows:

$$\begin{cases} \overline{e}_\xi &= \overline{\xi} - \xi \\ \underline{e}_\xi &= \xi - \underline{\xi} \end{cases} \tag{40}$$

which are given under a developed form:

$$\begin{cases} \overline{e}_\xi &= F_1(q_1, \dot{y}) + M_1^+ \overline{e}_z - M_1^- \underline{e}_z \\ &\quad + M_2^+(\overline{u} - u) - M_2^-(u - \underline{u}) \\ &\quad + M_3^+(\overline{\omega} - \omega) - M_3^-(\overline{\omega} + \omega) \\ &\quad + M_4^+(\overline{\dot{\delta}} - \dot{\delta}) - M_4^-(\overline{\dot{\delta}} + \dot{\delta}) \\ \underline{e}_\xi &= F_2(q_1, \dot{y}) + M_1^+ \underline{e}_z - M_1^- \overline{e}_z \\ &\quad + M_2^+(u - \underline{u}) - M_2^-(\overline{u} - u) \\ &\quad + M_3^+(\overline{\omega} + \omega) - M_3^-(\overline{\omega} - \omega) \\ &\quad + M_4^+(\overline{\dot{\delta}} + \dot{\delta}) - M_4^-(\overline{\dot{\delta}} - \dot{\delta}) \end{cases} \tag{41}$$

where $F_1(q_1, \dot{y}) = M_0(q_1 - \dot{y}) + (M_0^+ - M_0^-)\beta$ and $F_2(q_1, \dot{y}) = M_0(\dot{y} - q_1) + (M_0^+ - M_0^-)\beta$.

Replacing the term β defined in (8) into $F_1(.)$ and $F_2(.)$ leads to the following equalities

$$\begin{aligned} F_1(q_1, \dot{y}) &= M_0^+(\mu_1 \overline{\delta}^{\frac{1}{2}} + (q_1 - \dot{\psi})) \\ &\quad - M_0^-(\mu_1 \overline{\delta}^{\frac{1}{2}} - (q_1 - \dot{\psi})) \\ &\quad + M_0^+(\overline{\dot{\delta}} - \dot{\delta}) - M_0^-(\overline{\dot{\delta}} + \dot{\delta}) \end{aligned} \tag{42}$$

and

$$\begin{aligned} F_2(q_1, \dot{y}) &= M_0^+(\mu_1 \overline{\delta}^{\frac{1}{2}} - (q_1 - \dot{\psi})) \\ &\quad - M_0^-(\mu_1 \overline{\delta}^{\frac{1}{2}} + (q_1 - \dot{\psi})) \\ &\quad + M_0^+(\overline{\dot{\delta}} + \dot{\delta}) - M_0^-(\overline{\dot{\delta}} - \dot{\delta}) \end{aligned} \tag{43}$$

According to (4), it is deduced that $\forall t \geq T$,

$$\begin{aligned} \mu_1 \overline{\delta}^{\frac{1}{2}} - (q_1 - \dot{\psi}) \leq 2\mu_1 \overline{\delta}^{\frac{1}{2}}, \\ \mu_1 \overline{\delta}^{\frac{1}{2}} + (q_1 - \dot{\psi}) \leq 2\mu_1 \overline{\delta}^{\frac{1}{2}} \end{aligned} \tag{44}$$

and also that

$$\begin{aligned} F_1(q_1, \dot{y}) \geq 0, \\ F_2(q_1, \dot{y}) \geq 0 \end{aligned} \tag{45}$$

Now, let's define the unknown input observation error

$$E_\xi = \begin{bmatrix} \overline{e}_\xi \\ \underline{e}_\xi \end{bmatrix} \tag{46}$$

and let's introduce the following notations for clarity of presentation:

$$\Delta = \begin{bmatrix} \overline{\delta} \\ \overline{\delta} \end{bmatrix}, \dot{\Delta} = \begin{bmatrix} \overline{\dot{\delta}} \\ \overline{\dot{\delta}}, \end{bmatrix}, U = \begin{bmatrix} \overline{u} - \underline{u} \\ \overline{u} - \underline{u} \end{bmatrix},$$

$$\Omega = \begin{bmatrix} \overline{\omega} \\ \overline{\omega} \end{bmatrix}, \Theta = \begin{bmatrix} \mu_1 \overline{\delta}^{\frac{1}{2}} \\ \mu_1 \overline{\delta}^{\frac{1}{2}} \end{bmatrix}.$$

Then, (41) can be rewritten under a compact form as:

$$E_\xi \leq N_1 E_z + 2(\overline{\overline{N}}_2 U + \overline{\overline{N}}_3 \Omega + \overline{\overline{N}}_4 \dot{\Delta} + \overline{\overline{N}}_5 \Theta) \tag{47}$$

where

$$\begin{aligned} N_1 &= \begin{bmatrix} M_1^+ & -M_1^- \\ -M_1^- & M_1^+ \end{bmatrix}, N_2 = M_2^+ - M_2^-, \\ N_3 &= M_3^+ - M_3^-, N_4 = M_0^+ - M_0^- + M_4^+ - M_4^-, \\ N_5 &= M_0^+ - M_0^- \end{aligned}$$

By construction, the matrices N_i, $i = 1, \ldots, 5$ are nonnegative elementwise. Then, based on (18) and on theorem 3, it follows from (47) that $E_\xi \geq 0$ and property (38) holds.

Using the expression (30) with (47), we deduce that the boundedness of E_z implies the boundedness of E_ξ. Then, the total nonnegative error $\overline{e}_\xi - \underline{e}_\xi$ is asymptotically elementwise lower than the nonnegative vector:

$$\rho_\xi = -N_1 \overline{\overline{J}}^{-1} \overline{\overline{H}}^* + 2(\overline{\overline{N}}_2 U + \overline{\overline{N}}_3 \Omega + \overline{\overline{N}}_4 \dot{\Delta} + \overline{\overline{N}}_5 \Theta) \tag{48}$$

Then, property (39) holds. □

5. ILLUSTRATIVE EXAMPLE : FIXED-WING AIRCRAFT MODEL

In this section, some simulation results are presented to illustrate the proposed methodology. The case study corresponds to a 5th-order lateral axis model of a fixed-wing aircraft at cruise flight conditions taken from Edwards and Spurgeon [1998]. The actuator dynamics are neglected in this example. Moreover, it is assumed that the inputs are all unknown (no known inputs).

5.1 Observer design

Given the system (1) with:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -0.154 & -0.0042 & 1.54 & 0 \\ 0 & 0.249 & -1 & -5.2 & 0 \\ 0.0386 & -0.996 & -0.0003 & -0.117 & 0 \\ 0 & 0.5 & 0 & 0 & -0.5 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ -0.744 & -0.0320 \\ 0.337 & -1.12 \\ 0.02 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The initial state is

$$x_0 = [0.342 \ 0.32 \ 0.0178 \ -0.287 \ -0.9497]^T.$$

The state and measurement errors are given by some uniformly bounded noises such that $\delta(t) \in [-\overline{\delta}, \overline{\delta}]$ with $\overline{\delta} = [1e^{-2}, 1e^{-2}, 1e^{-2}, 1e^{-2}]^T$ and $\omega(t) \in [-\overline{\omega}, \overline{\omega}]$ with $\overline{\omega} = [1e^{-2}, 1e^{-2}, 1e^{-2}, 1e^{-2}, 1e^{-2}]^T$, $\forall t \geq 0$. Assume also that $\dot{\delta}(t) \in [-\overline{\dot{\delta}}, \overline{\dot{\delta}}]$ with $\overline{\dot{\delta}} = [1e^{-2}, 1e^{-2}, 1e^{-2}, 1e^{-2}]^T$. The unknown inputs are simulated by $\xi_1 = cos(t)$ and $\xi_2 = sin(t)$.
We consider here the same inputs as in Edwards and Spurgeon [1998], but here, to illustrate the proposed methodology, the input (deterministic) signals are assumed to be unknown. This assumption is made just to illustrate the mechanisation equations illustrative context. Note that, generally, the unknown inputs are not deterministic in nature, the only deterministic knowledge that can be available is on their bounds. Note also that in some situations, the unknown input could be harmonic signals which should be detected, for example the Oscillatory Failure Case in aircraft control surface servoloops Zolghadri et al. [2013].

The constants of the HOSM differentiator (3) are taken as $s = 1$, $\lambda_j^i = 100$ with $i = 1, \ldots, 4$ and $j = 0, 1$ and the parameter μ used in (8) is chosen as $\mu_1 = 1.1$. In Levant [2003], some numerical values are presented for the parameters of the HOSM differentiator that should work for the 6 first derivatives. In our work, only the μ_i can be obtained from experiments.

Hypothesis 4 is verified since $rank(CD) = rank(D) = 2$ and the output vector dimension is lower than the unknown input vector dimension. The matrices in hypothesis 5 are given by

$$W = 0_{5,4},\ E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 9.993e^{-1} & -2.011e^{-2} & 2.652e^{-2} & 0 \\ -2.011e^{-5} & -1 & -7.577e^{-4} & 0 \\ 2.652e^{-2} & -7.577e^{-4} & -7.044e^{-4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 7.038e^{-4} & 2.011e^{-5} & 2.652e^{-2} & 9.993e^{-1} \\ 0 & -2.011e^{-5} & 5.745e^{-7} & 7.577e^{-4} & 2.011e^{-5} \\ 0 & 2.652e^{-2} & -7.577e^{-4} & 9.993e^{-1} & -2.652e^{-2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The pair (PA, C) is observable, hence, hypothesis 5 is verified.

Based on hypothesis 6, a pole placement $\{-5,\ -6,\ -4,\ -3,\ -2,\ -1\}$ is chosen that leads to :

$$K = \begin{bmatrix} -6.30e^{-1} & 3.23 & -6.42e^{-4} & -8.46e^{-1} & -6.86e^{-1} \\ 1 & 6.37e^{-3} & 6 & 2.08e^{-4} & 6.29e^{-3} \\ -1.70e^{-1} & -8.40 & -8.41e^{-5} & 6.11 & -8.29 \\ 3.88 & 2.66 & 6.33e^{-4} & -8.35e^{-1} & 1.40 \end{bmatrix}.$$

The state transformation (12) ensuring the cooperativity property is given by

$$T = \begin{bmatrix} 4.02e^{-1} & 1.35 & 4.18e^{-3} & -5.51 & -1.42 \\ -7.90e^{-1} & 5.07e^{-1} & 8.13e^{-5} & -1.07e^{-1} & -5.38e^{-1} \\ -1.97e^{-1} & -1.24 & -8.24e^{-4} & 1.09 & 1.45 \\ 1.18 & 2.06 & -3.56e^{-3} & 4.70 & -8.73e^{-1} \\ 0 & 0 & 1 & 7.58e^{-4} & 0 \end{bmatrix}.$$

The matrices in (14) are deduced and the interval observer described by (15) is designed. It is assumed that $\overline{x}_0 = [1\ 1\ 1\ 1\ 1]^T$, $\underline{x}_0 = [-1\ -1\ -1\ -1\ -1]^T$. The initial observer state is given by $\overline{z}_0 = T^+\overline{x}_0 + T^-\underline{x}_0$ and $\underline{z}_0 = T^+\underline{x}_0 + T^-\overline{x}_0$. Finally, the unknown inputs bounds are deduced from (37).

5.2 Simulation results

The simulation results are presented with a sampling time $T_e = 1e^{-4}s$. The lower and upper state bounds are depicted in figures 1, 2, 3, 4 and 5. Furthermore, lower and upper unknown inputs bounds are illustrated in figures 6 and 7. The proposed interval observer converges asymptotically and gives accurate system state and unknown input bounds even under disturbances and noises presence.

6. CONCLUDING REMARKS

The problem studied in this paper is that of interval estimation of state and unknown input for LTI continuous-time systems. Set membership unknown input estimation can be very useful for instance in model-based prognosis. It enables to compute system fault evolution and to manage model uncertainties and environmental disturbances and their propagations. Basically, this can be achieved if the slow dynamic behaviour state representing the damage model can be predicted. The latter can be modelled as additional unknown inputs to the system. In this context, interval unknown input estimation can be an interesting solution to this problem.

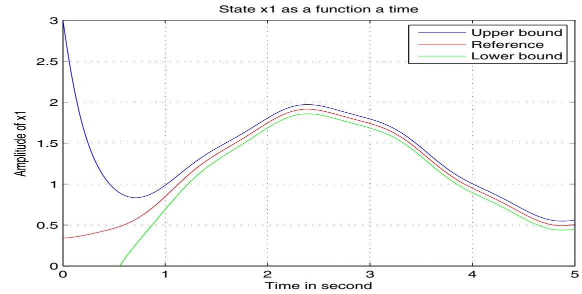

Fig. 1. Lower and upper bounds of state x_1

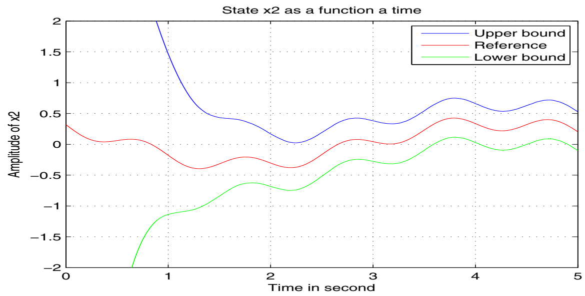

Fig. 2. Lower and upper bounds of state x_2

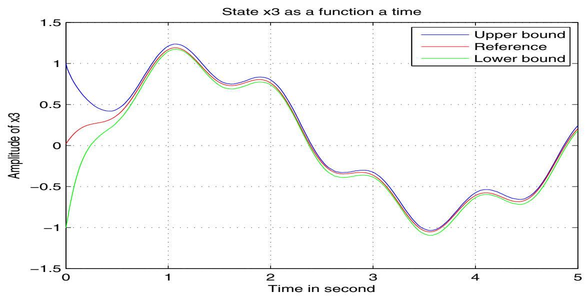

Fig. 3. Lower and upper bounds of state x_3

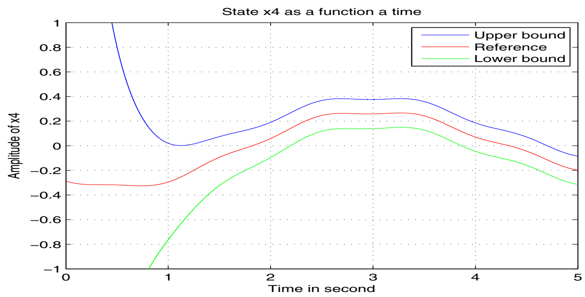

Fig. 4. Lower and upper bounds of state x_4

7. ACKNOWLEDGEMENT

This work is done within the MAGIC-SPS project (Guaranteed Methods and Algorithms for Integrity Control and Preventive Monitoring of Systems) funded by the

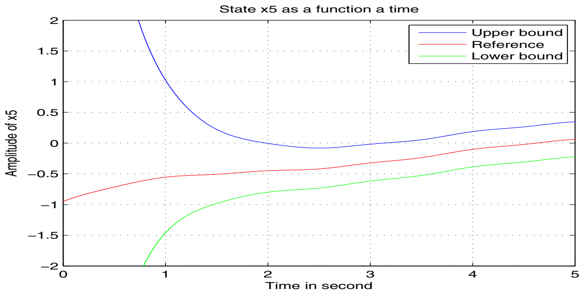

Fig. 5. Lower and upper bounds of state x_5

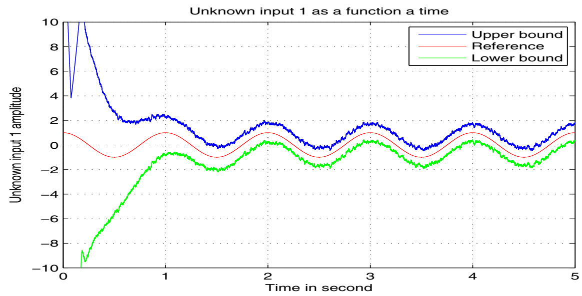

Fig. 6. Lower and upper bounds of unknown input ξ_1

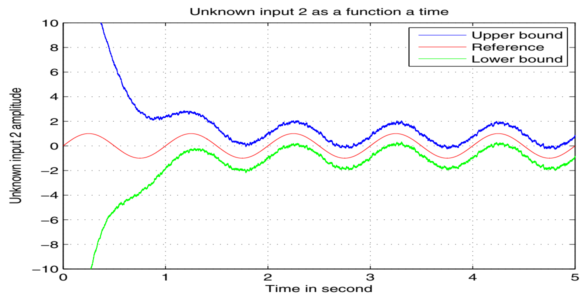

Fig. 7. Lower and upper bounds of unknown input ξ_2

French National Research Agency (ANR) under the decision ANR2011-INS-006.

REFERENCES

Alamo, T., Bravo, J., and Camacho, E. (2005). Guaranteed state estimation by zonotopes. *Automatica*, 41(6), 1035–1043.

Bernard, O. and Gouzé, J. (2004). Closed loop observers bundle for uncertain biotechnological models. *Journal of Process Control*, 14, 765–774.

Chisci, L., Garulli, A., and Zappa, G. (1996). Recursive state bounding by parallelotopes. *Automatica*, 32(7), 1049 – 1055.

Combastel, C. (2013). Stable interval observers in $\mathbb{C}$ for linear systems with time-varying input bounds. *IEEE Transactions on Automatic Control*, 58(2), 481–487.

Darouach, M., Zasadzinski, M., and Xu., S. (1994). Full-order observer for linear systems with unknown inputs. *IEEE Transactions on Automatic and Control*, 39(3), 606–609.

Edwards, C. and Spurgeon, S. (1998). *Sliding mode control: theory and applications*. Taylor & Francis.

Efimov, D., Fridman, L., Raissi, T., Zolghadri, A., and Seydou, R. (2012). Interval estimation for {LPV} systems applying high order sliding mode techniques. *Automatica*, 48(9), 2365 – 2371.

Fairman, F., Mahil, S., and Luk, L. (1984). Disturbance decoupled observer design via singular value decomposition. *IEEE Trans. Automat. Contr.*, 29, 84–86.

Gouzé, J., Rapaport, A., and Hadj-Sadok, M. (2000). Interval observers for uncertain biological systems. *Ecological Modelling*, 133, 46–56.

Guerra, P., Puig, V., and Witczak, M. (2008). Robust fault detection with unknown-input interval observers using zonotopes. 17th IFAC World Congress, Seoul, Korea.

Hou, M. and Müller, P. (1992). Design of observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, 37, 871–875.

Jaulin, L. (2002). Nonlinear bounded-error state estimation of continuous time systems. *Automatica*, 38(2), 1079–1082.

Kieffer, M. and Walter, E. (2006). Guaranteed nonlinear state estimation for continuous-time dynamical models from discrete-time measurements. ROCOND, Toulouse.

Kletting, M., Rauh, M., Aschemann, H., and Hofer, E. (2006). Interval observer design for nonlinear systems with uncertain time-varying parameters. 12th IEEE International Conference on Methods and Models in Automation and Robotics, Miedzyzdroje, Poland.

Levant, A. (1980). Observers for linear systems with unknown inputs. *IEEE Trans. Automat. Contr.*, 25, 113–115.

Levant, A. (2003). Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, 76(9), 924 – 941.

Mazenc, F. and Bernard, O. (2011). Interval observers for linear time-invariant systems with disturbances. *Automatica*, 47(1), 140 – 147.

Moisan, M., Bernard, O., and Gouzé, J. (2009). Near optimal interval observers bundle for uncertain bioreactors. *Automatica*, 45(1), 291–295.

Raïssi, T., Efimov, D., and Zolghadri, A. (2012). Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1), 260–265.

Raïssi, T., Ramdani, N., and Candau, Y. (2004). Set membership state and parameter estimation for systems described by nonlinear differential equations. *Automatica*, 40(10), 1771–1777.

Rapaport, A. and Gouzé, J. (2002). Parallelotopic and practical observers for nonlinear uncertain systems. *Int. Journal of Control*, 76, 237–251.

Smith, H. (1995). Monotone dynamical systems : An introduction to the theory of competitive and cooperative systems. In *Mathematical Surveys and Monographs*, volume 41. Providence, Rhode Island.

Zolghadri, A., Henry, D., Cieslak, J., Efimov, D., and Goupil, P. (2013). *Fault Diagnosis and Fault-Tolerant Control and Guidance for Aerospace Vehicles, from theory to application.* Springer, Series: Advances in Industrial Control.