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Long memory models: a first solution to the infinite energy storage ability of linear time-invariant fractional models

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Abstract

In this paper, it is shown that linear time-invariant fractional models do not reflect the reality of physical systems in terms of energy storage ability. It is first shown that this property may result from poorly chosen asymptotic behaviors. Another reason is that a fractional model can be viewed as a doubly infinite model. Indeed, its real state is of infinite dimension as it is distributed. Moreover, this state is distributed on an infinite domain. It is precisely this last feature that induces the ability to store an infinite energy, even if the fractional behavior is limited to a frequency band. As a consequence, even if fractional models permit to capture accurately the input-output dynamical behavior of many physical systems, the obtained models do not reflect the internal behavior of the modelled system which implies hard theoretical problems. Such problems may be avoided by the use of other models that exhibit the same input-output behavior but that do not have an infinite energy storage ability. As a first attempt to solve this issue, a new class of models is thus introduced in the paper.

Keywords: Fractional order system, initialization, infinite dimension, energy storage, long memory systems

1 - Introduction

Fractional modelling has found applications in assorted fields, like engineering (Sabatier et al, 2007), physics (Hilfer, 2000), finance (Scalas, 2006), chemistry (Nigmatullin and Le Mehauté, 2005) and bioengineering (Magin, 2006). Fractional order models are usually described by fractional differential equations or pseudo state space descriptions. In system identification or system modeling, fractional models are used to capture long memory phenomena that are encountered for instance in electrochemical devices or visco-elastic materials.

The objective of the present paper is to show that fractional models do not reflect the reality of physical systems in terms of energy storage ability. This property may result from poorly chosen asymptotic behaviors. Another reason is that a fractional model can be viewed as a doubly infinite model as revealed using the interpretation recently proposed in the literature (Sabatier et al, 2008) (Sabatier et al, 2010). Indeed, its real state is of infinite dimension as it is distributed. Moreover, this state is distributed on an infinite domain. It is precisely this last feature that induces the ability to store an infinite energy, even if the fractional behavior is limited to a frequency band. Using another interpretation, fractional models are not strictly speaking long memory but rather infinite memory models. As a consequence, even if fractional models permit to capture accurately the dynamical behavior of some systems, the use of fractional models implies some difficult mathematical problems that are not related to the physics of the modeled phenomena. Thus, fractional models analysis approaches based on internal (in state sense) behavior, such as initialization, internal stability, controllability, observability among others, are not linked to a physical reality but stem from the used models.

In this paper, an interpretation of fractional models that exhibits their double infinite nature is first reminded. It is first shown that model with fractional asymptotic behavior in low frequencies may store an infinite amount of energy. Then, it is shown that, even if the fractional behavior is limited to a frequency band, this infinite energy storage ability remains. As a first attempt to overcome these problems, a new class of models is introduced: the long memory models. Two examples illustrate their efficiency to describe long memory behaviors from an input output point of view, while not having the drawbacks described above.

2 – Representation of fractional systems

In recent years, an increasing number of studies have been done on fractional order models, namely models that can be described by differential equations that involve fractional derivatives or alternatively described by transfer functions that involve fractional powers of the Laplace variable s. In this paper, only linear time-invariant fractional differential equations are considered. Let a fractional differential equation linking the system input $u(t)\in\mathbb{R}$ and the system output $y(t)\in\mathbb{R}$ defined by:

$$\sum_{k=0}^{m} r_k \left(\frac{d}{dt}\right)^{\alpha_k} y(t) = \sum_{l=0}^{r} r_k \left(\frac{d}{dt}\right)^{\beta_l} u(t) \qquad (1)$$

where $\beta_{l+1} \geq \beta_l \geq 0$ and $\alpha_{k+1} \geq \alpha_k \geq 0$. $(d/dt)^{\alpha_k}$ and $(d/dt)^{\beta_l}$ denote fractional differential operators of orders $\alpha_k \in \mathbb{R}$ and $\beta_l \in \mathbb{R}$ respectively. Such operators are defined in (Oldham and Spanier, 1974) (Samko et al, 1993) (Miller and Ross, 1993) (Podlubny, 1999) and a detailed survey of the properties linked to these definitions can be found in (Oldham and Spanier, 1974).

Based on its impulse response, model (1) also admits the following state space description (Sabatier et al, 2008), (Sabatier et al, 2010):

$$\begin{bmatrix} \dot{v}(t) \\ \dot{w}(t,x) \end{bmatrix} = \begin{bmatrix} A & (0) \\ (0) & -x \end{bmatrix} \begin{bmatrix} v(t) \\ w(t,x) \end{bmatrix} + \begin{bmatrix} B \\ 1 \end{bmatrix} u(t) \tag{2}$$

$$y(t) = y_1(t) + y_2(t), \tag{3}$$

with

$$y_1(t) = Cv(t), \tag{4}$$

$$y_2(t) = \int_0^\infty \mu(x) w(t,x) dx\,. \tag{5}$$

In relations (2) and (4), A, B, C are state space description matrices associated to the systems poles that can be computed as described in (Sabatier et al, 2010), (Sabatier et al, 2012). In relation (5), function $\mu(x)$ is defined by (Matignon,1998):

$$\mu(x) = \frac{1}{\pi} \frac{\sum_{k=0}^{m} \sum_{l=0} r_k q_l \sin((\alpha_k - \beta_l)\pi) x^{\alpha_k + \beta_l}}{\sum_{k=0}^{m} r_k^2 x^{2\alpha k} + \sum_{0 \leq k < l < m} 2 r_k r_l \cos((\alpha_k - \alpha_l)\pi) x^{\alpha_k + \alpha_l}} \tag{6}$$

and

$$w(t,x) = \int_0^t e^{-(t-\tau)x} u(\tau) d\tau \qquad x \in \mathbb{R}^+\,. \tag{7}$$

Laplace transform of relation (5) (using Laplace transform of relation (7)) is given by:

$$y_2(s) = \int_0^\infty \frac{\mu(x)}{s+x} u(s) dx\,. \tag{8}$$

Such a representation is connected to the "diffusive representation" introduced by Montseny (Montseny, 2005) and Matignon (Matignon, 1998).

Using the change of variable $x = e^{-\chi}$ (Sabatier et al, 2010), relation (8) becomes

$$y_2(s) = \int_{-\infty}^{\infty} \frac{e^{-\chi} \mu(e^{-\chi})}{s + e^{-\chi}} u(s) d\chi\,. \tag{9}$$

In relation (3), $y_1(t)$ is the model poles response (model exponential part) and corresponds to the response of a classical integer model. Depending on the model, this part may be zero (no pole). $y_2(t)$ is the response of an always stable model. If the initial fractional model is unstable, the instability appears in $y_1(t)$.

Initial conditions are defined for relation (5) by $w(0,x) = \rho(x)$ and thus permit to give the exact expression of the system response with initial conditions (Sabatier et al, 2008):

$$\begin{aligned} y(t) = C&\left(z(0) e^{At} + \int_0^t e^{A(t-\tau)} u(\tau) \right) + \\ &\int_0^\infty \mu(x) \left(w(0,x) e^{-xt} + \int_0^t e^{-x(t-\tau)} u(\tau) d\tau \right) dx \end{aligned}\,. \tag{10}$$

Function $y_2(t)$ can also be written as (using spatial Fourier transform) (Sabatier al, 2010):

$$\begin{cases} \dfrac{\partial \phi(t,\zeta)}{\partial t} = \dfrac{\partial^2 \phi(t,\zeta)}{\partial \zeta^2} + u(t)\,\delta(\zeta) \\ y_2(t) = \displaystyle\int_{-\infty}^{\infty} m(\zeta)\,\phi(t,\zeta)\, d\zeta \end{cases} \tag{11}$$

with

$$m(\zeta) = \mathfrak{F}^{-1}\left\{4\pi^2 \zeta\, \mu\left(4\pi^2 \zeta^2\right)\right\}, \tag{12}$$

$$\phi(\zeta, 0) = \mathfrak{F}^{-1}\left\{\rho\left(4\pi^2 z^2\right)\right\} \quad \zeta \in \mathbb{R}\,. \tag{13}$$

As illustrated in figure 1, any fractional model can thus be seen as the association of an infinite dimensional system described by a diffusion equation (diffusion based submodel) and a classical linear (exponential) model (integer order submodel).

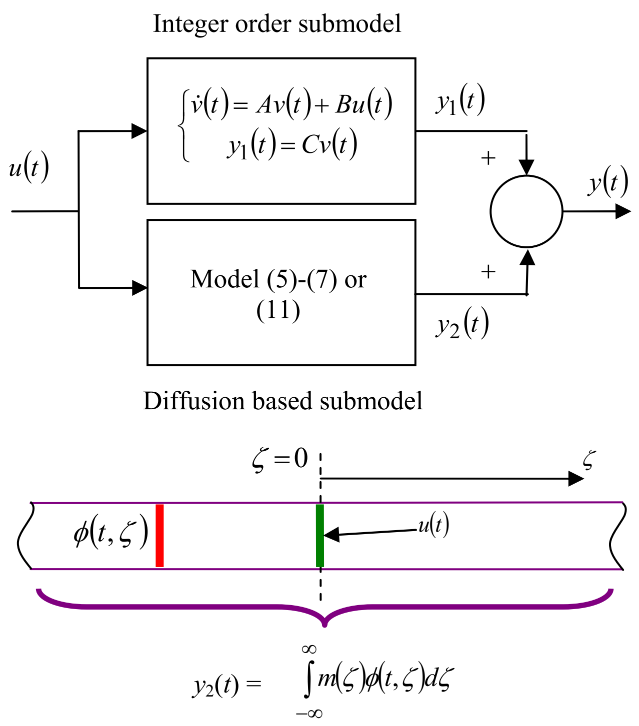

Figure 1 *- Representation of a fractional model as the association of a diffusion based submodel and an integer order submodel.*

3 – Infinite storage ability of fractional models

From existing fractional models found in the literature, this section shows the ability of fractional models to store an infinite amount of energy. Articles proposing these models are deliberately not referenced because authors cannot provide an exhaustive list of these papers, it would not be fair to mention some and not others. The concerned papers can be provided upon request to the authors.

3.1 – Infinite storage ability due to asymptotic behavior

We consider first a capacitor. A classical integer model of this electrical component is a resistance R (in order to model ohmic loses) in series with a capacitance C (in order to model charge accumulation). The model linking the current applied to the capacitor to the voltage is thus:

$$\frac{U(s)}{I(s)} = R + \frac{1}{Cs} . \tag{14}$$

If a constant voltage U_0 is applied to the capacitor, the current is thus defined by

$$I(t) = \frac{U_0}{R} e^{-\frac{t}{RC}} . \tag{15}$$

The energy stored by the capacitor is thus

$$E = \int_0^\infty U(t) I(t) dt = \int_0^\infty \frac{U_0{}^2}{R} e^{-\frac{t}{RC}} dt = C U_0{}^2 . \tag{16}$$

Even though this model is not perfect, it reflects a physical reality: the energy stored by the capacitor is of finite value.

Among the fractional models for capacitors that are available in the literature, one can find the following transfer functions

$$\frac{U(s)}{I(s)} = R + \frac{(Ts+1)^\alpha}{Cs^\beta}, \quad 0 < \alpha < \beta < 1 . \tag{17}$$

Depending on the papers $\alpha = \beta = 0.5$ or $\alpha = 0.3$ and $\beta = 0.98$. If a constant voltage U_0 is applied to such a model, the resulting current is

$$I(s) = \frac{Cs^\beta}{RCs^\beta + (Ts+1)^\alpha} \frac{U_0}{s} . \tag{18}$$

As time tends towards infinity, the current decreases as:

$$I(t) = \frac{CU_0}{\Gamma(1-\beta)} t^{-\beta} . \tag{19}$$

The energy stored is thus defined by

$$E = \int_0^\infty U(t) I(t) dt = E_0 + \int_T^\infty \frac{CU_0{}^2}{\Gamma(1-\beta)} t^{-\beta} dt , \tag{20}$$

which is infinite. This is not physically consistent.

The electric domain is not the only one to have produced models that can store an infinite energy. Literature proposes this kind of models for processes involving thermal transfers or viscoelastic materials. In this last case, the spring-spot element introduced in the Maxwell or Kelvin-Voigt model has the same drawback.

This infinite storage ability also appears clearly on the electrical realizations of a fractional integrator of figure 2. Whether it is a ladder network or parallel network, these realizations highlight an infinity number of capacitance without any possibility for the electrical charges to shunt the capacitance. Note that if a shunt resistance is added to these networks, the circuit no more behaves as a fractional integrator, but as a first kind fractional system (no more asymptotic behavior problem) (Oustaloup, 1983).

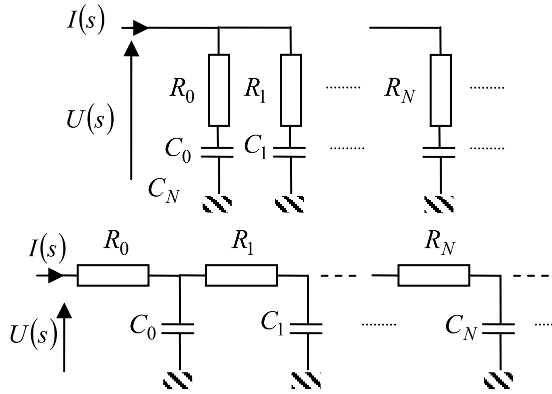

Figure 2 *– Electrical realizations of fractional integrator $U(s)/I(s)=1/s^n$*

Remark 1

If previous comments focus on the low-frequency asymptotic behavior, the high-frequency behavior is now studied. For this, the following thermal models, that can be found in the literature, are considered

$$\frac{T(s)}{\phi(s)} = H(s) = \frac{\sum_{i=0}^{N-1} b_i s^{\frac{i}{2}}}{\sum_{i=0}^{N-1} a_i s^{\frac{i}{2}} + s^{\frac{N}{2}}} . \tag{21}$$

This model links the temperature T(s) to the thermal flux $\phi(s)$. It is supposed that a constant flux ϕ_0 is applied to the system and the temperature variation is studied at time t=0. Using initial value theorem, then:

$$\left[\frac{dT(t)}{dt}\right]_{t=0} = \lim_{s\to\infty} \frac{s^2 H(s)\phi_0}{s} = \lim_{s\to\infty} s^{\frac{1}{2}} \phi_0 . \tag{22}$$

Relation (22) demonstrates that as soon as the flux is applied, the temperature increase in the model is infinite during an infinitesimal time. In fractional behaviors in low frequencies can produce physically inconsistent problems, this last analysis shows that high frequencies asymptotic fractional behaviors can produce models that are not physically compatible with the thermal behavior of a material supplied by a heat flow (heat transfer by electronic agitation).

■

The authors are well aware that a model is not intended to represent the reality of a system, but to gives an image for a specific use. The models that have been used as examples in this section have likely proven their accuracy in describing dynamic input-output behavior of the system while exhibiting a reduced number of parameters. However, they also imply a property that has nothing to do with the modeled system. If this problem can be easily solved using a convenient low frequency asymptotic behavior, as shown in the next paragraph, fractional models introduce the ability for the state to store an infinite amount of energy, whatever the model high and low frequencies asymptotic behavior.

3.2 – Infinite storage ability due to infinite space

Whatever the asymptotic behavior of model (1), equation (11) highlights that a fractional model is always defined on an infinite domain (integral on space from -∞ to +∞). If the continuous nature of this representation brings up an infinite number of time constants, the infinite nature of the definition domain brings up infinitely fast and infinitely slow time constants. Relation (8) indeed shows that the model time constants are *x* with $x \in \mathbb{R}^+$.

A consequence is the state infinite energy storage ability of fractional models. This can be highlighted on the following example. Consider the diffusion based submodel of a fractional model as indicated on figure 1. Using relations (5) and (7), the output $y_2(t)$ of this submodel is also given by

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = -xw(t,x) + u(t) \\ y_2(t) = \displaystyle\int_0^\infty \mu(x) w(t,x) dx \end{cases}, \quad \text{with} \quad x \in \mathbb{R}^+. \tag{23}$$

In (23), *w*(*t*, *x*) can be viewed as the state of the diffusion based submodel (state of the whole fractional model being given by *w*(*t*, *x*) along with the state *v*(*t*) of the integer submodel). System (23) is stable as all the roots of the dynamical part of (23) are in $\mathbb{R}^-$. It is now supposed that the system input (of finite energy) is defined by

$$u(t) = H(t) - H(t-T), \tag{24}$$

H(*t*) being the Heaviside function. *w*(*t*, *x*) in response to *u*(*t*) is thus given by

$$w(t,x) = \frac{1}{x}\left(1 - e^{-xt}\right)H(t) - \frac{1}{x}\left(1 - e^{-x(t-T)}\right)H(t-T), \tag{25}$$

with $x \in \mathbb{R}^+$. Energy stored by *w*(*t*, *x*) is given by the relation

$$E_T = \int_0^\infty\int_0^\infty [w(t,x)]^2 dtdx = \int_0^\infty\int_0^T \frac{1}{x^2}\left(1 - e^{-xt}\right)^2 dtdx - \int_0^\infty\int_T^\infty \frac{1}{x^2}\left(e^{-x(t-T)} - e^{-xt}\right)^2 dtdx, \tag{26}$$

or, after the integral on time

$$E_T = \int_0^\infty \frac{-e^{xT} + 1 + xTe^{xT}}{x^3 e^{xT}} dx. \tag{26}$$

A limited Taylor expansion of the integrant in (26) permits to show that

$$\frac{-e^{xT} + 1 + xTe^{xT}}{x^3 e^{xT}} \underset{x=0}{\approx} \frac{-\left(1 + xT + \frac{(xT)^2}{2}\right) + 1 + xT(1 + xT)}{x^3} = \frac{T^2/2}{x} \tag{27}$$

and permits to conclude that E_T tends towards infinity $\forall T \in \mathbb{R}^+$. The internal energy of the state is infinite for a finite energy input signal, which is not a consistent physical behavior for a stable system. This infinite energy storage ability is the results of infinitely low time constants. The infinitely low time constants provide to these models an infinite memory. This infinite memory is also physically not compatible with the behavior of physical systems (thermal, electrical, or other) that may only exhibit a long but finite memory.

If a model is always an approximation of reality, the infinite memory of fractional models thus gives them radically different properties from those of the modeled system. In relation to the literature in the field, the infinite history has given rise to works on initialization problem (Lorenzo and Hartley, 2001) (Lorenzo and Hartley, 2007) (Sabatier et al, 2010). These difficult problems are thus not linked to a physical reality but stem from the used models. Moreover, due to the infinite history, fractional models have properties such as observability (Sabatier et al, 2012), or controllability which are not necessarily those of the modeled system. It is interesting to note that the stability property is not affected, since the stability only depends on integer order submodel whose output is $y_1(t)$ (classical integer order model), diffusion based submodel whose output is $y_2(t)$ being always stable. However, as evidenced by the literature, in most cases the (fictitious) infinite history of fractional models (linked to the infinite domain definition that has created initialization problems), induce large difficulties in their properties analysis (stability with Lyapunov based method, observability, controllability…).

3.3 – On the need of new class of models to handle a long memory behavior

In spite of the above mentioned limitations, the authors acknowledge that fractional models are nice conceptualization tools. There are numerous examples of that in the literature. Crone control (Sabatier et al, 2002), Crone suspension (Oustaloup et al, 1996) concepts, signal (Das et pan, 2011) or image filtering (Mathieu et al, 2003), … have shown that some properties of fractional operators can advantageously be used to ensure the robustness or more discriminating filter. But the implementation of these concepts always involves an approximation step (Oustaloup et al, 2000). The problems that have arisen in recent years (initialization, observability, …) do not concern the aforementioned approaches.

In contrast, during modeling or identification by fractional model, a real system is represented by a model that has infinite energy storage ability. This system is indeed a model of infinite dimension (that is not a problem) on an infinite space (double infinity). If from an input output point of view and in system modeling/identification context this is not a problem for many reasons (limited sampling periods, limited test durations, limited effects of the infinite reaction velocity and an infinite memory on the global model response), the double infinity feature have given rise to problems that have no physical reality and are simply induced by the handled models. These models are thus adapted to study a system input-output behavior, but not its internal properties (initialization, internal stability, controllability, observability among others). Another conclusion is also that variable $w(t, x)$ in relation (23) along with $v(t)$ in relation (2), that can be considered as the model state, has no connection with the internal real state of the modeled system.

The question is in fact what is the need? The need is modeling and identification of systems with long memory behaviors (not infinite). Fractional models meet the need with their infinite memory but introduce inconsistencies on the system state, some of which being described above. The goal of the next paragraph is thus to show that the need can be met by the introduction of a new class of models, without involving infinite domains.

4 – A new class of models: long memory models

4.1 – Model definition

To overcome the problem of infinite memory inherent to the diffusion based submodel, a truncated version of relation (5) or relation (8) may be used. However, using a truncated version of representation (11), function $m(\zeta)$ is difficult to compute (inverse Fourier transform) in order to establish a link with fractional models. Using a truncated version of relation (5):

- it is difficult to combine several models under this representation;
- a physical interpretation is not easy to derive as x is homogenous to a frequency.

Moreover, integrals in relations (5) or (8) do not help in the analysis of model properties.

Thus a new representation is required. Before proposing one, it must be first noticed that there exists in the literature results showing that some classes of partial differential equations behave as real or complex fractional integrators in a given frequency band. The first results on this topic are available in (Oustaloup, 1983), (Oustaloup and Sabatier, 1995), (Levron et al, 1999). They are limited to a single class of equation and a first generalization has been recently proposed in (Sabatier et al, 2013).
Here we propose to generalize one more time this approach, through the representation:

$$\begin{aligned} &\beta(\chi)\frac{\partial z(\chi,t)}{\partial \chi}+\gamma(\chi)\frac{\partial^2 z(\chi,t)}{\partial t\partial \chi}=Bu(t) \\ &y(t)=C\left(z(Z_1,t)-z(-Z_2,t)\right) \end{aligned} . \quad (28)$$

where $Z_1 \in \mathrm{R}$, $Z_2 \in \mathrm{R}$, $\chi \in [Z_1, Z_2]$, $\beta(\chi) \in \mathrm{R}$, $\gamma(\chi) \in \mathrm{R}$, $B \in \mathrm{R}$, $C \in \mathrm{R}$.

To show the ability of this class of model to capture the long memory of a system, let

$$\frac{\partial z(\chi,t)}{\partial \chi}=f(\chi,t) \quad (29)$$

and suppose that Z_1 and Z_2 tend towards infinity. Output $y(t)$ is thus defined by

$$\begin{aligned} y(t)&=C\int_{-\infty}^{\infty} f(\chi,t)d\chi \\ &=C\int_{-\infty}^{\infty}\frac{\partial z(\chi,t)}{\partial \chi}d\chi=C\left(z(\infty,t)-z(-\infty,t)\right) \end{aligned} . \quad (30)$$

Laplace transform of relations (28) and (30) thus leads to

$$y(s)=C\int_{-\infty}^{\infty}\left[\beta(\chi)+\gamma(\chi)s\right]^{-1}\left(\gamma(\chi)f(\chi,0)+Bu(s)\right)d\chi \quad (31)$$

or with B=C=1

$$y(s)=\int_{-\infty}^{\infty}\gamma^{-1}(\chi)\left[s+\gamma^{-1}(\chi)\beta(\chi)\right]^{-1}\left(\gamma(\chi)f(\chi,0)+u(s)\right)d\chi . \quad (32)$$

If initial conditions are supposed equal to 0, relation (32) becomes:

$$y(s)=\int_{-\infty}^{\infty}\frac{\gamma^{-1}(\chi)}{s+\gamma^{-1}(\chi)\beta(\chi)}d\chi u(s) . \quad (33)$$

and using

$$\gamma^{-1}(\chi)=\mu\left(e^{-\chi}\right)e^{-\chi} \quad \text{and} \quad \gamma^{-1}(\chi)\beta(\chi)=e^{-\chi} , \quad (34)$$

relation (31) becomes:

$$\frac{Y(s)}{U(s)}=\int_{0}^{\infty}\frac{\mu\left(e^{-\chi}\right)e^{-\chi}}{s+e^{-\chi}}d\chi . \quad (35)$$

This corresponds to the Laplace transform of relations (5) and (7) and permits to conclude that representation (28), that will be denoted "long memory" representation in the sequel behaves as a fractional model in a frequency band defined by Z_1, Z_2, and functions $\beta(\chi)$ and $\gamma(\chi)$. In system (28) and according to (Sabatier et al, 2013) and relation (35), the system lowest time constant is defined by

$$\beta(-Z_2)\gamma^{-1}(-Z_2) \quad (36)$$

and the system highest time constant is defined by

$$\beta(Z_1)\gamma^{-1}(Z_1) . \quad (37)$$

Thus, with this class of model:

- a physical interpretation can be done in many domains (electrical, thermal, mechanical, …) and implementations can be deduced in all these domains (in the mechanical domain $\beta(\chi)$ and $\gamma(\chi)$ can be connected to the damping coefficient and the stiffness of a material) ;
- the domain definition is finite and the long memory behaviors can be captured;
- behaviors that were captured with complex fractional differentiation can also be captured with this class of models;
- many tools existing for partial differential equations analysis and control could be adapted to study this class of models;
- reduced number of parameters property induced by a fractional modeling is maintained with these models.

4.2 – Examples

Suppose a long memory system whose dynamic behavior can be described by the fractional model whose transfer function is:

$$H(s)=\frac{1}{s^{\nu}+a}. \quad (38)$$

Such a model can also be represented by the partial differential equation

$$\begin{aligned}&\beta(\chi)\frac{\partial z(\chi,t)}{\partial \chi}+\gamma(\chi)\frac{\partial^2 z(\chi,t)}{\partial t\partial \chi}=Bu(t)\\&y(t)=C\left(z(Z_1,t)-z(-Z_2,t)\right)\end{aligned}, \quad (39)$$

with B=1, C=1 and with

$$\beta(\chi)=\frac{\pi}{\sin(\nu\pi)}\left(e^{\nu\chi}a^2+2a\cos(\nu\pi)+e^{-\nu\chi}\right), \quad (40)$$

$$\gamma(\chi)=\frac{\pi}{\sin(\nu\pi)}\left(e^{\nu\chi}a^2+2a\cos(\nu\pi)+e^{-\nu\chi}\right)e^{\chi} \quad (41)$$

and $Z_1=Z_2=20$.

Laplace transform of relations (39) to (41) leads to the transfer function $G(s)=Y(s)/U(s)$ with

$$G(s)=\frac{\sin(\nu\pi)}{\pi}\int_{-20}^{20}\frac{e^{-\nu\chi}e^{-\chi}\left(s+e^{-\chi}\right)^{-1}}{\left(a^2+2ae^{-\nu\chi}\cos(\nu\pi)+e^{-2\nu\chi}\right)}d\chi. \quad (42)$$

A comparison of the bode diagrams of transfer functions (38) and (42) with a=1 and ν=0.4 is proposed in figure 3. It highlights the ability of the introduced class of models to behave, on a frequency band, as a fractional model. This thus also highlights their ability to catch long memory dynamical behaviors. Note that transfer function (42) frequency response has been obtained thought an integral discretization method (Euler method).

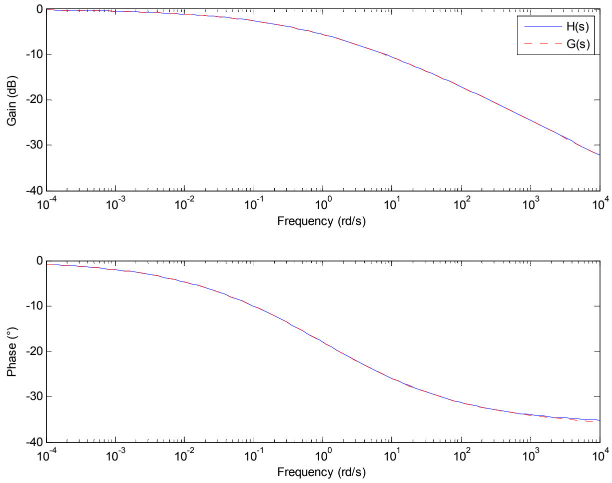

Figure 3 – Bode diagram comparison

Using the change of variable $x=e^{-\chi}$, and according to relations (5), (7) and (23), the response to an input $u(t)$ of the system characterized by $G(s)$ is given by:

$$\begin{cases}\dfrac{\partial w(t,\chi)}{\partial t}=-e^{-\chi}w(t,\chi)+u(t)\\ y(t)=\dfrac{\sin(\nu\pi)}{\pi}\displaystyle\int_{-20}^{20}\frac{e^{-\nu\chi}e^{-\chi}w(t,\chi)}{\left(a^2+2ae^{-\nu\chi}\cos(\nu\pi)+e^{-2\nu\chi}\right)}d\chi\end{cases}. \quad (43)$$

with $\chi\in[-20,20]$. For the input

$$u(t)=\frac{1}{T}\left(H(t)-H(t-T)\right), \quad (44)$$

energy stored by $w(t,\chi)$ is given by the relation

$$E_T(T)=\frac{1}{T^2}\int_{-20}^{20}\frac{-e^{e^{\chi}T}+1+e^{\chi}Te^{e^{\chi}T}}{e^{3\chi}e^{xT}}dx. \quad (45)$$

The variations of $E_T(T)$ are represented by figure 4 and highlights that $E_T(T)$ tends towards a finite value as T tends towards 0, namely as the input behaves as a Dirac pulse.

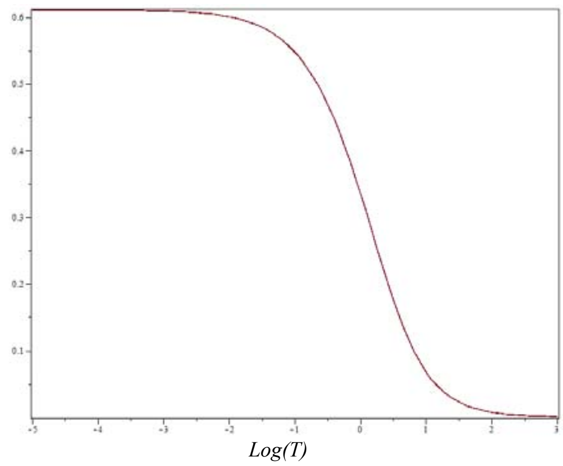

Figure 4 – Variations of $E_T(T)$

Now, consider a system modeled by this (implicit) transfer function:

$$H(s)=\frac{1}{(s+a)^{\nu}}. \quad (46)$$

Such a system can also be represented by partial differential equation (28) with B=1, C=1 and with

$$\beta(\chi)=\frac{\pi}{\sin(\nu\pi)}\left(e^{-\chi}+a\right)e^{(1-\nu)\chi}\quad \gamma(\chi)=\frac{\pi}{\sin(\nu\pi)}e^{(1-\nu)\chi} \quad (47)$$

and $Z_1=Z_2=20$.

Laplace transform of relations (28) and (44) leads to the transfer function $G(s)=Y(s)/U(s)$ with

$$G(s)=\frac{\sin(\nu\pi)}{\pi}\int_{-20}^{20}\frac{e^{-\chi}}{e^{-\nu\chi}\left(s+e^{-\chi}+a\right)}d\chi. \quad (48)$$

These examples thus show the ability of the introduced class of model to represent systems that are usually modeled by implicit fractional order transfer functions. However the drawback linked to the doubly infinite dimension of fractional systems disappears with the introduced class of models.

5 – Conclusions

In this paper, it is shown that linear time-invariant fractional models do not reflect the reality of physical systems in terms of energy storage ability. Sometimes, it is the consequence of a bad choice of the low frequency asymptotic behavior of these models. This is shown in this paper on several examples. This problem can nevertheless be easily solved by a proper choice of the low frequency behavior. However, the doubly infinite property of fractional models confers to fractional models the ability to store internally an infinite amount of energy. Using an interpretation reminded in this paper, it is indeed shown that fractional models can be viewed as a distributed parameter models on an infinite domain. This infinite domain induces infinitely fast and slow time constants and thus an infinitely fast dynamical behavior and an infinite memory.

Even if these infinitely fast dynamical behavior and this infinite memory is not a problem in system modeling/identification for many reasons (limited sampling periods, limited test durations, limited effects of the infinite reaction velocity and an infinite memory on the global model response), they have given rise to problems that have no physical reality and are simply induced by the handled models. These models are thus adapted to study a system input-output behavior, but not its internal properties (initialization, internal stability, controllability, observability among others).

As a first attempt to overcome the above mentioned problems, and taking into account that a need is for models which can catch long memory (not infinite) input-output dynamical behaviors (which led to use fractional models), the authors propose a new class of models: the long memory models. These models are described by partial differential equations on a limited space domain. They are a generalization of models previously introduced by the authors that have shown the ability to behave as frac

tional integrators on a limited frequency band (Oustaloup, 1983), (Oustaloup and Sabatier, 1995), (Levron et al, 1999), (Sabatier et al, 2013). Analysis of these new models properties will be the topic of future work by the authors.

To conclude, the authors think that the new class of models introduced, partially solve the question of the real nature of the internal state of systems that exhibit fractional behaviors. A partial differential equation is a solution to fit the input-output behavior of a system, and tackle the infinite energy storage ability problem, but it perhaps hides something more essential about the fractional models nature.

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