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Alternative Stability Conditions for Switched Discrete Time Linear Systems

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Abstract: Alternative necessary and sufficient conditions for absolute exponential stability are presented, for switched discrete–time linear systems. To reach these results, we exploit concepts from set theory and in specific the forward reachability set mappings. The developed theorem can be utilized to construct iterative numerical procedures for verifying stability. Two examples illustrate the effectiveness of the proposed methodology.

Keywords: stability, switched linear systems, discrete-time systems.

1. INTRODUCTION

Discrete–time linear switched systems [Liberzon and Morse, 1999, Shorten et al., 2007, Lin and Antsaklis, 2009, Sun, 2010] are abundant in control applications. Consider for example closed–loop linear systems under multiple linear state feedback controllers that satisfy different performance criteria, which is the typical case in process control, automotive industry, and power systems. Furthermore, the corresponding stability analysis problem is by far non–trivial, as demonstrated by [Blondel and Tsitsiklis, 2000].

The dynamics of switched linear systems is set-valued, giving rise to two different notions of stability, namely the corresponding absolute (strong) and weak notion. From the control theory perspective, absolute stability is stability under arbitrary switching. On the other hand, when the switching is not arbitrary and the switching signal is considered to be a decision variable, the weak stability property coincides with the stabilizability property. In this article, we focus on the absolute stability property.

Most available analysis methods correspond to the absolute stability property. They can be grouped in the ones that consider the construction of a Lyapunov function [Brayton and Tong, 1979, Molchanov and Pyatnitskii, 1986, Blanchini, 1994, Polański, 2000, Yfoulis and Shorten, 2004, Lazar, 2010, Sun, 2010] and the approximation of the joint spectral radius and extremal norms [Barabanov, 1988, Gurvits, 1995, Gripenberg, 1996, Barabanov, 2005, Theys, 2005, Jungers, 2009, Chang and Blondel, 2011]. Also, it is worth to consider the survey article [Margaliot, 2006]. A novel Lyapunov–type approach to stability analysis is presented in the recent work [Lazar et al., 2013], where non–conservatism of set–induced finite–time Lyapunov functions was established for homogeneous difference equations. The weak stability property is also studied in numerous works, see for example [Shorten et al., 2007, Section 5], [Lin and Antsaklis, 2009, Section III] and the references therein. The recent contribution [Fiacchini and Jungers, 2013], establishes necessary and sufficient stabilizability conditions, when the switching signal is a decision variable, using set–induced Lyapunov–type functions. The weak instability property can been studied utilizing Lyapunov theory and set theory, by modifying appropriately the result in [Blanchini, 1994] and [Blanchini and Miani, 2008]. For the continuous–time case, marginal instability has been studied in [Polanski, 2000, Sun, 2008, Chitour et al., 2012].

In this article we exploit tools from set theory, using the forward dynamics of the system and its operation on sets. Alternative necessary and sufficient conditions for global absolute exponential stability are established, in the form of set inclusions. More specifically, we show that absolute stability is ensured if and only if for an arbitrary convex and compact set that includes the origin in its interior, there exists an integer k such that the k-step forward reachability map lies in the interior of that set. It is worth noticing that these theoretical results are derived directly by exploiting the radial convexity of the sets involved and the homogeneity of the dynamics, without having to resort to a set-induced Lyapunov function framework. By exploiting the relation between the stability of switched linear systems and linear systems under polytopic uncertainties, three corollary results are established, which offer a numerically tractable method to verify absolute stability. The required operations for verification of stability, when choosing proper $\mathcal{C}$-polytopic sets, involve matrix multiplications with vectors, computation of convex hulls of a finite number of vertices and verification of set inclusion between polytopic $\mathcal{C}$-sets, which is equivalent to verifying a set of linear algebraic relations. Consequently, from a computational point of view, the results offer an alternative to the well known algorithm established in [Blanchini, 1994], which makes use of the preimage map intersected with a proper $\mathcal{C}$–set, leading to the construction of a contractive set and a corresponding set–induced polyhedral Lyapunov function. In specific, while the computations in [Blanchini, 1994] utilize the half-space polytopic description, in this article the vertex description is used. Finally, the proposed stability conditions can be extended to dynamics described by homogeneous of order one set–valued maps, in a similar fashion to the one presented in [Lazar et al., 2013].

In Section 2, the necessary notation and definitions are given. The main results follow in Section 3. Two numerical examples

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illustrating the established results are presented in Section 4, while the conclusions are drawn in Section 5.

2. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ denote the field of real numbers, the set of non-negative reals and the set of nonnegative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c} := \{k \in \Pi : k \geq c\}$, and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{N}_\Pi := \mathbb{N} \cap \Pi$. The vector with all elements equal to one is denoted by $1_n \in \mathbb{R}^n$, while the zero vector is denoted by $0_n \in \mathbb{R}^n$.

A compact set is a closed and bounded set. A $\mathcal{C}$–set $\mathcal{S} \subset \mathbb{R}^n$ is a compact, convex set which contains the origin. A proper $\mathcal{C}$–set $\mathcal{S} \subset \mathbb{R}^n$ is a $\mathcal{C}$–set which contains the origin in its interior. A set $\mathcal{S} \subset \mathbb{R}^n$ is a radially convex set [Rubinov and Yagubov, 1986] with respect to a vector $y \in \mathbb{R}^n$ if $x \in \mathcal{S}$ implies $\alpha(x-y) \in \mathcal{S}$, for all $\alpha \in \mathbb{R}_{[0,1]}$. For simplicity, radially convex sets with respect to the zero vector are called radially convex sets. Notice that radially convex sets can be unbounded, or they can have the origin on their boundary. Given a set $\mathcal{S} \subset \mathbb{R}^n$ and a real matrix $A \in \mathbb{R}^{n\times n}$ (A is a number for $n = 1$), the set $A\mathcal{S}$ is defined by $A\mathcal{S} := \{x \in \mathbb{R}^n : (\exists y \in \mathcal{S} : x = Ay)\}$. An arbitrary norm in $\mathbb{R}^n$ is denoted by $\|\cdot\|$. The unit ball of an arbitrary norm is denoted by $\mathcal{B} := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. The convex hull of a set $\{X_i\}_{i\in\mathbb{N}_{[1,M]}}$, $X_i \in \mathbb{R}^{n\times m}$, will be denoted by $\mathrm{conv}(\{X_i\}_{i\in\mathbb{N}_{[1,M]}})$. The boundary and the interior of a set $\mathcal{S} \subset \mathbb{R}^n$ are denoted by $\partial\mathcal{S}$ and $\mathrm{interior}(\mathcal{S})$ respectively.

A polytope is the bounded intersection of a finite number of closed half–spaces. Proper $\mathcal{C}$–polytopic sets are described by vertex or half–space representations [Ziegler, 2007]. The vertex representation of an arbitrary proper $\mathcal{C}$–polytopic set $\mathcal{S}$ corresponds to

$$\mathcal{S} := \mathrm{conv}(\{v^i\}_{i\in\mathbb{N}_{[1,q]}}), \tag{1}$$

for some $q \in \mathbb{N}_{\geq n+1}$. The matrix $V := [v_1, v_2, \ldots, v_q] \in \mathbb{R}^{n\times q}$ has as columns the vertices of $\mathcal{S}$ and is of full row–rank.

A mapping, possibly set–valued, $g(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, is called a positively homogeneous mapping of order one, or simply, a homogeneous map, if for any pair $(\alpha, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, it holds that $g(\alpha x) = \alpha g(x)$.

Let $\mathcal{A} := \{A_i\}_{i\in\mathbb{N}_{[1,N]}}$ be a set of matrices. Then, it holds that $\mathcal{A}^1 := \mathcal{A}$. Moreover, for any $l \in \mathbb{N}_{\geq 2}$,

$$\mathcal{A}^l := \left\{ \prod_{i=1}^{l} A_{j_i} : (j_1, \ldots, j_l) \in \mathbb{N}^l_{[1,N]} \right\} \tag{2}$$

denotes the set which contains all possible products of the elements of the set $\mathcal{A}$ of length l.

We consider the discrete-time autonomous linear inclusions

$$x_{t+1} \in \mathcal{A}x_t, \tag{3}$$

where $x_t \in \mathbb{R}^n$ is the state vector and $t \in \mathbb{N}$ is the time variable. Let $\Phi(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, where

$$\Phi(x) := \mathcal{A}x.$$

Given an integer k, the k-th iterated mapping $\Phi^k(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, is defined as follows. For $k = 0$, it holds that $\Phi^0(x) := x$. For $k = 1$, it holds that $\Phi^1(x) := \Phi(x)$. For $k \in \mathbb{N}_{\geq 2}$, it follows that $\Phi^k(x) := \Phi(\Phi^{k-1}(x))$.

Definition 1. The system (3) is called globally absolutely/strongly exponentially stable (GAES) if and only if there exists a pair $(\Gamma, \varepsilon) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that for all $x_0 \in \mathbb{R}^n$ it holds that

$$\|x_t\| \leq \Gamma\varepsilon^t\|x_0\|, \quad \forall t \in \mathbb{N}. \tag{4}$$

Remark 1. In the recent work [Lazar et al., 2013, Corollary V.3], it was shown that for dynamics described by single–valued homogeneous maps, and thus, for switched linear systems with state–dependent switching as well, asymptotic stability is equivalent to exponential stability. Moreover, local exponential stability implies global exponential stability. An extension of the results in [Lazar et al., 2013] to dynamics described by arbitrary switching between a finite number of single–valued homogeneous maps is possible, i.e., by imposing the corresponding conditions in the *absolute* sense. As such, the subsequent results that concern global absolute exponential stability are not restrictive, compared to the results concerning global absolute asymptotic stability.

Given a set of matrices $\mathcal{A} \subset \mathbb{R}^{n\times n}$, the set $\mathcal{A}\mathcal{S}$ is defined by

$$\mathcal{A}\mathcal{S} := \{x \in \mathbb{R}^n : (\exists (y, A) \in \mathcal{S} \times \mathcal{A} : x = Ay)\}.$$

The set–valued map $\overline{\Phi}(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by

$$\overline{\Phi}(\mathcal{S}) := \mathcal{A}\mathcal{S}, \tag{5}$$

for any $\mathcal{S} \subset \mathbb{R}^n$. Given an integer k, the set–valued k–th iterated map is defined as follows. For $k = 0$, it holds that $\overline{\Phi}^0(\mathcal{S}) := \mathcal{S}$. For $k = 1$, it holds that $\overline{\Phi}^1(\mathcal{S}) := \overline{\Phi}(\mathcal{S})$. For $k \in \mathbb{N}_{\geq 2}$, it follows that $\overline{\Phi}^k(\mathcal{S}) := \overline{\Phi}(\overline{\Phi}^{k-1}(\mathcal{S}))$. The zero singleton set $\{0_n\}$ is the unique fixed point of any positive k–th iteration, i.e., $\overline{\Phi}^k(\{0_n\}) = \{0_n\}$ for all $k \in \mathbb{N}_{\geq 1}$.

3. MAIN RESULTS

The following facts will be used to derive the main theoretical results of the article.

Fact 1. For any pair $(\rho, k) \in \mathbb{R}_{[0,1)} \times \mathbb{N}_{\geq 1}$, there exists a pair $(M, \lambda) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[\rho,1)}$ such that

$$\rho^{\lfloor \frac{i}{k} \rfloor} \leq M\lambda^i, \quad \forall i \in \mathbb{N}. \tag{6}$$

Proof Relation (6) is satisfied with $M := \rho^{\frac{-k+1}{k}}$, $\lambda := \rho^{\frac{1}{k}}$. ■

Fact 2. For any proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$, there exists a pair $(c_1, c_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{(0,c_1]}$ such that

$$c_2\mathcal{B} \subseteq \mathcal{S} \subseteq c_1\mathcal{B}. \tag{7}$$

Proof Relation (7) is satisfied with

$$c_1 := \max_{x\in\mathcal{S}} \|x\|, \quad c_2 := \min_{x\in\partial\mathcal{S}} \|x\|. \blacksquare$$

Fact 3. Let $\mathcal{S} \subset \mathbb{R}^n$ be a radially convex set. Then, the set $\overline{\Phi}^i(\mathcal{S})$ is a radially convex set, for all $i \in \mathbb{N}$.

Proof Suppose that $\overline{\Phi}(\mathcal{S})$ is not radially convex. Then, there exists a pair $(y, \alpha) \in \overline{\Phi}(\mathcal{S}) \times \mathbb{R}_{[0,1]}$ such that $\alpha y \notin \overline{\Phi}(\mathcal{S})$. By definition, there exists a vector $x \in \mathcal{S}$ such that $y \in \Phi(x)$. Then, since $\alpha \in \mathbb{R}_{[0,1]}$ and $\mathcal{S}$ is radially convex, $\alpha x \in \mathcal{S}$. Since $\Phi(\cdot)$ is homogeneous, it holds that $\Phi(\alpha x) = \alpha\Phi(x)$, or $\alpha y \in \Phi(\alpha x)$. Thus, there exists a $y^* \in \overline{\Phi}(\mathcal{S})$ such that $y^* = \alpha y$, which is a contradiction to the hypothesis that $\overline{\Phi}(\mathcal{S})$ is not radially convex. Next, suppose that $\overline{\Phi}^l(\mathcal{S})$ is a radially convex set. Then $\overline{\Phi}^{l+1}(\mathcal{S}) = \overline{\Phi}(\overline{\Phi}^l(\mathcal{S}))$, is also radially convex. Consequently, $\overline{\Phi}^i(\mathcal{S})$ is radially convex, for all $i \in \mathbb{N}$.

The first main result, regarding the global absolute exponential stability property is presented next.

Theorem 1. The system (3) is globally absolutely exponentially stable if and only if for every proper $\mathcal{C}$–set $\mathcal{S} \subset \mathbb{R}^n$, there exists a pair $(k, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that

$$\overline{\Phi}^k(\mathcal{S}) \subseteq \rho\mathcal{S}. \quad (8)$$

Proof Suppose that (8) holds. For any $N \in \mathbb{N}_{\geq 1}$, it holds that

$$\overline{\Phi}^{Nk}(\mathcal{S}) = \overline{\Phi}^{(N-1)k}(\overline{\Phi}^k(\mathcal{S})) \subseteq \overline{\Phi}^{(N-1)k}(\rho\mathcal{S}) = \rho\overline{\Phi}^{(N-1)k}(\mathcal{S}) \subseteq \ldots \subseteq \rho^N\mathcal{S}.$$

For any $i \in \mathbb{N}$, there exists a pair $(N, j) \in \mathbb{N} \times \mathbb{N}_{[0,k-1]}$ such that $i := kN + j$. Then, $\overline{\Phi}^i(\mathcal{S}) = \overline{\Phi}^{kN+j}(\mathcal{S}) \subseteq \rho^N\overline{\Phi}^j(\mathcal{S})$. The set $\mathcal{S}$ is a bounded set and the map $\overline{\Phi}(\cdot)$ is bounded in any bounded set in $\mathbb{R}^n$. Thus, $\overline{\Phi}(\mathcal{S})$ is bounded. Consequently, there exists a number $c_3 \in \mathbb{R}_{\geq 1}$ such that

$$\overline{\Phi}^j(\mathcal{S}) \subseteq c_3\mathcal{S}, \quad \forall j \in \mathbb{N}_{[1,k-1]}.$$

Thus, $\overline{\Phi}^i(\mathcal{S}) \subseteq \rho^N c_3\mathcal{S}$, and taking into account Fact 1, there exists a pair $(M, \lambda) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[\rho,1)}$ such that

$$\overline{\Phi}^i(\mathcal{S}) \subseteq M\lambda^i c_3\mathcal{S}, \ \forall i \in \mathbb{N}. \quad (9)$$

From Fact 2, there exist positive numbers $c_1, c_2 \in \mathbb{R}_+$ such that (7) holds. For any $x_0 \in \partial\mathcal{S}$, $x_i \in \overline{\Phi}^i(\mathcal{S})$, and from (9), it holds that $x_i \in M\lambda^i c_3 c_1\mathcal{B}$, or, equivalently, $\|x_i\| \leq M\lambda^i c_3 c_1$, $\forall i \in \mathbb{N}$. For any $x_0 \in \partial\mathcal{S}$, $\|x_0\| \geq c_2$, or, $1 \leq c_2^{-1}\|x_0\|$. Thus,

$$\|x_i\| \leq M\lambda^i c_3 c_1 c_2^{-1}\|x_0\|, \ \forall i \in \mathbb{N}, \quad (10)$$

for any $x_0 \in \partial\mathcal{S}$. Since $\mathcal{S}$ is a proper $\mathcal{C}$–set, for any $x_0 \in \mathbb{R}^n$, there exists a pair $(\alpha, x_0^\star) \in \mathbb{R}_+ \times \partial\mathcal{S}$ such that $x_0 = \alpha x_0^\star$. Then, for all $i \in \mathbb{N}$ and for each solution $x_i \in \mathbb{R}^n$, there exists a vector $x_i^\star \in \Phi^i(x_0^\star)$ such that $x_i = \alpha x_i^\star$, because $\Phi^i(\cdot)$ is homogeneous of order one, for all $i \in \mathbb{N}$. Thus, for any $x \in \mathcal{S}$,

$$\|x_i\| = \|\alpha x_i^\star\| \leq \alpha M\lambda c_3 c_1 c_2^{-1}\|x_0^\star\| = M\lambda^i c_3 c_1 c_2^{-1}\|\alpha x_0^\star\| = M\lambda^i c_3 c_1 c_2^{-1}\|x_0\|.$$

Thus, relation (4) is satisfied with $\Gamma := Mc_3c_1c_2^{-1}$, $\varepsilon := \lambda$, and the system (3) is GAES.

Conversely, suppose that the system (3) is GAES. Then, there exists a pair $(\Gamma, \varepsilon) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that $\|x_i\| \leq \Gamma\varepsilon^i\|x_0\|$, for all $x_0 \in \mathbb{R}^n$, for all $x_i \in \Phi^i(x_0)$, for all $i \in \mathbb{N}$. From Fact 2, for any proper $\mathcal{C}$–set $\mathcal{S} \subset \mathbb{R}^n$, there exists a positive number c_1 such that $\|x_0\| \leq c_1$, for any $x_0 \in \mathcal{S}$. Then, for all $x_0 \in \mathcal{S}$ it follows that

$$\|x_i\| \leq \Gamma\varepsilon^i\|x_0\| \leq \Gamma\varepsilon^i c_1.$$

If there exists a finite integer $k \in \mathbb{N}_{\geq 1}$, such that $x_0 \in \mathcal{S}$ implies $x_k \in \rho\mathcal{S}$, for some $\rho \in \mathbb{R}_{(0,1)}$, then relation (8) can be verified. This is true if $x_k \in \rho c_2\mathcal{B}$, or, $\|x_k\| \leq \rho c_2$. Thus, it has to be verified that there exists a k such that $\|x_k\| \leq \Gamma\varepsilon^k c_1 \leq \rho c_2$, or[1], $k\log\varepsilon \leq \log(\frac{\rho c_2}{\Gamma c_1})$, or, $k \geq \left\lceil \frac{\log\frac{\rho c_2}{\Gamma c_1}}{\log\varepsilon} \right\rceil$. Such a finite integer always exists for the quintuple $(\rho, c_2, \Gamma, c_1, \varepsilon) \in \mathbb{R}_{(0,1)} \times \mathbb{R}_{(0,c_1]} \times \mathbb{R}_{\geq 1} \times \mathbb{R}_+ \times \mathbb{R}_{(0,1)}$. Thus, for every proper $\mathcal{C}$–set $\mathcal{S}$, there exists a pair $(k, \rho) \in \mathbb{N}_{\geq 1} \times R_{(0,1)}$ such that relation (8) holds. ■

Remark 2. It is worth observing that only the homogeneity property and the boundedness of the mapping $\Phi(\cdot)$ are used in both the necessary and sufficient part of the proof of Theorem 1. Consequently, the result is valid for the more general class of homogeneous inclusions of order one. This remark comes in accordance with the theoretical results reported in [Lazar et al., 2013] which provided a converse theorem of existence of finite–time sublinear Lyapunov functions for stable homogeneous difference equations.

Proposition 1. Consider a proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$ and a pair $(k, \lambda) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{(0,1)}$ such that relation $\overline{\Phi}^k(\mathcal{S}) \subseteq \mathcal{S}$ holds. Then, for any proper $\mathcal{C}$-set $\mathcal{M} \subset \mathbb{R}^n$, there exists a pair $(\hat{k}, \hat{\rho}) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{(0,1)}$ such that $\overline{\Phi}^{\hat{k}}(\mathcal{M}) \subseteq \hat{\rho}\mathcal{M}$.

Proof Since $\mathcal{S}$, $\mathcal{M}$ are proper $\mathcal{C}$-sets, by setting $\alpha_1 := \max_\alpha\{\alpha \in \mathbb{R}_+ : \alpha\mathcal{S} \subseteq \mathcal{M}\}$ and $\alpha_2 := \max_\alpha\{\alpha \in \mathbb{R}_+ : \alpha\mathcal{M} \subseteq \alpha_1\mathcal{S}\}$, relation

$$\alpha_2\mathcal{M} \subseteq \alpha_1\mathcal{S} \subseteq \mathcal{M} \quad (11)$$

holds. Then,

$$\overline{\Phi}^k(\alpha_2\mathcal{M}) \subseteq \overline{\Phi}^k(\alpha_1\mathcal{S}) = \alpha_1\overline{\Phi}^k(\mathcal{S}) \subseteq \alpha_1\rho\mathcal{S} \subseteq \rho\mathcal{M}.$$

Applying the set map $\overline{\Phi}(\cdot)$ kN times, for $N \in \mathbb{N}_{\geq 1}$, it follows that $\overline{\Phi}^{Nk}(\alpha_2\mathcal{M}) \subseteq \rho^N\mathcal{M}$, or, $\overline{\Phi}^{Nk}(\mathcal{M}) \subseteq \frac{\rho^N}{\alpha_2}\mathcal{M}$. There exists a pair $(\hat{k}, \hat{\rho}) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that $\overline{\Phi}^{\hat{k}}(\mathcal{M}) \subseteq \hat{\rho}\mathcal{M}$ holds if

$$\hat{k} = kN, \quad \frac{\rho^N}{\alpha_2} \leq \hat{\rho},$$

or, equivalently, if

$$\hat{k} \geq \left\lceil \frac{\log\hat{\rho}\alpha_2}{\log\rho} \right\rceil k.$$

Such an integer $\hat{k}$ always exists for the quadruple $(\hat{\rho}, \alpha_2, \rho, k) \in \mathbb{R}_{(0,1)} \times \mathbb{R}_{(0,1]} \times \mathbb{R}_{(0,1)} \times \mathbb{N}_{\geq 1}$. The proof is complete. ■

Proposition 1 establishes that it is sufficient to verify for a single, arbitrary, proper $\mathcal{C}$-set the set inclusion (8) in order to verify global absolute exponential stability of (3).

On the other hand, it is well known, see e.g. [Lin and Antsaklis, 2009, Proposition 1], that the absolute asymptotic stability property of the system (3) coincides with the absolute asymptotic stability property of the system

$$x_{t+1} \in \text{conv}(\{A_i\}_{i\in\mathbb{N}_{[1,N]}})x_t. \quad (12)$$

Let the set–valued map $\overline{\Phi}_c(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, induced by the system (12), be defined by $\overline{\Phi}_c(\mathcal{S}) := \text{conv}(\{A_i\}_{i\in\mathbb{N}_{[1,N]}})\mathcal{S}$.

Fact 4. Given a proper $\mathcal{C}$–set $\mathcal{S} \subset \mathbb{R}^n$ and a set of matrices $\mathcal{A} := \{A_i\}_{i\in\mathbb{N}_{[1,N]}}$, the following relation holds

$$\text{conv}(\{A_i\}_{i\in\mathbb{N}_{[1,N]}})\mathcal{S} = \text{conv}(\{A_i\mathcal{S}\}_{i\in\mathbb{N}_{[1,N]}}). \quad (13)$$

Proof For any $x \in \text{conv}(\{A_i\}_{i\in\mathbb{N}_{[1,N]}})\mathcal{S}$, there exists a vector $y \in \mathcal{S}$ and scalars $\lambda_i \in \mathbb{R}_+$, $i \in \mathbb{N}_{[1,N]}$, such that $x = \sum_{i=1}^N A_i\lambda_i y$ and $\sum_{i=1}^N \lambda_i = 1$. Setting $y_i := A_i y$, $i \in \mathbb{N}_{[1,N]}$, it holds that $y_i \in A_i\mathcal{S}$. Thus, since $x = \sum_{i=1}^N \lambda_i y_i$, it follows that $x \in \text{conv}(\{A_i\mathcal{S}\}_{i\in\mathbb{N}_{[1,N]}})$. Since x is arbitrary, $\text{conv}(\{A_i\}_{i\in\mathbb{N}_{[1,N]}})\mathcal{S} = \text{conv}(\{\mathcal{A}_i\mathcal{S}\}_{i\in\mathbb{N}_{[1,N]}})$. ■

The next result follows immediately from Theorem 1 and Proposition 1.

Corollary 1. Let $\mathcal{S} \subset \mathbb{R}^n$ be an arbitrary, proper $\mathcal{C}$-set. Then, the system (3) is GAES if and only if there exists a pair $(k, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that

$$\overline{\Phi}_c^k(\mathcal{S}) \subseteq \rho\mathcal{S}. \quad (14)$$

The value of Corollary 1 and Fact 4 lies in the fact that the sets $\overline{\Phi}_c^i(\mathcal{S})$, $i \in \mathbb{N}_{[1,k]}$ are $\mathcal{C}$-sets and can be easily computed,

[1] For the logarithm function log in this article, consider a basis $b \in \mathbb{R}_{\geq 1}$.

for example when $\mathcal{S}$ is a proper $\mathcal{C}$-polytopic set. Moreover, it is worth observing that all subsequent results that concern the verification of absolute stability of system (3) apply also to the equivalent linear system under polytopic uncertainties (12).

Remark 3. The main result in [Bauer et al., 1993] for absolute stability, also reported in [Lin and Antsaklis, 2009, Lemma 1] and generalized for any norm in [Lazar et al., 2013, Corollary V.7], can be recovered directly from Corollary 1, by choosing $\mathcal{S} := \mathcal{B}_\infty$, where $\mathcal{B}_\infty$ is the unit ball of the infinity norm, according to the following result.

Lemma 1. Let $\mathcal{A} \subset \mathbb{R}^n$ be a finite set of matrices. Then,

$$\|\mathcal{A}^k\|_\infty := \max_{A\in\mathcal{A}^k} \|A\|_\infty = \max_{i\in\mathbb{N}_{[1,q]}} \|z^i\|_\infty, \tag{15}$$

where z^i, $i \in \mathbb{N}_{[1,q]}$, are the q vertices of the set $\mathrm{conv}(\mathcal{A}^k\mathcal{B}_\infty)$.

Proof For any single matrix $A \in \mathbb{R}^{n\times n}$, it holds by definition of the induced norm that

$$\|A\|_\infty = \max_{i\in\mathbb{N}_{[1,n]}} \sum_{j=1}^{n} |[A]_{ij}|. \tag{16}$$

Consider the vectors $e_i \in \mathbb{R}^n$, $i \in \mathbb{N}_{[1,2^n]}$ which are all the possible realizations of the vectors which have their elements equal to $+1$ or -1. It is straightforward to see that $\mathcal{B}_\infty = \mathrm{conv}(\{e_i\}_{i\in\mathbb{N}_{[1,2^n]}})$. From (16) it follows that

$$\begin{aligned}\|A\|_\infty &= \max_{i\in\mathbb{N}_{[1,n]}} \max_{l\in\mathbb{N}_{[1,2^n]}} \sum_{j=1}^{n}[A]_{ij}[e_l]_j = \\ &= \max_{l\in\mathbb{N}_{[1,2^n]}} \|Ae_l\|_\infty = \max_{l\in\mathbb{N}_{[1,2^n]}} \|y_l\|_\infty,\end{aligned} \tag{17}$$

where the vectors y_l, $l \in \mathbb{N}_{[1,2^n]}$ constitute the (possibly redundant) set of the vertices of the set $A\mathcal{B}_\infty$. Let $\{A_i\}_{i\in\mathbb{N}_{[1,N^k]}}$ denote the elements of the set $\mathcal{A}^k$. Consequently,

$$\|\mathcal{A}^k\|_\infty = \max_{A\in\mathcal{A}^k} \|A\|_\infty = \max_{i\in\mathbb{N}_{[1,N^k]}} \max_{l\in\mathbb{N}_{[1,2^n]}} \|y_l^i\|_\infty, \tag{18}$$

where y_l^i, $l \in \mathbb{N}_{[1,2^n]}$ denote the vertices of each set $A_i\mathcal{B}_\infty$, for all $i \in \mathbb{N}_{[1,N^k]}$. The result then follows directly. ■

Two alternative necessary and sufficient conditions for global absolute exponential stability, based on Corollary 1, and which can lead to a more efficient numerical verification of GAES, are stated below.

Corollary 2. Consider an arbitrary proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$. Then, the system (3) is GAES if and only if there exists a triplet $(j, i, \rho^\star) \in \mathbb{N}_{\geq 1} \times \mathbb{N}_{[0,j-1]} \times \mathbb{R}_{[0,1)}$ such that

$$\overline{\Phi}_c^j(\mathcal{S}) \subseteq \rho^\star \overline{\Phi}_c^i(\mathcal{S}). \tag{19}$$

Proof The necessity part follows directly from Corollary 1. Conversely, suppose there exists a triplet $(j, i, \rho^\star) \in \mathbb{N}_{\geq 1} \times \mathbb{N}_{[0,j-1]} \times \mathbb{R}_{[0,1)}$ such that (19) holds. Since the set $\mathcal{S}$ is bounded and the mapping $\overline{\Phi}(\cdot)$ is bounded for any bounded set in $\mathbb{R}^n$, it follows from (19) that there exists a positive number $M \in \mathbb{R}_{>0}$ such that $\overline{\Phi}_c^j(\mathcal{S}) \subseteq \rho^\star \overline{\Phi}_c^i(\mathcal{S}) \subseteq \rho^\star M\mathcal{S}$. Then, there exists an integer $r \in \mathbb{N}_{\geq 1}$ such that $\overline{\Phi}_c^{rj}(\mathcal{S}) \subseteq \ldots \subseteq \rho^{\star r} M\mathcal{S}$ and, moreover, $\rho^{\star r} M \in \mathbb{R}_{[0,1)}$. Thus, relation (14) of Corollary 1 holds with $\rho := \rho^{\star r} M$ and $k = rj$ and the system (3) is GAES. ■

Next, we focus on the particular family of proper $\mathcal{C}$-polytopic sets, in order to present a consequent stability verification result.

Corollary 3. Let $\mathcal{S} \subset \mathbb{R}^n$ be an arbitrary proper $\mathcal{C}$-polytopic set. The system (3) is GAES if and only if there exists a pair $(k, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that

$$\overline{\Phi}_c^k(\mathcal{S}) \subseteq \rho\,\mathrm{conv}\Big(\bigcup_{i=0}^{k-1} \overline{\Phi}_c^i(\mathcal{S})\Big). \tag{20}$$

Proof The necessity part follows directly from Corollary 1. Conversely, suppose that there exists a pair $(k, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{(0,1)}$ such that (20) holds. Let $\mathcal{S}^\star := \mathrm{conv}(\cup_{i=0}^{k-1}\overline{\Phi}_c^i(\mathcal{S}))$. Then, it holds that

$$\begin{aligned}\overline{\Phi}_c^1(\mathcal{S}^\star) &= \overline{\Phi}_c^1(\mathrm{conv}(\cup_{i=0}^{k-1}\overline{\Phi}_c^i(\mathcal{S}))) \\ &= \overline{\Phi}_c^1(\cup_{i=0}^{k-1}\overline{\Phi}_c^i(\mathcal{S})) \\ &= \mathrm{conv}(\cup_{i=1}^{k}\overline{\Phi}_c^i(\mathcal{S})) \\ &\subseteq \mathrm{conv}(\cup_{i=0}^{k-1}\overline{\Phi}_c^i(\mathcal{S}) \cup \rho\mathcal{S}^\star) \\ &= \mathrm{conv}(\mathrm{conv}(\cup_{i=0}^{k-1}\overline{\Phi}_c^i(\mathcal{S})) \cup \rho\mathcal{S}^\star) \\ &= \mathrm{conv}(\mathcal{S}^\star \cup \rho\mathcal{S}^\star) = \mathcal{S}^\star.\end{aligned}$$

Thus, the set $\mathcal{S}^\star$ is positively invariant with respect to the system (12), and moreover, the set $\alpha\mathcal{S}^\star$ is positively invariant, for any $\alpha \in \mathbb{R}_{>0}$ [Blanchini and Miani, 2008]. Next, we show that every vertex of the set $\mathcal{S}^\star$ enters $\rho\mathcal{S}^\star$ in a finite number of steps. To this end, for any vertex $y \in \mathcal{S}^\star$, there exists an integer $i \in \mathbb{N}_{[0,k-1]}$ such that $y \in \overline{\Phi}_c^i(\mathcal{S})$. Moreover, there exists an integer $j^\star \in \mathbb{N}_{[1,k-i]}$ such that $\Phi_c^j(y) \in \rho\mathcal{S}^\star$, for all $j \in \mathbb{N}_{\geq j^\star}$, because $\rho\mathcal{S}^\star$ is positively invariant and relation (20) holds. Next, we show that any initial condition that lies in $\mathcal{S}^\star$ is transferred to $\rho\mathcal{S}^\star$ in a finite number of steps. To this end, since $\mathcal{S}^\star$ is a proper $\mathcal{C}$-polytopic set, for any $x_0 \in \mathcal{S}^\star$, by Caratheodory's theorem there exist scalars $\lambda_i \in \mathbb{R}_{>0}$, $i \in \mathbb{N}_{[1,n+1]}$ and vertices y_0^i, $i \in \mathbb{N}_{[1,n+1]}$ such that $x_0 = \sum_{i=1}^{n+1} \lambda_i y_0^i$, and moreover, $\sum_{i=1}^{n+1} \lambda_i = 1$. Then, $\Phi_c^k(x_0) = \Phi_c^k(\sum_{i=1}^{n+1} \lambda_i y_0^i) = \sum_{i=1}^{n+1} \lambda_i \Phi_c^k(y_0^i) = \sum_{i=1}^{n+1} \lambda_i y_k^i$, and since $y_k^i \in \rho\mathcal{S}^\star$, for all $i \in \mathbb{N}_{[1,n+1]}$, for all $y_k^i \in \Phi_c^k(y_0^i)$ it holds that $\Phi_c^k(x_0) \subseteq \rho\mathcal{S}^\star$. Consequently, it holds that $\overline{\Phi}_c^k(\mathcal{S}^\star) \subseteq \rho\mathcal{S}^\star$, and by Corollary 1, the system (3) is GAES. ■

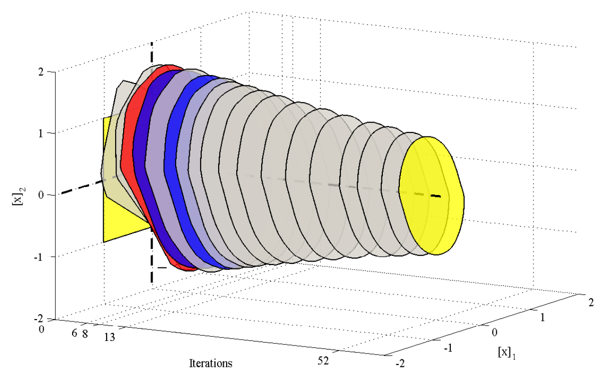

Fig. 1. Example 1. Elements of the set sequence $\{\overline{\Phi}^i(\mathcal{S})\}_{i\in\mathbb{N}_{[0,52]}}$. The sets $\overline{\Phi}_c^8(\mathcal{S}), \overline{\Phi}_c^{13}(\mathcal{S})$, are shown in blue, the sets $\mathcal{S}, \overline{\Phi}_c^{52}(\mathcal{S})$ are shown in yellow and the set $\overline{\Phi}_c^7(\mathcal{S})$ is shown in red color.

Taking into account Corollary 1, Corollary 2 and Corollary 3, an iterative procedure can be established to characterize a switched linear system (3) with GAES. Since any proper $\mathcal{C}$–set can be used as an initial condition of the algorithm, a simple proper $\mathcal{C}$-polytopic set $\mathcal{S}$ (1) can be chosen. For example, for high-dimensional systems, it is preferable to choose a set $\mathcal{S}$ which belongs to the 1–norm polytopic family of sets, whose number of vertices scales linearly with respect to the state space dimensions (for more details see [Athanasopoulos and Lazar, 2013]). In this case, computing the set $\overline{\Phi}_c(\mathcal{S})$ is equivalent to computing the vertices $\{v_{i,j}\}_{(i,j)\in\mathbb{N}_{[1,N]}\times\mathbb{N}_{[1,q]}}$, where $v_{i,j} := A_i v^j$, for all $(i,j) \in \mathbb{N}_{[1,N]} \times \mathbb{N}_{[1,q]}$. Then,

$$\overline{\Phi}_c(\mathcal{S}) := \mathrm{conv}(\{v_{i,j}\}_{(i,j)\in\mathbb{N}_{[1,N]}\times\mathbb{N}_{[1,q]}}).$$

Removal of the redundant vertices in order to obtain a minimal vertex representation of $\overline{\Phi}_c(\mathcal{S})$ is possible by application of any standard vertex elimination algorithm [Ziegler, 2007]. Since $\overline{\Phi}_c(\mathcal{S})$ is a $\mathcal{C}$–polytopic set when $\mathcal{S}$ is a proper $\mathcal{C}$–polytopic set, it follows that all sets $\overline{\Phi}_c^i(\mathcal{S})$, $i \in \mathbb{N}$ are $\mathcal{C}$–polytopic sets. Then, the related set inclusions are equivalent to verifying a set of linear algebraic conditions, obtained by application of the dual generalized Farkas' Lemma.

Remark 4. Theorem 1, and consequently, Corollary 1, Corollary 2 and Corollary 3, offer an alternative method to verify GAES from the well known algorithm established in [Blanchini, 1994], which makes use of the preimage map intersected with a proper $\mathcal{C}$–set. It is worth noting that while the computations in [Blanchini, 1994] utilize the half–space polytopic description, in this article the vertex description is used.

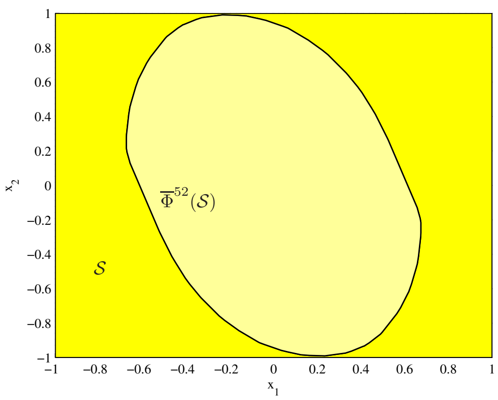

Fig. 2. Example 1, the sets $\mathcal{S}$ (yellow) and $\overline{\Phi}^{52}(\mathcal{S})$ (light yellow). Corollary 1 is satisfied for $k = 52$.

4. NUMERICAL EXAMPLES

Example 1. We consider a second order switched discrete–time system (3) with $\mathcal{A} := \{A_i\}_{i\in\mathbb{N}_{[1,4]}}$, where

$$A_1 = \begin{bmatrix} 1.0 & 2.5 \\ -0.3 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 1.0 & 0.2 \\ -0.45 & 0.85 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0.78 & 0.3 \\ -0.45 & -0.85 \end{bmatrix}, A_4 = \begin{bmatrix} -0.78 & 0.3 \\ -0.15 & -0.5 \end{bmatrix}.$$

The proper $\mathcal{C}$–set $\mathcal{S}$ was set to be the unit sublevel set of the infinity norm, i.e.,

$$\mathcal{B}_\infty = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}.$$

Utilizing Corollary 1, relation (14) is satisfied for $k = 52$. The conditions (19) of Corollary 2 are satisfied for $j = 13$ and $i = 8$, while the condition (20) of Corollary 3 is satisfied for $k = 7$. A graphical depiction of the aforementioned set inclusions is in Figures 2,3 and 4 respectively, while the evolution of the set iterations $\overline{\Phi}^i(\mathcal{S})$, $i \in \mathbb{N}_{[0,52]}$ is shown in Figure 1.

It is worth comparing the computational requirements of the method proposed in this article with the result mentioned in Remark 3. For this example, the corresponding stability condition $\|A\|_\infty < 1$, for all $A \in \mathcal{A}^i$, would be verified for the first time for $i = 52$. This would require the computation of the infinity norm for all elements of the set $\mathcal{A}^{52}$. The cardinality of this set is $4^{52} \cong 2 \times 10^{31}$. Furthermore, applying the algorithm in [Blanchini, 1994], as mentioned in Remark 4, an invariant set was computed in 12 iterations, verifying global Lyapunov stability for the system.

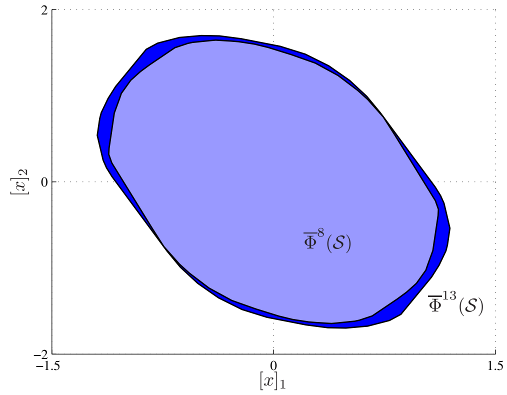

Fig. 3. Example 1, the sets $\overline{\Phi}^{13}(\mathcal{S})$ (blue) and $\overline{\Phi}^{8}(\mathcal{S})$ (light blue). Corollary 2 is satisfied with $j = 13$, $i = 8$.

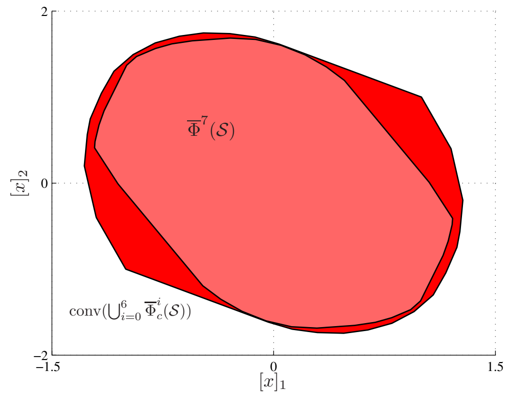

Fig. 4. Example 1, the sets $\mathrm{conv}(\bigcup_{i=0}^{6} \overline{\Phi}_c^i(\mathcal{S}))$ (red) and $\overline{\Phi}^{7}(\mathcal{S})$ (light red). Corollary 3 is satisfied for $k = 7$.

Example 2. We consider the stability analysis problem for a second order system (3), where the set of matrices $\mathcal{A} \subset \mathbb{R}^{2\times 2}$ consists of the two matrices

$$A_1 = \begin{bmatrix} 1 & 2.5 \cdot 10^{-4} \\ -5 \cdot 10^{-4} & 9.9975 \cdot 10^{-1} \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1 & 2.5 \cdot 10^{-4} \\ -2.2425 \cdot 10^{-3} & 9.9975 \cdot 10^{-1} \end{bmatrix}.$$

The example is taken from [Blanchini and Miani, 2008, Example 5.23], where it was used to illustrate the complexity of the stability analysis problem for switched discrete-time linear systems. Using the necessary and sufficient conditions from

Corollary 3 and setting $\mathcal{S} := \mathcal{B}_\infty$, relation (20) is satisfied for $k = 5998$ and $\rho = 0.9998$. In Figure 5 the set $\mathcal{S}^\star := \text{conv}(\bigcup_{i=1}^{k-1} \overline{\Phi}_c^i(\mathcal{S}))$ is depicted in blue color, the set $\overline{\Phi}_c^k(\mathcal{S})$ is depicted in red color, while the set $\mathcal{S}$ is shown in black.

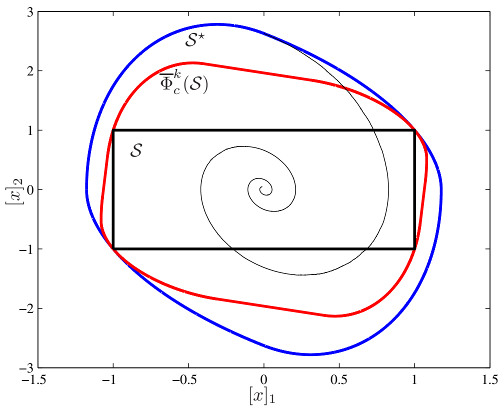

Fig. 5. Example 2. The set $\mathcal{S}^\star := \text{conv}(\bigcup_{i=1}^{k-1} \overline{\Phi}_c^i(\mathcal{S}))$ (blue), the set $\overline{\Phi}_c^k(\mathcal{S})$ (red), the set $\mathcal{S}$ (black), and a solution of the switched system (3) starting from the boundary of $\mathcal{S}^\star$ (thin black line).

5. CONCLUSIONS

An alternative set of necessary and sufficient conditions for absolute exponential stability was established for switched linear systems, using tools from set theory. Three consequent results that concern equivalent stability conditions were proposed, which can be utilized to verify stability. The developed methodology was illustrated on several examples, including a benchmark from the literature.

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