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Decentralised Observation Using Higher Order Sliding Mode Techniques

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Abstract: In this paper, a decentralised observer scheme is proposed for a class of nonlinear interconnected systems based on higher order sliding mode techniques. It is not required that the nominal subsystems or the isolated nominal subsystems are linearizable. Under the assumption that the isolated nominal subsystems have uniform relative degree, local coordinate transformations are used to transform these systems to a new interconnected system which facilitates higher order sliding mode design. Observers composed of a set of decoupled dynamical systems are designed such that the observation error dynamical systems converge to zero exponentially. The designed observer is continuous and chattering can be avoided. The configuration is convenient for practical implementation as it is a decentralised scheme.

Keywords: Higher order sliding mode, Interconnected system, decentralised observer, nonlinear systems.

1. INTRODUCTION

Interconnected systems are often characterised by geographical separation. Issues such as economic cost and reliability of communication links have to be considered thus providing impetus to use a decentralised scheme. Although many results concerning decentralised schemes for interconnected systems have been obtained (Bakule, 2008), results applying decentralised higher order sliding mode (HOSM) strategies to nonlinear interconnected systems are very limited. Moreover, the state variables of a system are often incompletely known and thus control schemes based on state feedback cannot be directly implemented. In order to exploit such control strategies, an appropriate estimate of the states may be constructed for use in the original control law. Therefore, observer design is an important problem in practice (Spurgeon, 2008; Yan et al., 2013).

Unlike decentralised control, contributions to the development of decentralised observation schemes are very rare because each observer dynamical subsystem cannot employ information from the other subsystems, which may produce complex network interactions in the observation error dynamics. The interactions existing in the observation error dynamics cannot be rejected by using only local information. Therefore, nearly all of the existing observation schemes for nonlinear interconnected systems are not decentralised. As early as 1977, Sundareshan (1977) studied decentralised observation where it is clearly stated that the observers should be decoupled in a decentralised scheme. A class of linear interconnected systems are considered in Sundareshan (1977) and Pillosu et al. (2011) where the proposed observers are coupled together through networked interconnections and thus are not decentralised. Later, for a class of nonlinear interconnected systems, a variable structure observer is proposed in Yan et al. (2003) and a first order sliding mode observer is proposed in Yan and Edwards (2008b) to enhance robustness to uncertainties. However, in both Yan and Edwards (2008b) and Yan et al. (2003), interconnection terms are employed in the observer design and thus the designed observers are not decentralised. Decentralised observers are proposed for linear large scale systems in Gopal and Ghodekar (1985) where it is required that the interconnections can be reconstructed by dynamical systems. Linear systems with nonlinear interconnections are studied in Tlili and Braiek (2009) where all the nonlinear terms are considered as uncertainties and the designed observer is not decentralised. Although observer design for interconnected systems has been studied using sliding mode techniques (Lin and Wang (2010)), methods for decentralised sliding mode observation appear an open research question.

Note that observers based on first order sliding mode (Spurgeon (2008); Yan and Edwards (2008a)) require that the considered system is relative degree one (interconnections and nonlinear terms may be considered as unknown inputs). This restriction can be removed using higher order sliding mode differentiators (see, for example, Fridman et al. (2011)). Step-by-step vector-state reconstruction using super-twisting first order robust exact differentiators has been presented in Floquet and Barbot (2007) in which the systems are transformed to a triangular or the Brunovsky form and then the states are estimated based on the equivalent output error injection. These observers theoretically ensure finite time convergence for all system states. Although super-twisting first order robust exact differentiators provide the best possible asymptotic accuracy of the derivative estimation at each single realization step (Levant (1998)), the accuracy is proportional to the sampling step δ for discrete realisation in the absence of noise, and to the square root of input noise magnitude if discretisation error is negligible. Since the step-by-step and hierarchical observers take the output of super-twisting algorithm as noisy input at the next step, the overall observation accuracy is of the order $\delta^{1/2^{r-1}}$ where r is the observability index of the system. Similarly in the presence of measurement noise with magnitude ϵ, the estimation accuracy is proportional to $\epsilon^{1/2^r}$ which requires magnitude of measurement noises less than 10^{-16} for a fourth-order observer implementation to achieve an accuracy of 10^{-1}.

HOSM differentiators developed in Levant (2003) for exact observation design for nonlinear systems with unknown inputs has been employed in Fridman et al. (2008). The observation scheme proposed in Fridman et al. (2008) is based on two steps: i) transformation of the system to the Brunovsky canonical form and ii) the application of HOSM differentiators for each component of the output error vector. The proposed scheme ensures exact finite-time state estimation of the observable variables and asymptotic exact estimation of the unobservable variables for the case when the system has stable internal dynamics. This work has shown that the estimation accuracies increase to d and $\epsilon^{1/r}$ respectively.

In this paper, HOSM observer is employed, for the first time, in decentralised observer design for a class of nonlinear interconnected systems. This paper can be considered as an initial exploration of the problem of decentralised observation for nonlinear interconnected systems using HOSM differentiator. It should be emphasised that both the isolated subsystems and the interconnections are nonlinear in the considered systems. The main contribution of this paper is that the designed observer is completely decentralised, which is more convenient for real implementation than a centralised scheme. The use of HOSM differentiator provides the best possible asymptotic accuracy in terms of noise and discretisation step. Under the assumption that the isolated subsystems have uniform relative degree, a nonlinear geometric coordinate transformation is employed to explore the structure of the isolated systems. Then, the considered systems are transformed to new nonlinear interconnected systems which facilitates the application of the HOSM technique. A set of sufficient conditions is developed to guarantee the convergence of the designed observer. The estimation is split into two parts: one part converges in finite time and the other exponentially. The designed observer is continuous thereby preserving the advantages of sliding mode control and largely eliminating the chattering.

Notation: In this paper, $\mathcal{R}^+$ denotes the nonnegative set of real numbers $\{t \mid t \geq 0\}$. For a square matrix $A \in \mathcal{R}^{n\times n}$, the expression $A > 0$ represents that the matrix A is symmetric positive definite and the symbol $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) represents its maximum (minimum) eigenvalue. Suppose function $g : \mathcal{R}^n \mapsto \mathcal{R}$ is differentiable, and $f := (f_1(\cdot), f_2(\cdot), \cdot, f_n(\cdot))^T : \mathcal{R}^n \mapsto \mathcal{R}^n$. The notation $L_f g(x)$ denotes the derivative of $g(x)$ along f defined by $L_f g(x) := \sum_{i=1}^n \frac{\partial g}{\partial x} f(x)$ and $L_f^k g(x)$ represents a recursion defined by $L_f^k g(x) := \frac{\partial L_f^{k-1} g}{\partial x} f(x)$ with $L_f^0 := g(x)$. For vectors $x = (x_1, x_2, \ldots, x_{n_1})^T \in \mathcal{R}^{n_1}$ and $y = (y_1, y_2, \ldots, y_{n_2})^T \in \mathcal{R}^{n_2}$, the expression $f(x, y)$ denotes a function $f(x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2})$ defined on $\mathcal{R}^{n_1+n_2}$. Finally, $\|\cdot\|$ denotes the Euclidean norm or its induced norm.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider a nonlinear interconnected system composed of n n_i-th order subsystems described by

$$\dot{x}_i = f_i(x_i) + g_i(x_i)\left(u_i + \xi_i(t, x_i)\right) + \psi_i(x) \tag{1}$$

$$y_i = h_i(x_i), \qquad i = 1, 2, \ldots, n, \tag{2}$$

where $x := \mathrm{col}(x_1, \ldots, x_n) \in \mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$, and $x_i \in \mathcal{X}_i \subset \mathcal{R}^{n_i}$ ($\mathcal{X}_i$ is a neighborhood of the origin), $u_i \in \mathcal{U}_i \subset \mathcal{R}^{m_i}$ and $y_i \in \mathcal{Y}_i \subset \mathcal{R}^{m_i}$ are state variables, inputs and outputs of the i-th subsystem respectively. The terms $\xi_i(\cdot)$ represent the uncertainty in the input channel of the i-th subsystems which satisfy

$$\|\xi_i(t, x_i)\| \leq \gamma_{\xi_i}, \qquad i = 1, 2, \ldots, n$$

for some positive constants γ_{ξ_i}. The terms $\psi_i(x)$ are interconnections of the i-th subsystem. The function matrices $g_i(x_i) := [g_{i1}(x_i), g_{i2}(x_i), \ldots, g_{im_i}(x_i)]$ describe the input distribution, and $h_i(x_i) := [h_{i1}(x_i), h_{i2}(x_i), \cdots, h_{im_i}(x_i)]$ the output distribution. All the vector fields $f_i(x_i) \in \mathcal{R}^{n_i}$, $h_i(x_i) \in \mathcal{R}^{m_i}$ and $g_{il}(\cdot) \in \mathcal{R}^{n_i}$ are assumed to be smooth enough for $i = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, m_i$. It is assumed that all the control signals are bounded, that is, there exist constants γ_{u_i} such that $\|u_i\| \leq \gamma_{u_i}$, which is consistent with engineering practice.

Definition 1. Consider system (1)–(2). The system

$$\dot{x}_i = f_i(x_i) + g_i(x_i)(u_i + \xi_i(t, x_i))$$

$$y_i = h_i(x_i), \qquad i = 1, 2, \ldots, n,$$

is called the i-th isolated subsystem of system (1)–(2), and the system

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i \tag{3}$$

$$y_i = h_i(x_i), \qquad i = 1, 2, \ldots, n, \tag{4}$$

is called the i-th nominal isolated subsystem of system (1)–(2).

In this paper, the objective is to design n n_i-th order dynamical systems

$$\dot{\hat{x}}_i = \Phi_i(t, \hat{x}_i, y_i, u_i), \quad i = 1, 2, \ldots, n \tag{5}$$

where $\hat{x}_i \in \mathcal{R}^{n_i}$, such that the solutions $\hat{x}_i(t)$ of system (5) are convergent to $x_i(t)$ exponentially for $i = 1, 2, \ldots, n$, that is, there exist constants $\alpha_i > 0$ and $\beta_i > 0$ such that

$$\|x_i(t) - \hat{x}_i(t)\| \leq \alpha_i \exp\{-\beta_i t\}, \quad i = 1, 2, \ldots, n$$

where $x_i(t)$ are the solutions of the interconnected systems (1)–(2). The systems in (5) comprise an exponential observer for the interconnected system (1)–(2).

It should be noted that in the context of decentralised control, only local information x_i can be used if state feedback is considered, and only local output information y_i can be used for the case of output feedback. Now, consider the dynamical observer (5). The i-th dynamical system in (5) is only dependent on the time t and the local information $\hat{x}_i$, y_i and u_i, and does not involve states x_j, inputs u_j or outputs y_j $(j \neq i)$ of the other dynamical systems. Thus, the n dynamical systems in (5) are clearly decoupled from each other. Such a set of dynamical systems (5) is called a decentralised observer for the system (1)–(2). Such a decentralised observation scheme is illustrated in Figure 1 by using an interconnected network with 3 subsystems.

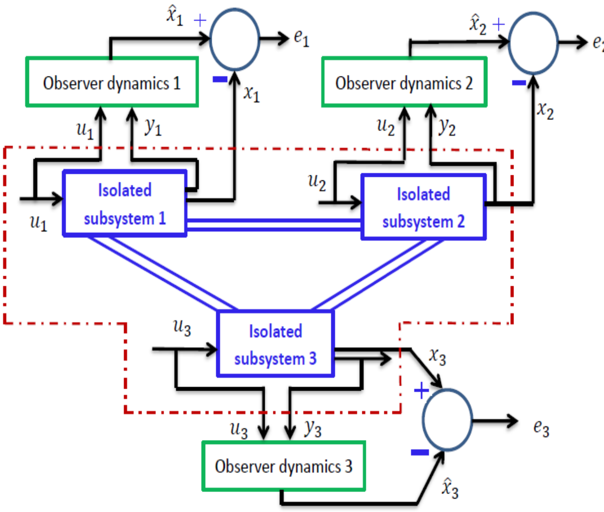

Fig. 1. Block-diagram of the proposed decentralised observation scheme

From Figure 1, it is clear that the inputs of the i-th observer dynamics are u_i and y_i and the output is $\hat{x}_i$. The figure clearly shows the decentralised nature of the framework.

3. BASIC ASSUMPTIONS AND STRUCTURE ANALYSIS

Consider the interconnected system (1)–(2). Note that the input function matrices $g_{i1}(x_i), g_{i2}(x_i), \dots\ g_{im_i}(x_i)$ are smooth in the domain $\mathcal{X}_i$ for $i = 1, 2, \dots, n$. The distributions generated by $g_{i1}, g_{i2}, \dots, g_{im_i}$ are smooth in $\mathcal{X}_i$ and are denoted by

$$\mathcal{G}_i(x_i) := \text{span}\{g_{i1}(x_i), g_{i2}(x_i), \dots, g_{im_i}(x_i)\} \qquad (6)$$

for $i = 1, 2, \dots, n$. The following assumptions are imposed on the system (1)–(2).

Assumption 1. The i-th nominal isolated subsystem (3)–(4) has a uniform relative degree vector

$$(\rho_{i1}, \rho_{i2}, \cdots, \rho_{im_i})$$

and the distribution $\mathcal{G}_i(x_i)$ is involutive in the domain $\mathcal{X}_i$ for $i = 1, 2, \dots, n$.

Let $\rho_i := \sum_{j=1}^{m_i} \rho_{ij}$ for $i = 1, 2, \dots n$. From the definition of relative degree, ρ_{ij} are nonnegative constants and $\rho_i \leq n_i$ (see, Isidori (1995)). Under Assumption 1, the differentials $dh_{ij}(x_i)$, $dL_{f_i}h_{ij}(x_i)$, $\cdots$, $dL_{f_i}^{\rho_{ij}-1}h_{ij}(x_i)$ are linearly independent for $j = 1, 2, \dots, m_i$ and $i = 1, 2, \dots, n$. Let

$$z_{ij} = \begin{bmatrix} h_{ij}(x_i) \\ L_{f_i}h_{ij}(x_i) \\ \vdots \\ L_{f_i}^{\rho_{ij}-1}h_{ij}(x_i) \end{bmatrix} := z_{ij}(x_i), \quad j = 1, 2, \dots \rho_i \qquad (7)$$

for $i = 1, 2, \dots, n$. Since the distribution $\mathcal{G}_i(x_i)$ with $i = 1, 2, \dots, n$ is involutive, there always exist $n_i - \rho_i$ functions w_{i1}, w_{i2}, $\cdots$, $w_{i(n_i-\rho_i)}$ defined in $\mathcal{X}_i$ such that the Jacobian matrix of the mapping

$$T_i : x_i \mapsto \text{col}(z_{i1}, \cdots, z_{i\rho_i}, w_{i1}, \cdots, w_{i(n_i-\rho_i)}) \qquad (8)$$

is nonsingular in $\mathcal{X}_i$. Thus the transformations $\text{col}(z_i, w_i) = T_i(x_i)$ defined by

$$T_i : \begin{cases} z_{i1} = z_{i1}(x_i) \\ z_{i2} = z_{i2}(x_i) \\ \quad\vdots \\ z_{i\rho_i} = z_{i\rho_i}(x_i) \\ w_i = w_i(x_i) \end{cases}$$

are diffeomorphisms in $\mathcal{X}_i$, where $z_{ij} \in \mathcal{R}^{\rho_{ij}}$ with $j = 1, 2, \dots, \rho_{im_i}$ are defined in (7) and $w_i := \text{col}(w_{i1}(x_i), w_{i2}(x_i), \cdots, w_{i(n_i-\rho_i)})$ for $i = 1, 2, \dots, n$. Let

$$T(x) := \begin{pmatrix} T_1(x_1) \\ T_2(x_2) \\ \vdots \\ T_n(x_n) \end{pmatrix} \qquad (9)$$

It is clear that $T(x)$ defines a new coordinate system $\text{col}(z_1, w_1, z_2, w_2, \cdots, z_n, w_n)$. Let $\mathcal{Z}_i \times \mathcal{W}_i := T_i(\mathcal{X}_i)$ where $\mathcal{Z}_i \in \mathcal{R}^{\rho_i}$ and $\mathcal{W}_i \in \mathcal{R}^{n_i-\rho_i}$ for $i = 1, 2, \dots, n$.

Assumption 2. The interconnection terms $\psi_i(x)$ satisfy the following, for any $x \in \mathcal{X}$, $j = 1, 2, \dots, m_i$ and $i = 1, 2, \dots, n$,

i)

$$L_{\psi_i(x)}h_{ij}(x_i) = 0 \qquad (10)$$

$$L_{\psi_i(x)}L_{f_i(x_i)}h_{ij}(x_i) = 0 \qquad (11)$$

$$\cdots\cdots$$

$$L_{\psi_i(x)}L_{f_i(x_i)}^{\rho_{ij}-2}h_{ij}(x_i) = 0 \qquad (12)$$

ii) there exist constants γ_{ψ_i} such that for any $x \in \mathcal{X}$

$$\left\| L_{\psi_i(x)}L_{f_i(x_i)}^{\rho_{ij}-1}h_{ij}(x_i) \right\| \leq \gamma_{\psi_i}, \qquad (13)$$

iii) $\left[\frac{\partial T_i(x_i)}{\partial x_i}\psi_i(x)\right]_{x_i = T_i^{-1}(z_i, w_i)} = \begin{bmatrix} \star \\ \Phi_i(z_i, w_i) \end{bmatrix}$ where $\Phi_i(\cdot) \in \mathcal{R}^{(n_i - \sum_{j=1}^{m} \rho_{ij})}$ are Lipschitz with respect to the variables w_i in $\mathcal{W}_i$ uniformly for $z_i \in \mathcal{Z}_i$ and $\star$ represents uninterested entries of appropriate dimension.

Remark 1 From the fact that $\Phi_i(\cdot)$ are Lipschitz in the condition iii) in Assumption 2, it follows that for any $w_i \in \mathcal{W}_i$ and $\hat{w}_i \in \mathcal{W}_i$

$$\|\Phi_i(z_i, w_i) - \Phi_i(z_i, \hat{w}_i)\| \leq \mathcal{L}_{\Phi_i}(z_i)\|w_i - \hat{w}_i\| \qquad (14)$$

for $z_i \in \mathcal{Z}_i$ where $\mathcal{L}_{\Phi_i}(\cdot)$ are the generalised Lipschitz constants of the functions $\Phi_i(\cdot)$ with respect to w_i and uniformly for z_i for $i = 1, 2, \ldots, n$.

By direct computation, it follows that under Assumptions 1-2, the interconnected system (1)–(2) can be described by

$$\begin{aligned}
\dot{z}_{i1} &= A_{i1} z_{i1} + B_{i1}\big(u_{i1} + \eta_{ij}(t, z_i) \\
&\quad + L_{\psi_i(x)} L_{f_i(x_i)}^{\rho_{i1}-1} h_{i1}(x_i)\big) && (15)\\
\dot{z}_{i2} &= A_{i2} z_{i1} + B_{i2}\big(u_{i2} + \eta_{ij}(t, z_i) \\
&\quad + L_{\psi_i(x)} L_{f_i(x_i)}^{\rho_{i2}-1} h_{i2}(x_i)\big) && (16)\\
&\vdots \\
\dot{z}_{im_i} &= A_{im_i} z_{im_i} + B_{im_i}\big(u_{im_i} + \eta_{ij}(t, z_{im_i}) \\
&\quad + L_{\psi_i(x)} L_{f_i(x_i)}^{\rho_{im_i}-1} h_{im_i}(x_i)\big) && (17)\\
\dot{w}_i &= q_i(z_i, w_i) + \Phi_i(z_i, w_i) && (18)\\
y_{ij} &= C_{ij} z_{ij} && (19)
\end{aligned}$$

where $z_i := \text{col}\,(z_{i1}, z_{i2}, \cdots, z_{im_i})$ with $z_{ij} \in \mathcal{R}^{\rho_{ij}}$ and $w_i := \text{col}\,\big(w_{i1}, w_{i2}, \cdots, w_{i(n_i-\rho_i)}\big) \in \mathcal{R}^{n_i-\rho_i}$, the triples (A_{ij}, B_{ij}, C_{ij}) have the Brunovsky standard form as follows

$$A_{ij} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{\rho_{ij}\times\rho_{ij}}, \quad B_{ij} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\rho_{ij}\times 1} \tag{20}$$

$$C_{ij} = [1 \quad 0 \quad \cdots 0]_{1\times\rho_{ij}} \tag{21}$$

for $j = 1, 2, \ldots, m_i$, and

$$\begin{bmatrix} \eta_{i1}(t, z_i) \\ \vdots \\ \eta_{im_i}(t, z_i) \end{bmatrix} := [\xi_i(t, x_i)]_{x_i = T_i^{-1}(z_i, w_i)} \tag{22}$$

where $\eta_{il}(\cdot) \in \mathcal{R}$ for $l = 1, 2, \ldots, m_i$ and the interconnection terms $\Phi_i(\cdot)$ is defined in iii) in Assumption 2 for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, \rho_{m_i}$.

Remark 2. It should be noted that the interconnected system (15)–(18) is the expression of the system (1)–(2) in the new coordinate system $\text{col}(z_1, w_1, z_2, w_2, \cdots, z_n, w_n)$. Saif and Guan (1992) pointed out that if interconnections are not used in the observer design, the observer performance is usually unsatisfactory and the convergence of the observation error dynamics cannot be guaranteed. Here, in order to get a completely decentralised scheme, the conditions on the interconnection as given in Assumption 2 are required to guarantee the exponential stability of the observation error dynamics.

Remark 3. The functions w_{ij} for $j = 1, 2, \ldots, n_i - \rho_i$ with $i = 1, 2, \ldots, n$ in (8) can be obtained by solving the partial deferential equation $L_{g_i} w_{ij}(x_i) = 0$ for $j = 1, 2, \ldots, n_i - \rho_i$ and $i = 1, 2, \ldots, n$. Once w_{ij} is obtained, then the coordinate transformation $z = T(x)$ is available and thus system (15)-(19) is well defined. Specifically, if $\rho_i = n_i$, then the coordinate transformation can be directly obtained from (7) and in this case, the nominal isolated subsystem is linearisable.

4. HOSM-BASED DECENTRALISED OBSERVER DESIGN

In this section, a hybrid decentralised sliding mode observer will be presented using HOSM techniques. Consider system (15)–(19). The i-th subsystem of (15)-(17) can be written as

$$\begin{aligned}
\dot{z}_{ij} &= A_{ij} z_i + B_{ij}\big(u_{ij} + \eta_{ij}(t, z_i) \\
&\quad + L_{\psi_i(x)} L_{f_i(x_i)}^{\rho_{ij}-1} h_{ij}(x_i)\big), && (23)\\
y_{ij}(t) &= z_{ij1}(t) := h_{ij}(x_i(t)) && (24)
\end{aligned}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m_i$, where $z_{ij} := \text{col}(z_{ij1}, z_{ij2}, \cdots, z_{ij\rho_{ij}}) \in \mathcal{R}^{\rho_{ij}}$, $y_i := \text{col}(y_{i1},\ y_{i2},\ \cdots,\ y_{im_i})$ and the pairs (A_{ij}, B_{ij}) are defined in (20). Since u_i is bounded, it follows from (13) in Assumption 2 that there exist constants $L_i > 0$ such that the inequalities

$$\left| u_{ij} + \eta_{ij}(t, z_i) + L_{\psi_i(x)} L_{f_i(x_i)}^{\rho_{ij}-1} h_{ij}(x_i) \right| \le L_i,$$

hold in $x \in \mathcal{X}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m_i$.

For the i-th subsystem (23)–(24), consider the higher order sliding surfaces defined by

$$s_{ij} = L_{g_i} s_{ij} = L_{g_i}^2 s_{ij} = \cdots = L_{g_i}^{\rho_{ij}} s_{ij} = 0$$

where the scalar $s_{ij}(t) := \hat{z}_{ij1}(t) - y_{ij}(t)$. The variables y_{ij} represent the j-th component of the outputs of the i-th subsystem, and the scalar $\hat{z}_{ij1}$ is determined by the following HOSM differentiator algorithm from Levant (2003)

$$\begin{aligned}
\dot{\hat{z}}_{ij1} &= \nu_{ij1} && (25)\\
\nu_{ij1} &= -\lambda_{ij1} \left|s_{ij}\right|^{\frac{\rho_{ij}-1}{\rho_{ij}}} \text{sign}(s_{ij}) + z_{ij1} && (26)\\
\dot{\hat{z}}_{ij2} &= \nu_{ij2} && (27)\\
\nu_{ij2} &= -\lambda_{ij2} \left|\hat{z}_{ij2} - \nu_{ij1}\right|^{\frac{\rho_{ij}-2}{\rho_{ij}-1}} \text{sign}(z_{ij1} \\
&\quad -\nu_{ij0}) + z_{ij2} && (28)\\
&\vdots \\
\dot{\hat{z}}_{ij(\rho_{ij}-1)} &= \nu_{ij(\rho_{ij}-1)} && (29)\\
\nu_{ij(\rho_{ij}-1)} &= -\lambda_{ij(\rho_{ij}-1)} \left|\hat{z}_{ij(\rho_{ij}-1)} - \nu_{ij(\rho_{ij}-2)}\right|^{\frac{1}{2}} \\
&\quad \text{sign}(z_{ij(\rho_{ij}-1)} - \nu_{ij(\rho_{ij}-2)}) + z_{ij\rho_{ij}} && (30)\\
\dot{\hat{z}}_{ij\rho_{ij}} &= -\lambda_{ij\rho_{ij}} \text{sign}(\hat{z}_{ij\rho_{ij}} - \nu_{ij(\rho_{ij}-1)}) && (31)
\end{aligned}$$

where λ_{ijk} are positive parameters for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m_i$ and $k = 1, 2, \ldots, \rho_{ij}$.

From Levant (2003), it follows that by choosing appropriate parameters λ_{ijk}, z_{ijk} will converge to the $k-th$ derivative of $y_{ij}(t)$, $y_{ij}^{(k)}$ in finite time T_{ij}.

Choose $T_0 > T_{ij}$ and let $\hat{z}_{ij} := \text{col}\,\big(\hat{z}_{ij1}, \hat{z}_{ij2}, \cdots, \hat{z}_{ij\rho_{ij}}\big)$ for $j = 1, 2, \ldots, m_i$ and $i = 1, 2, \ldots, n$. Considering the structure of (A_{ij}, B_{ij}) in (23)–(24), when $t \ge T_0$,

$$\hat{z}_{i1} = z_{i1}, \quad \hat{z}_{i2} = z_{i2}, \quad \cdots, \quad \hat{z}_{im_i} = z_{im_i}$$

for $i = 1, 2, \ldots n$.

Consider system (15)–(17). The analysis above shows that $\hat{z}_{ij}$ produced by the differentiator (25)–(31), is an estimate of z_{ij}. The objective now is to estimate the variables w_i. The following assumptions are imposed on (18).

Assumption 3 The nonlinear functions $q_i(z_i, w_i)$ satisfy the following, for $i = 1, 2, \ldots, n$

i) $q_i(z_i, w_i)$ are Lipschitz with respect to the variables w_i in $\mathcal{W}_i$ uniformly for $z_i \in \mathcal{Z}_i$;
ii) there exist $P_i > 0$ ($P_i \in \mathcal{R}^{(n_i-\rho_i)\times(n_i-\rho_i)}$) and positive functions $k_i(z_i)$ such that for any varaibles $\vartheta_i \in \mathcal{R}^{n_i-\rho_i}$, $z_i \in \mathcal{Z}_i$ and $w_i \in \mathcal{W}_i$
$$\vartheta_i^T P_i \frac{\partial q_i(z_i, w_i)}{\partial w_i} \vartheta_i \le -k_i(z_i)\|\vartheta_i\|^2 \tag{32}$$
where $\frac{\partial q_i(z_i,w_i)}{\partial w_i}$ denote the Jacobian matrices of $q_i(\cdot)$ with respect to the variables w_i.

Remark 4 The condition ii) in Assumption 2 and the condition i) in Assumption 3 are fundamental in the local case and will hold in any bounded compact set due to the smoothness of the associated nonlinear functions and the continuity of $q_i(z_i, w_i)$. The condition ii) of Assumption 3 has been employed by Tsinias (1989) and Jo and Seo (2000). If the matrix $\frac{\partial q_i(z_i,w_i)}{\partial w_i}$ at $\mathrm{col}(z_i, w_i) = 0$ is Hurwitz, then the condition ii) in Assumption 3 holds in a neighbourhood of the origin $\mathrm{col}(z_i, w_i) = 0$ (see, Jo and Seo (2000)).

Construct the following dynamical systems
$$\dot{\hat{w}}_i = q_i(\hat{z}_i, \hat{w}_i) + \Phi_i(\hat{z}_i, \hat{w}_i), \quad i = 1, 2, \cdots, n \tag{33}$$
where $\hat{z}_i := (\hat{z}_{i11}, \hat{z}_{i12}, \cdots, \hat{z}_{i1\rho_{i1}}, \cdots, \hat{z}_{im_i1}, \cdots, \hat{z}_{im_i\rho_{i1}})$ and $\hat{z}_{ijk}$ are given by (25)–(31). Clearly the n systems defined in (33) are decoupled from each other. Let $e_i(t) = w_i(t) - \hat{w}_i(t)$. It follows from Assumption 2 that the error dynamics are described by
$$\dot{e}_i = q_i(z_i, w_i) - q_i(\hat{z}_i, \hat{w}_i) + \Phi_i(z_i, w_i) - \Phi_j(\hat{z}_i, \hat{w}_i) \tag{34}$$
for $i = 1, 2, \ldots, n$. The following result can be presented.

Theorem 1. Under Assumptions 1–3, the error dynamical system (34) is exponentially stable if for $i = 1, 2, \ldots, n$
$$\inf_{z_i} \Big\{ k_i(z_i) - \|P_i\| \, \mathcal{L}_{\Phi_i}(z_i) \Big\} := \beta_i > 0 \tag{35}$$
where P_i and $k_i(z_i)$ satisfy (32) and $\Phi_i(z_i)$ are given in Assumption 2.

Proof: Since the solutions of the system (1)–(2) are continuous, and the coordinate transformation defined by (9) is a diffeomorphism, the solutions of systems (18) and (33) are continuous. Therefore, the solutions to system (34) are continuous and thus $e_{ij}(t)$ are bounded in $t \in [0, T_0]$. This shows that system (34) has no finite escape time . Note that after $t \ge T_0$, $z_{ij}(t) = \hat{z}_{ij}(t)$. Now, consider the system (33) when $t \ge T_0$.

The analysis above shows that when $t \ge T_0$, $\hat{z}_i = z_i$. The error dynamical system (33) can be described by
$$\dot{e}_i = q_i(z_i, w_i) - q_i(z_i, \hat{w}_i) + \Phi_i(z_i, w_i) - \Phi_j(z_i, \hat{w}_i) \tag{36}$$
For system (36), consider the candidate Lyapunov function
$$V(e_1, e_2, \cdots, e_n) = \sum_{i=1}^{n} e_i^T(t) P_i e_i(t) \tag{37}$$
where $P_i > 0$ are defined in Assumption 3. Then, the time derivative of $V(\cdot)$ along the trajectories of system (36) is given by
$$\begin{aligned}\dot{V} &= 2\sum_{i=1}^{n} e_i^T P_i \left(q_i(z_i, w_i) - q_i(z_i, \hat{w}_i)\right) \\ &\quad + 2\sum_{i=1}^{n} e_i^T P_i \Big(\Phi_i(z_i, w_i) - \Phi_j(z_i, \hat{w}_i)\Big)\end{aligned} \tag{38}$$
From Assumption 3, it follows that for $i = 1, 2, \ldots, n$
$$\begin{aligned}&e_i^T P_i \left(q_i(z_i, w_i) - q_i(z_i, \hat{w}_i)\right) \\ &= e_i^T P_i \Big(\frac{\partial q_i(z_i, w_i + \theta_i(w_i - \hat{w}_i))}{\partial w_i} (w_i - \hat{w}_i) \Big) \\ &= e_i^T P_i \frac{\partial q_i(z_i, w_i + \theta_i e_i)}{\partial w_i} e_i \\ &\le -k_i(z_i)\|e_i\|^2\end{aligned} \tag{39}$$
where θ_i are scalar parameters relating to the variables e_i and satisfy $0 \le \theta_i \le 1$.

From (14)
$$\begin{aligned}&\sum_{i=1}^{n} e_i^T P_i \Big(\Phi_i(z_i, w_i) - \Phi_j(z_i, \hat{w}_i)\Big) \\ &\le \sum_{i=1}^{n} \|e_i^T\| \, \|P_i\| \, \mathcal{L}_{\Phi_i}(z_i) \, \|w_i - \hat{w}_i\| \\ &= \sum_{i=1}^{n} \|P_i\| \, \mathcal{L}_{\Phi_i}(z_i) \, \|e_i\|^2\end{aligned} \tag{40}$$
Now, substituting (39) and (40) into (38) yields
$$\begin{aligned}\dot{V} &\le -2\sum_{i=1}^{n} \Big(k_i(z_i) - \|P_i\| \, \mathcal{L}_{\Phi_i}(z_i)\Big) \|e_i\|^2 \\ &\le -2\beta \sum_{i=1}^{n} \|e_i\|^2 = -2\beta\|e(t)\|^2\end{aligned} \tag{41}$$
where $\min_i\{\beta_i\} := \beta > 0$ and $e(t) := \mathrm{col}(e_1(t), e_2(t), \ldots, e_n(t))$. From the definition of V in (37),
$$\begin{aligned}&\min_i\{\lambda_{\min}(P_i)\} \sum_{i=1}^{n} \|e_i\|^2 \le \lambda_{\min}(P_i)\|e_i(t)\|^2 \\ &\le V(e(t)) \le \sum_{i=1}^{n} \lambda_{\max}(P_i)\|e_i(t)\|^2 \\ &\le \max_i\{\lambda_{\max}(P_i)\} \sum_{i=1}^{n} \|e_i\|^2\end{aligned}$$
Therefore,
$$\min_i\{\lambda_{\min}(P_i)\}\|e\|^2 \le V(e(t)) \le \max_i\{\lambda_{\max}(P_i)\}\|e\|^2 \tag{42}$$

From (41) and (42), it follows that for $t \ge T_0$
$$\dot{V}(t) \le -\frac{2\beta}{\max_i\{\lambda_{\max}(P_i)\}} V(t)$$
which implies that
$$V(t) \le V(e(T_0)) \exp\{-\frac{2\beta}{\max_i\{\lambda_{\max}(P_i)\}}(t - T_0)\} \tag{43}$$

Therefore, from (43) and (42),
$$\|e(t)\| \le \sqrt{\frac{1}{\min_i\{\lambda_{\min}(P_i)\}} V(e(t))}$$

$$\leq \sqrt{\frac{V(e(T_0))}{\min\limits_i\{\lambda_{\min}(P_i)\}}} \exp\Big\{-\frac{\beta}{\max_i\{\lambda_{\max}(P_i)}(t-T_0)\Big\}$$

$$\leq \sqrt{\frac{\max\limits_i\{\lambda_{\max}(P_i)\}}{\min\limits_i\{\lambda_{\min}(P_i)\}}}\,\|e(T_0)\|$$

$$\cdot \exp\Big\{-\frac{\beta}{\max_i\{\lambda_{\max}(P_i)\}}(t-T_0)\Big\}$$

Hence, the error dynamical system (34) is exponentially stable. □

Remark 5 From the structure of the dynamical system (25)–(31) and the dynamical system (33), it is straightforward to see that the designed dynamics are decoupled and thus they are decentralised observers which compares favourably with the existing results (Sundareshan, 1977; Sundareshan and Elbanna, 1990; Pillosu et al., 2011; Yan and Edwards, 2008b; Yan et al., 2003).

Remark 6 From the analysis above, it is clear that the observable variables $\hat{z}_i$ converge to $z_i := \mathrm{col}\,(z_{i1}, z_{i2}, \cdots, z_{im_i})$ in finite time for $i = 1, 2, \ldots, n$. Theorem 1 shows that the unobservable variables $\hat{\omega}_i$ converges to ω_i exponentially for $i = 1, 2, \ldots, n$.

5. CONCLUSIONS

In this paper, a decentralised observer scheme has been proposed for a class of nonlinear interconnected systems based on HOSM techniques. By employing structural characteristics, the system is transformed to a new nonlinear interconnected system for which HOSM differentiator can be used for design. A set of sufficient conditions is developed such that the error dynamical system relating to the interconnected systems and the designed dynamical systems converges to zero. The result developed in this paper shows that for nonlinear large scale interconnected systems, a completely decentralised observation scheme is possible if HOSM properties are appropriately employed. Further research will focus on exploring structures of interconnection terms so that the developed results can be expanded to a larger class of systems.

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