Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

On Synthesis of Stabilizing Distributed Controllers with an Application to Power Systems

E.P. van Horssen [*,**] **M. Lazar** [**] **S. Weiland** [**]

[*] *Control System Technology Group of the Department of Mechanical Engineering*
[**] *Control Systems Group of the Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, e-mail: e.p.v.horssen@tue.nl, m.lazar@tue.nl, s.weiland@tue.nl*

Abstract: This paper presents synthesis results for distributed controllers for interconnected linear time-invariant systems. The setting of the paper is in discrete time. A parameterization of the closed-loop system is used for interconnected systems and distributed static state feedback controllers with an interconnection structure that can be chosen arbitrary in the design phase. Based on conditions for closed-loop stability using centralized static state feedback, computationally tractable synthesis procedures are derived that yield a distributed controller. The synthesis procedures involve convex optimization problems in the form of linear matrix inequalities (LMIs) which are solved in a centralized way. Suggestions are made to reduce the number of independent variables in the problems. An algorithm is presented that uses these synthesis procedures to find a controller with a distributed structure or eliminates candidate controller structures using heuristics. A complexity analysis for the synthesis procedures is incorporated. The synthesis algorithm is illustrated on examples of electric power systems, proving feasibility of the synthesis procedures for real-life applications.

Keywords: Controller synthesis, Convex optimization, Discrete time systems, Distributed control, Electric power systems, Interconnected systems, Stability, State feedback

1. INTRODUCTION

Interconnected systems become increasingly important as a central theme of study in control theory. This focus is understandable since the construction and control of networks of dynamical systems brings a broad range of technical and scientific challenges in control, communication and computation. Examples of such networks are electric power systems (Saadat, 1999), where generators and loads are distributed over the power grid, water networks, such as irrigation channels (Li and De Schutter, 2011), where water flows are connected, IT systems (Napoli and Bamieh, 2001), where data is exchanged between servers, communication networks, where nodes communicate over interconnecting channels and transportation networks (Raza and Ioannou, 1996; Zecevic and Siljak, 2005), where communication between vehicles can provide faster and safer transportation.

The need for distributed control of networked systems is usually motivated by the practical consideration that centralized communication among the network components is not desirable or is practically infeasible. Indeed, the logistic configuration of the network may prevent from a full and centralized communication between a controller and all systems in the network. Also, a model based synthesis of a centralized controller may imply a modeling effort that leads to complex or computationally infeasible models. The decentralized control paradigm amounts to finding controllers for individual systems in the network, but typically ignores the dynamical interactions over communication channels. As such, decentralized control algorithms fall short in proving global stability and robustness properties of the network. Instead, a distributed control architecture allows multiple controllers to exchange information in a well defined manner so as to accomplish a desirable behaviour of the network.

The focus of this paper is on networks of interconnected linear discrete time systems. We aim to derive a computationally tractable procedure for the synthesis of distributed controllers with a predefined structure for the controller network, such that closed-loop stability is guaranteed. Application of the synthesis procedure either results in the construction of a controller or proves the non-existence of a controller with that structure, corresponding to necessary and sufficient conditions for controller existence. The focus is on a simple, but practical, problem set-up. In particular, the focus is on the synthesis of stabilizing static state feedback controllers. The main results include procedures for the synthesis of such controllers based on linear matrix inequalities (LMIs). Application of the theory to power systems is studied. The focus on discrete time systems

[⋆] The first author is supported by the Dutch Technology Foundation (STW) under grant number 12697 "Control based on Data-Intensive Sensing"

allows to also consider delays in the interconnections in a natural way.

Distributed systems have been studied extensively in the past. An overview of various decentralized control schemes, initialized by (Siljak, 1991), has been described in (Bakule, 2008). A low-rank centralized correction was added to all systems in a decentralized scheme by (Zecevic and Siljak, 2005), such that global stabilization was possible when decentralized control was insufficient. The interaction between subsystems has been explicitly taken into account in distributed control research, where the interaction between subcontrollers is an important part of the control strategy. Various studies have been made to apply dissipativity and small gain approaches to these systems. These approaches have not succeeded in solving the problem in a decentralized way and even solve a problem that is more complex than the centralized problem. Based on (D'Andrea and Dullerud, 2003; Langbort et al., 2004) and (Dullerud and D'Andrea, 2004), an LMI-based tool for the synthesis of distributed dynamic output feedback controllers achieving a bounded $\mathcal{H}_\infty$ performance has been developed for a distributed system over an arbitrary graph by (Jokic et al., 2012) and adapted for discrete time by (Van Horssen and Weiland, 2013). Recent work by (Bobiti et al., 2013) shows that these approaches, even when only considering stability analysis, quickly increase the problem complexity when the distributed system grows larger, potentially rendering the problem computationally infeasible.

The goal of this paper is to present a computationally feasible solution to the synthesis problem of stabilizing distributed static state feedback controllers for interconnected discrete time systems. Currently, this problem is not completely solved. A set of necessary conditions for the existence of a distributed controller is employed to search for distributed controllers that satisfy a set of sufficient, but not necessary conditions. In this way the gap between the necessary conditions on one hand and the sufficient but not necessary ones on the other hand, can be reduced. The synthesis procedures have been brought together in a synthesis algorithm. All procedures are convex optimization problems which have to be solved in a centralized way. An assessment of the complexity of each of these procedures has been made. The application of the theory to power systems is studied to ascertain feasibility for systems with increasing complexity, proving feasibility of the synthesis procedures for relevant real-life applicationsfor which other methods are not tractable.

This paper is organized as follows. In Section 2, the parameterization of a closed-loop distributed system is explained in detail. A formal problem formulation is given in Section 3. LMI-based synthesis methods are presented in Section 4. An analysis of the complexity of the methods and an algorithm to use the methods is given. Simulations are presented and discussed in Section 5. Conclusions and comments on future work are given in Section 6.

1.1 Notation

Let $\mathbb{R}$ and $\mathbb{Z}$ denote the set and field of real numbers and the set of integers, respectively. Let $\mathbb{R}^n$ denote the real vectors of dimension n and let $\mathbb{R}^{m\times n}$ denote real matrices of dimension $m \times n$. Let $\mathbb{R}^n_S \subset \mathbb{R}^{n\times n}$ denote the set of symmetric matrices in $\mathbb{R}^{n\times n}$. For every $\Pi \subseteq \mathbb{R}$ and $a, b, c \in \Pi$ define $\Pi_{(a,b]} := \{k \in \Pi \mid a < k \leq b\}$, and let $\Pi\backslash\{c\}$ denote the set Π excluding the element $\{c\}$.

For matrices $A, B \in \mathbb{R}^n_S$, the matrix inequality $A \prec B$ (respectively, $A \preceq B$) means that $B - A$ is positive definite (positive semi-definite, respectively). The transpose of a vector or matrix is denoted by the superscript $\top$. For matrices A_1, A_2, let $\mathbf{diag}(A_1, A_2)$, $\mathbf{col}(A_1, A_2)$, and $\mathbf{row}(A_1, A_2)$ denote block-diagonal, block-column, and block-row matrices, respectively, with matrices A_1, A_2 on its block-diagonal, stacked as a column, and filed in a row, respectively. Similarly, $\mathbf{diag}_{i\in\mathbb{Z}_{[k,\ell]}} A_i$, $\mathbf{col}_{i\in\mathbb{Z}_{[k,\ell]}} A_i$, and $\mathbf{row}_{i\in\mathbb{Z}_{[k,\ell]}} A_i$ denote block-diagonal, block-column, and block-row matrices for matrices $A_k, \ldots, A_\ell$, respectively. In this paper, when discussing matrices, the 'block' property is assumed to be implied and therefore often not mentioned explicitly.

2. CLOSED-LOOP PARAMETERIZATION

In this section we present a parameterization of a closed-loop (CL) system resulting from the interconnection of a distributed system and a distributed controller.

2.1 Distributed systems

A LTI distributed system with L subsystems (with control inputs u and outputs y and interconnection inputs v and outputs w), can be described by

$$\begin{aligned}
&\forall i \in \mathbb{N}_{[1,L]} \quad (i = 1, \ldots, L) \\
&\forall j \in \mathbb{N}_{[1,L]}\backslash\{i\} \quad (j = 1, \ldots, i-1, i+1, \ldots, L) \\
x^i(k+1) &:= A^i x^i(k) + \sum_j F^{ij} v^{ij}(k) + B^i u^i(k) \qquad (1)\\
v^{ij}(k) &:= W^{ij} w^{ji}(k) = W^{ij} x^j(k) \qquad (2)\\
y^i(k) &= x^i(k) \qquad (3)
\end{aligned}$$

where state $x^i(k) \in \mathbb{R}^{n^i_x}$ and input $u^i(k) \in \mathbb{R}^{n^i_u}$. Matrices W^{ij} are pass-through matrices determining if the states of subsystem j affect the states of subsystem i. Matrices W^{ij} are zero or identity depending on whether subsystem i and j are connected, i.e. $W^{ij} = I$ if and only if subsystem i and j are connected, otherwise $W^{ij} = 0$. The elements of $F^{ij} \in \mathbb{R}^{n^i_x \times n^j_x}$ are multiplications of the output gain of the transmitting subsystem (j) and the input gain of the receiving subsystem (i). If state p from a connected subsystem j does not affect state q of subsystem i, this will be reflected in F^{ij} as a zero in the (q,p)-th entry. In this system definition, each system outputs its full state to all other systems. Each receiving system has the information on whether to accept the states or not through W^{ij}. How the accepted states affect the subsystem is specified in F^{ij}.

In the remainder of this paper, when considering system (1)–(3), we drop the time indicator (k) for notational brevity and use $x^+ := x(k+1)$ for the notation of the next time instance.

For completeness, we assume there also exist matrices W^{ii} and F^{ii} which only contain zero elements such that there are no self-connections, i.e. $W^{ii} = 0$ and $F^{ii} = 0$. Note that any such connection can be incorporated in A^i.

The full open loop distributed system can now be described by

$$x^+ := A_O x + Bu = [A + FW]x + Bu \tag{4}$$

where state $x \in \mathbb{R}^{n_x}$ with $n_x = \sum_i n_x^i$ and input $u \in \mathbb{R}^{n_u}$ with $n_u = \sum_i n_u^i$. The system matrices are created from the subsystem matrices, for $i, j \in \mathbb{Z}_{[1,L]}$, according to the parameterization

$$A := \operatorname*{\mathbf{diag}}_i A^i \tag{5}$$

$$F := \operatorname*{\mathbf{diag}}_i F^i \qquad \text{where} \qquad F^i := \operatorname*{\mathbf{row}}_j F^{ij} \tag{6}$$

$$W := \operatorname*{\mathbf{col}}_i W^i \qquad \text{where} \qquad W^i := \operatorname*{\mathbf{diag}}_j W^{ij} \tag{7}$$

$$B := \operatorname*{\mathbf{diag}}_i B^i, \tag{8}$$

where $F \in \mathbb{R}^{n_x \times L * n_x}$ and $W \in \mathbb{R}^{L * n_x \times n_x}$. In particular,

$$FW = \operatorname*{\mathbf{col}}_i F^i W^i \tag{9}$$

and

$$A_O = \begin{bmatrix} A^1 & F^{12}W^{12} & \dots & & F^{1N}W^{1N} \\ F^{21}W^{21} & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & F^{(N-1)N}W^{(N-1)N} \\ F^{N1}W^{N1} & \dots & F^{N(N-1)}W^{N(N-1)} & & A^N \end{bmatrix}. \tag{10}$$

2.2 Distributed controllers

Since control output $y^i = x^i$ is the full subsystem state, a distributed static state feedback controller for such a distributed system with L subcontrollers can be represented by

$$\forall i \in \mathbb{N}_{[1,L]},\ \forall j \in \mathbb{N}_{[1,L]} \backslash \{i\}$$

$$u^i := K^i x^i + \sum_j G^{ij} v_K^{ij} \tag{11}$$

$$v_K^{ij} := W_K^{ij} w_K^{ji} = W_K^{ij} x^j \tag{12}$$

where K^i and G^{ij} are the controller gain matrices and W_K^{ij} are pass-through matrices, similar to the subsystem matrices W^{ij}, determining if subcontroller j affects subcontroller i, i.e. the subcontrollers are connected. This representation is similar in structure to the subsystem representation of the distributed system (1)-(3).

Because W_K^{ij} is restricted to being identity or zero, the multiplication order of the matrices G^{ij} and W_K^{ij} can be changed ($GW_K \to W_K G$, only the dimensions of W_K^{ij} need to be changed from $\mathbb{R}_S^{n_x^i}$ to $\mathbb{R}_S^{n_u^i}$). This will prove useful in synthesis when searching for unknown G^{ij}.

The full distributed controller can now be described by

$$u = K_C x = [K + W_K G]x \tag{13}$$

where for $i, j \in \mathbb{Z}_{[1,L]}$

$$G := \operatorname*{\mathbf{col}}_i G^i \qquad \text{where} \qquad G^i := \operatorname*{\mathbf{diag}}_j G^{ij} \tag{14}$$

$$W_K := \operatorname*{\mathbf{diag}}_i W_K^i \qquad \text{where} \qquad W_K^i := \operatorname*{\mathbf{row}}_j W_K^{ij} \tag{15}$$

$$K := \operatorname*{\mathbf{diag}}_i (K^i) \tag{16}$$

where $G \in \mathbb{R}^{L * n_u \times n_x}$ and $W_K \in \mathbb{R}^{n_u \times L * n_u}$, such that

$$K_C = \begin{bmatrix} K^1 & W_K^{12}G^{12} & \dots & & W_K^{1N}G^{1N} \\ W_K^{21}G^{21} & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & W_K^{(N-1)N}G^{(N-1)N} \\ W_K^{N1}G^{N1} & \dots & W_K^{N(N-1)}G^{N(N-1)} & & K^N \end{bmatrix}. \tag{17}$$

Note that K_C is a parameterized structure and can therefore represent a centralized controller (all elements W_K^{ij} are identity) or a decentralized controller (all elements W_K^{ij} are zero) or anything in between, which can be considered a distributed controller. In particular it can represent a classical distributed controller if the identity and zero elements of W_K coincide with the identity and zero elements of W.

When looking for a controller, we assume that all W_K^{ij} are known, such that the *controller structure* W_K is known.

2.3 Controlled distributed systems

A full closed-loop system, built from the interconnection of a distributed controller (13) with a distributed system (4), can be written as

$$x^+ = \begin{bmatrix} \Phi^1 & \Gamma^{12} & \cdots & \Gamma^{1N} \\ \Gamma^{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Gamma^{(N-1)N} \\ \Gamma^{N1} & \cdots & \Gamma^{N(N-1)} & \Phi^N \end{bmatrix} x \tag{18}$$

$$= [A + BK + FW + BW_K G]x \tag{19}$$

$$= A_C x \tag{20}$$

where $\Phi^i := A^i + B^i K^i$ and $\Gamma^{ij} := F^{ij}W^{ij} + B^i W_K^{ij} G^{ij}$.

3. PROBLEM FORMULATION

Given the parameterization of the closed-loop system, we can now present our main problem formulation.

Problem 1. Given a distributed system (4), give a procedure to synthesize a controller (13), with a given structure, that results in global exponential stability (GES) of the closed-loop system (20), or prove that such a controller does not exist and find a different controller structure which results in GES of the closed-loop system (20).

The envisioned solution to the problem provides necessary and sufficient conditions for the existence of a distributed controller (13) with a particular structure, with the option of explicitly reconstructing this controller when these conditions are satisfied. For the solution to be computationally tractable, we aim to find convex conditions in the form of LMIs.

4. SYNTHESIS

This section presents methods to synthesize a distributed controller (13) for a distributed system (4) satisfying sufficient conditions for GES of the CL system (20). The methods provide a solution to Problem 1 for construction of a controller. We need the following assumption:

Assumption 1. The pair (A_O, B) as in (4) is stabilizable.

In other words, we assume that there exist a centralized controller that stabilizes the distributed system, which is a necessary condition for the existence of a distributed controller.

4.1 Full stability conditions

For a system (20) the necessary and sufficient conditions for GES (Hahn and Baartz, 1967) are

Criterion 1. $\exists P \succ 0$ such that

$$A_C^\top P A_C - P \prec 0. \tag{21}$$

Such that $V(x) = x^\top P x$ is a Lyapunov function (LF) for the CL system. Recall that $A_C = A + BK + FW + BW_K G$. For a given K and G and unknown P, these LMI conditions can be easily verified. However, these conditions pose a non-convex problem for unknown P, K, and G, and can therefore not be solved directly. By Schur's complement (Boyd et al., 1994), Criterion 1 is equivalent to

$$\begin{bmatrix} P^{-1} & A_C \\ A_C^\top & P \end{bmatrix} \succ 0. \tag{22}$$

By pre- and postmultiplication of $\mathbf{diag}(I, P^{-1})$ and using substitution $X = P^{-1}$, we get

$$\begin{bmatrix} P^{-1} & A_C P^{-1} \\ P^{-1} A_C^\top & P^{-1} \end{bmatrix} = \begin{bmatrix} X & A_C X \\ X A_C^\top & X \end{bmatrix} \succ 0. \tag{23}$$

We can now present the following necessary and sufficient conditions for the existence of a distributed controller (13) that stabilizes a distributed system (4).

Theorem 2. Given a distributed system (4), the distributed controller (13) achieves GES for (20) with parameterized gain matrices K and G and interconnection structure W_K *if and only if* there exists $X \succ 0$, such that

$$\begin{bmatrix} X & A_C X \\ X A_C^\top & X \end{bmatrix} \succ 0 \tag{24}$$

where

$$A_C X = AX + BKX + FWX + BW_K GX. \tag{25}$$

Now the problem is a bilinear matrix inequality (BMI) in, on the one hand X, and on the other hand K and G. For a given X, i.e. a given candidate LF, the conditions in Theorem 2 give a feasibility test in the form of LMIs in K and G. Successful evaluation of the feasibility test gives a controller with the desired structure.

In general we do not know a LF that will allow for a distributed controller, and thus X remains unknown. The classical way to transform the synthesis problem from BMI conditions to LMI conditions is through a change of variables ($KX = Y$ and $GX = Z$):

$$\begin{aligned} A_C X &= AX + BKX + FWX + BW_K GX \qquad (26) \\ &= AX + BY + FWX + BW_K Z. \qquad (27) \end{aligned}$$

Since X is invertible, matrices K and G can be explicitly reconstructed by $K = YX^{-1}$ and $G = ZX^{-1}$. Unfortunately, this procedure results in a loss of the structure of K and G and will therefore, in general, not give the desired distributed controller structure.

We can present the following necessary conditions for the existence of a distributed controller (13).

Theorem 3. Given a distributed system (4), the distributed controller (13) achieves GES for (20) with parameterized gain matrices K and G and interconnection structure W_K *only if* there exist $X \succ 0$, Y with dimension of KX, and Z with dimension of GX, such that

$$\begin{bmatrix} X & A_C X \\ X A_C^\top & X \end{bmatrix} \succ 0 \tag{28}$$

where

$$A_C X = AX + BY + FWX + BW_K Z. \tag{29}$$

Theorem 3 presents a convex problem in the form of LMI conditions. In the remainder of the paper we will indicate the evaluation of these conditions as *Method 1.*

We can see that the conditions are over-parameterized. Setting $Z = 0$ yields necessary and sufficient conditions for the existence of a centralized controller which can be reconstructed. From Assumption 1 we know that a centralized controller exists. Therefore, for full matrices X, Y, and Z, Method 1 is always feasible, but will likely not produce a distributed controller since matrices Y and Z contain more independent variables than K and G. A logical approach is to give Y and Z the same structure, and thus the same number of independent variables, as K and G. Unfortunately, this does not yield sufficient conditions for controller reconstruction. The multiplication of a matrix with a certain structure (such as Y and Z) by a full matrix X^{-1} does not (in general) yield a matrix with the same structure (which we require for K and G).

Method 1 does not directly yield a practical result for controller synthesis since it can only prove non-existence of a controller, as posed in Problem 1, using a method that is more complex than the centralized synthesis problem. However, the structure of the conditions in Theorem 3 is useful as we will show in the remainder of this section.

We have shown that we can present necessary and sufficient conditions for the existence and construction of a distributed controller in the form of BMIs. Also, we have shown that we can present necessary conditions for the existence of a distributed controller in the form of LMIs at the cost of losing sufficient conditions for controller construction. This shows that there is a gap between the necessary conditions, which are related to a full X, and sufficient conditions, related to the structure of K and G.

4.2 Structured stability conditions

In Section 4.1, a way to retain the controller structure in Method 1 was suggested by giving Y and Z the same structure as K and G, respectively. Because X is a full matrix the reconstruction did not yield a controller of the desired structure. However, if we allow for a specific structure in X, in particular require X to have a block diagonal structure (Zecevic and Siljak, 2010), the reconstruction does yield a controller with the desired structure. We can present the following sufficient LMI conditions for the existence of a distributed controller (13).

Theorem 4. Given a distributed system (4), the distributed controller (13) achieves GES for (20) with parameterized gain matrices K and G and interconnection structure W_K *if* there exist $X \succ 0$, Y with dimension of KX, and Z with dimension of GX, such that

$$\begin{bmatrix} X & A_C X \\ X A_C^\top & X \end{bmatrix} \succ 0 \tag{30}$$

where

$$A_C X = AX + BY + FWX + BW_K Z \tag{31}$$

and X has block diagonal structure of compatible dimension to K, and Y and Z have block structures conform with the block structures of K and G, respectively. In that case, the controller matrices K and G can be reconstructed through

$$K = YX^{-1} \tag{32}$$

$$G = ZX^{-1}. \tag{33}$$

The proof of the theorem follows directly from the manipulations of Criterion 1 in Section 4.1 and elementary matrix inversion algebra. Theorem 4 presents a convex problem in the form of LMI conditions. In the remainder of the paper we will indicate the evaluation of these conditions as *Method 2*.

Using a block diagonal X corresponds to a decentralized controller synthesis problem. The parameterization presented in Section 2 allows for synthesis of a distributed controller with a particular interconnection structure. If Method 2 yields a solution we immediately have a stabilizing distributed controller and a LF for the CL system. As an additional benefit of the method, the number of unknown variables is reduced. However, for a given system there may only exist a stabilizing controller for which only a full (or merely non-block-diagonal) matrix X can prove stability. This limitation is addressed in the next subsection. Method 2 gives a solution to Problem 1 if the CL system only admits block diagonal quadratic LFs.

4.3 Fixed control Lyapunov function

The results given by Method 1 and Method 2 have limitations. Method 1 uses a full matrix X for the candidate LF, but is unable to recover the desired controller structure. Method 2 can recover the structure of the controller, but poses structure on X and giving a smaller set of candidate LFs. Here we propose a synthesis method that uses a full matrix X for the candidate LF and allows for construction of a controller with the desired interconnection structure.

We have mentioned that, for a given X, we can evaluate Theorem 2 as a feasibility test and find parameterized gain matrices K and G. The problem lies in having a good choice for the candidate LF. As a good choice, we suggest taking the Lyapunov matrix X from Method 1, which represents a control Lyapunov function (CLF) for the system. This leads to a search for a distributed controller (13) that achieves the same convergence rate for the states of the system as a centralized controller can achieve. We argue that this is a useful result, since implementation of a distributed controller with a particular structure has various benefits over the implementation of an unstructured centralized controller. In the remainder of the paper, solving the matrix inequalities in Theorem 2 for Lyapunov matrix X inferred from Method 1 and structured matrix variables K and G, such that the conditions are LMIs in K and G, will be indicated as *Method 3a*.

Remark 1. A condition for this procedure to work is that a X can be found in Method 1, but computational complexity is a limitation for the procedure. As a computationally more tractable approach to find a X in Method 1, variables corresponding to elements of X can be set to zero. Practical suggestions for choosing these variables are the (i, j)-th and (j, i)-th blocks where subsystem i and j are far apart in terms of graph distance, or choosing X to have the same structure as K_C. The same procedure can be followed for Y and Z. Since a full X corresponds to particular convergence conditions for a centralized controller which can be infeasible for a distributed controller, having a non-full CLF can relax the necessary conditions enough to make the sufficient conditions feasible, yielding a solution in Method 3a.

Remark 2. A computationally more tractable procedure for finding a CLF X than to solve Method 1, is to solve the centralized synthesis problem. Or equivalently, solving Method 1 with the variables in Z eliminated.

Alternatively or supplementary to Method 3a, one may search for a full matrix K in Method 3a and minimize the off-diagonal elements. This is done by minimizing a cost function consisting of the sum of the squares of the off-diagonal elements in K which are decision variables of the convex problems. In the remainder of the paper this method will be indicated as *Method 3b*. This relaxation in the structure of K_C will always yield a feasible solution as it is an extension of the centralized synthesis problem. If the resulting K is not diagonal, it will show which off-diagonal elements are problematic. This gives an indication as to which additional interconnections are needed in the controller network, i.e. suggestions to change elements of W_K from zero to identity.

Remark 3. Unallowable interconnections can be prevented by not introducing the corresponding off-diagonal elements in K.

Remark 4. If some connections are more preferred than others, this can be reflected in a variation in cost per variable. A practical choice is to relate this variation to the graph distance.

Remark 5. Eliminating the non-zero off-diagonal elements in K can be a good alternative to adding interconnections, i.e. changing W_K, even if the elements are not small. The inferred controller may still yield a stable CL system. This can be easily checked with a stability test, e.g. an eigenvalue test, on the CL system.

Methods 3a and 3b do not give a complete solution to Problem 1, since for a given X, a distributed controller with a particular structure may not exist, while there exists a controller and another LF that satisfy Theorem 2.

4.4 Complexity assessment

The complexity of the presented synthesis methods can be analyzed through the number of independent variables in the convex optimization problems that need to be solved. In Table 1 an overview of the complexity of each of the presented methods is given as well as the case of synthesis of a centralized controller for comparison. For each method the unknown elements, the variables that are used in the LMI problem, and the number of independent variables is given.

Table 1. Complexity of Synthesis Problems.

Synthesis Problem (Method)	Unknown elements	Variables (Structure)	#Variables
Centralized	X, K	X (full, sym), Y (full)	$\frac{n_x \times (n_x+1)}{2} + n_u \times n_x$
Distributed Necessary (M. 1)	X, K, G	X (full, sym), Y (full), Z (full)	$\frac{n_x \times (n_x+1)}{2} + n_u \times n_x + L \times n_u \times n_x$
Distributed Diagonal (M. 2)	X, K, G	X (diag, sym), Y (diag), Z (diag)	$\sum_i \frac{n_x^i \times (n_x^i+1)}{2} + n_u \times n_x$
Fixed X (M. 3a)	K, G	K (diag), G (diag)	$n_u \times n_x$
Fixed X, minimize K (M. 3b)	K, G	K (full), G (diag)	$n_u \times n_x$

The number of variables shows that Method 1 does indeed add more variables to the centralized conditions. In Method 2 the number of variables is reduced compared to the centralized case. In the case of Method 3a, Method 1 has to be solved first and is therefore more involved. However, the controller synthesis part of Method 3a has a greatly reduced number of variables since variable X is known. When the off-diagonal elements of K are reintroduced in Method 3b, the number of variables grows again.

Some elements of G or Z do not affect the stability conditions, since they are always multiplied by zero elements in W_K. These element can be removed from the procedure (note that some LMI solvers do this automatically). In the case of minimizing K, the elements corresponding to an identity element in W_K are removed, since adapting G can then always reduce them to zero. This can be done for Methods 1, 2, and 3a.

Remark 6. One may argue that a centralized solution may not be computable, but alternative approaches, such as dissipativity or small-gain approaches, are prone to reach computational limits quickly as well (Bobiti et al., 2013). To date, no distributed solution has been found for solving a set of LMIs with coupling constraints and existing results solve a problem that is more complex than the centralized problem (Van Horssen and Weiland, 2013). The proposed approach starts from the centralized problem and suggests methods of reducing the number of variables.

The conditions in this section are only sufficient, since there can exist another X which proves stability for a distributed controller, while the X resulting from the centralized problem does not. However, they can be solved for large problems as shown in what follows.

4.5 Synthesis procedure

Here we propose an algorithm for the application of the synthesis methods. A graphical representation of the algorithm is shown in Fig. 1. The idea of the algorithm is to use the complexity of the methods to order them. The number of variables is a relevant metric for distributed systems of growing complexity. The number of variables is also related to the complexity of the resulting controller.

First, Method 2 is evaluated with a given controller structure W_K. If the procedure is successful we are done and have a LF corresponding to X, and controller gains K and G. If the procedure fails, i.e. there exists no block diagonal LF X for the chosen controller structure, we solve Method 1 to get a CLF corresponding to X which we use as a candidate LF in Method 3a. Successful evaluation again gives X, K, and G. If the procedure fails we can choose to re-evaluate Method 1 with a new structure for X (and Y and Z), and repeat the procedure. Or we can use Method 3b and find a full K which has off-diagonal

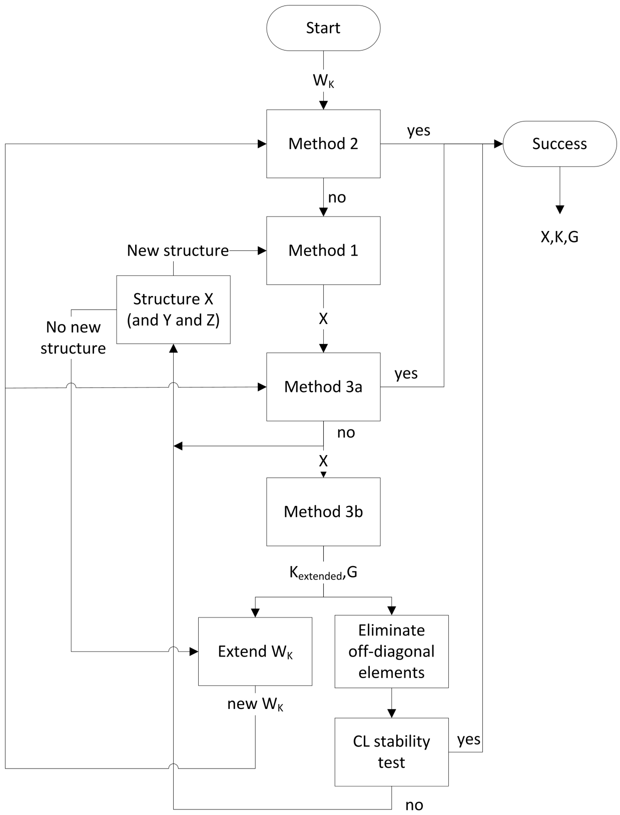

Fig. 1. Algorithm flowchart.

elements. We can then choose to introduce new control interconnections by extending the structure of W_K by introducing more/other identity elements. After extending the structure of W_K we can restart with Method 2. Or we can eliminate the off-diagonal elements in K and test if the resulting CL system is stable. If the system is stable we have K and G and solve a convex optimization problem to find a LF corresponding to X for the CL system. If the resulting CL system is not stable, we again change the allowed set of control LFs in Method 1 by changing the structure of X (and Y and Z). If we have exhausted all possible structures for X we have to look for an extended structure of W_K.

Note that the algorithm always gives a solution, but the resulting controller structure can differ from the originally desired structure, depending on choices made in the algorithm by the user. The presented algorithm uses a heuristic approach to solving Problem 1 using a trade-off between problem complexity and conservatism. The candidate solution set will be rich, since for each interconnection structure the algorithm can iterate through a large number of CLFs for the CL system.

5. DISTRIBUTED CONTROLLER SYNTHESIS FOR POWER SYSTEMS

To show the applicability of the developed synthesis methods for a relevant real-life application, this section focuses on synthesis of stabilizing distributed controllers for power systems. Power systems, see, e.g., (Kundur, 1994; Saadat, 1999) are becoming more demanding due to the decentralization introduced by the reorganization from monopolistic to liberal markets and introduction of renewable power generation technologies. In this context, guaranteeing stability of the network is not a trivial problem any more.

5.1 Modeling of power systems

Typically, two different control layers can be distinguished in power systems. An upper market–based layer and a lower control layer, see details in (Jokić, 2007). Typically, the upper control layer determines slowly varying generation profiles for each power plant while for the lower control layer, relatively small changes with respect to these profiles are considered. This particular case study focuses on the lower control layer, where the most important parameter for control is the frequency of the network.

Basically, a power system is composed of N generator-load components and a finite number of simple loads q, which are interconnected through a network of tie lines. We denote this as a $(N+q)$–bus system. Therefore, we start by describing the model of a subsystem in a power system. An accurate model for the i–th generator dynamics, $i \in \mathbb{N}_{[1,N]}$ is given by (Pai, 1981) the linearized equations

$$\begin{aligned} \dot{\delta}_i(t) &= \omega_i(t) \\ \dot{\omega}_i(t) &= \tfrac{1}{H_i}\big(P_{\mathrm{T}_i}(t) - D_i\omega_i(t) - \textstyle\sum_{j=1}^{N}[\Upsilon]_{i,j}\delta_j(t)\big) \\ \dot{P}_{\mathrm{T}_i}(t) &= \tfrac{1}{\tau_{\mathrm{T}_i}}\big(P_{\mathrm{G}_i}(t) - P_{\mathrm{T}_i}(t)\big) \\ \dot{P}_{\mathrm{G}_i}(t) &= \tfrac{1}{\tau_{G_i}}\big(-P_{\mathrm{G}_i}(t) - \tfrac{1}{r_i}\omega_i(t)\big) \end{aligned}, \qquad (34)$$

with $t \in \mathbb{R}_+$ and where δ_i [rad] is the rotor phase angle, ω_i [rad/s] the rotor frequency, P_{T_i} [MW] the turbine state and P_{G_i} [MW] the governor state of the i–th generator, respectively. The corresponding parameters are the inertia, H_i, the damping coefficient D_i, the turbine and governor time constants, i.e., τ_{T_i} and τ_{G_i} and the regulation constant of the primary loop r_i.

The interconnections in the power system are described by a weighted adjacency matrix $\Upsilon \in \mathbb{R}^{N\times N}$, where the elements of the matrix Υ have the unit $[\Omega^{-1}]$ and define the virtual inductive reactance between bus i and bus j. The virtual inductive reactance is obtained via an elimination of the buses (Hermans, 2012) that do not contain generators but only external loads such that $\Upsilon := (B_{11} - B_{12}B_{22}^{-1}B_{21}) \in \mathbb{R}^{N\times N}$ where $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} := C - \mathrm{diag}(C\mathbf{1}_N)$. The matrix $C \in \mathbb{R}^{(N+q)\times(N+q)}$ contains the real inductive reactances of the $(N+q)$–bus system.

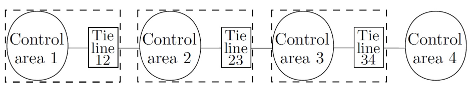

Fig. 2. 4–Generators [(Jokic et al., 2012)].

The paper considers four benchmark power systems, i.e. 4–Generators (Jokic et al., 2012), i.e., CIGRÉ–7, see, e.g.,

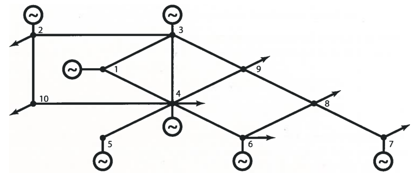

Fig. 3. CIGRÉ–7 [(Hermans, 2012)].

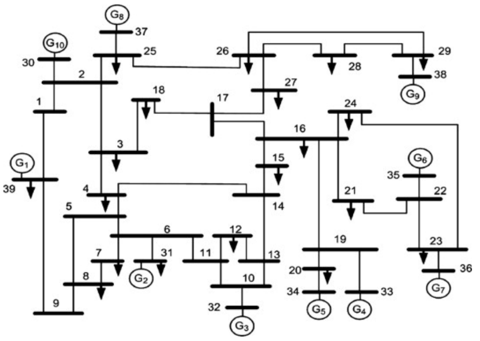

Fig. 4. New England [(Jokić, 2007)].

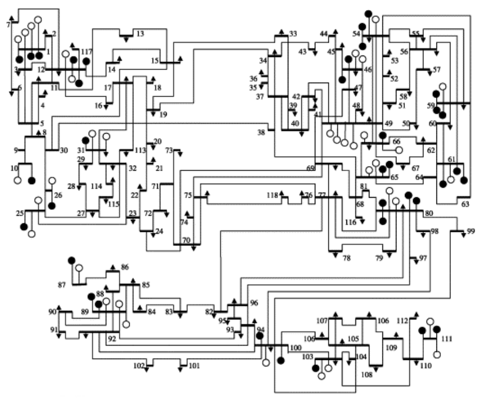

Fig. 5. 118–bus [(Yamin and Shahidehpour, 2003)].

(Hermans, 2012), the 39–Bus New England test system, see, e.g., (Jokić, 2007; Pai, 1981) and the IEEE 118–bus. For the 4–Generators model the parameters are taken from (Venkat et al., 2008). For the other three power systems the corresponding parameters are taken to be in the ±20% interval from the values given by (Saadat, 1999) on page 545, as done by Bobiti et al. (2013). In Table 2 the number of subsystems, states, and the number of inputs for each system are shown in increasing order of dimension.

Table 2. Number of subsystems, states, and number of inputs of the power system model.

Power system	#Subsystems L	#States n_x	#Inputs n_u
4–Generators	4	15	4
CIGRÉ–7	7	28	7
New England	10	40	10
IEEE 118–bus	54	216	54

The discretization of the networks is performed with Euler forward method and a sampling time of 0.01s, such that it complies with the Shannon–Nyquist sampling theorem in the context of the considered power systems, which have the time constant of the governor around the value of 0.2s. Note that Euler forward can be used on the system as a whole. Other discretization methods such as zero–order hold do not retain the interconnection structure of the system.

5.2 Synthesis results

Each synthesis method has been implemented for each of the power systems. A controller structure equivalent to the system structure was chosen. All computations are done on a PC with a Intel i7-3770 processor @3.4GHz with 16GB RAM and are carried out in Matlab using freely available software Yalmip (Lofberg, 2004) and SeDuMi (Sturm, 1999).

4–Generators: Methods 1 and 2 were solved successfully and without errors. Method 3a failed on numerical problems in the solver and was not able to find a K with diagonal structure. Method 3b provided a solution where the off-diagonal elements were small (order 10^3 smaller compared to the other elements). Eliminating them still gave a stable CL system, which was proven by the same X. This shows that even though Method 3a might fail on numerical difficulties in the solver, a distributed solution may exist, which can be found with Method 3b.

CIGRÉ–7: Methods 1, 2, and 3b were solved successfully and without errors. Method 3a gave a warning on numerical difficulties, but was able to find a controller with diagonal structure.

New England: Equivalent results to the CIGRÉ–7 system.

IEEE 118–bus: Solving the centralized synthesis problem for this system resulted in a memory error from the solver, due to the large amount of variables involved. Obviously Method 1 was not solvable either. Method 2 was successful and resulted in a stabilizing controller. This shows that the reduction of variables in Method 2 is useful. Because the 118-bus system did not have a full centralized solution, Method 1 was solved for a new structure of the variables. A full X, and Y and Z with structure of K and G were used. This reduced the number of variables sufficiently to find a X. Note that this involved a long computation time (12-24 hours). In the next step, Method 3a gave numerical problems. Method 3b still involved a lot of variables, and again took a long computation time (12-24 hours), but finds a solution as well. The resulting K has off-diagonal elements that are non-negligible. However, setting the off-diagonal elements to zero still resulted in a stable CL system as proven by a CL stability test. The CLF of the centralized problem was not a LF for the closed loop system. To reduce the computation time, the centralized synthesis problem was also solved with both X and Y with structure of K_C to find X. Note that this is equivalent to searching for Y with structure of K and Z with structure of G in Method 1. This gave similar results to using a full X but resulted in a lower computation time.

For the IEEE 118–bus system, the response to an initial condition of '1' for each state for controllers resulting from different synthesis procedures is shown in Fig. 6, Fig. 7, Fig. 8, and Fig. 9.

From the plots we can see that all closed-loop systems are stable, which is confirmed by checking the eigenvalues of the stability conditions, the Lyapunov matrix, and the

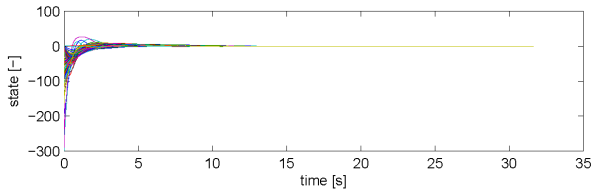

Fig. 6. Response to initial conditions for IEEE 118-bus system and controller from Method 2.

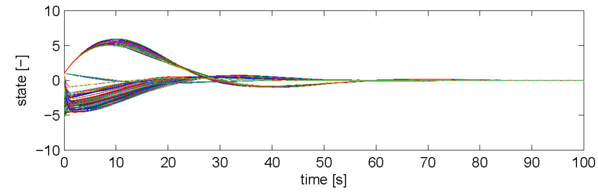

Fig. 7. Response to initial conditions for IEEE 118-bus system and centralized controller using full X and Y with structure of K_C.

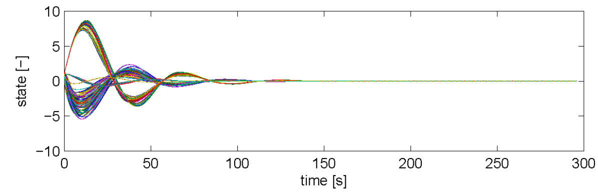

Fig. 8. Response to initial conditions for IEEE 118-bus system and controller from Method 3b using full X and $Y + Z$ with structure of K_C, where the off-diagonal elements of K have been eliminated.

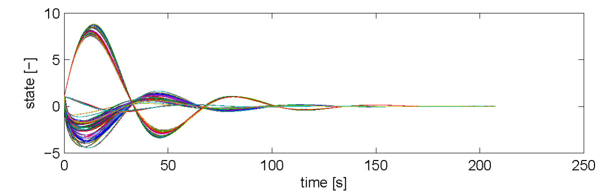

Fig. 9. Response to initial conditions for IEEE 118-bus system and controller from Method 3b using X and $Y + Z$ with structure of K_C.

eigenvalues of the CL system. No comparison between the rates of convergence can be made since no condition to optimize this is incorporated. To optimize the rate of convergence we could add a constraint $X \succ \alpha I$ and maximize $\alpha > 0$, such that the maximum eigenvalue of Lyapunov matrix P is minimized. Since no such constraint is added in the used methods, and since no condition to keep the controller gain small is added, we see a variation in the responses. In particular, Fig. 6 shows fast convergence, but has a large controller gain (order $10^1 - 10^2$ larger) compared to the other controllers.

For comparison we give the number of variables involved in the problems for various cases on the IEEE 188-bus system. A full X has 23426 variables, whereas a X with the structure of K_C has only 3052 variables. A full Y has 11664 variables, whereas a Y with the structure of K_C has only 1472. A full G, i.e. all interconnections are allowed, introduces 11664 variables, whereas $W_K \times G$ reduces the number of variables to 1256. This shows that more complex problems may be solvable by reducing the number of independent variables, which can be done in a logical way as explained in Section 4.3.

6. CONCLUSIONS AND COMMENTS

In this paper we presented a number of synthesis results for stabilizing distributed static state feedback controllers for interconnected discrete time dynamical systems. The results follow from a parameterization of the closed-loop system. The interconnection structure of the controller is an explicit design choice, such that the distributed controller can have the structure of a decentralized or a centralized controller. The main results include feasibility tests involving linear matrix inequalities (LMIs) of necessary conditions and sufficient conditions for stability in controller synthesis. The synthesis results have been brought together in a synthesis algorithm. With modern numerical tools in semi-definite programming the presented procedures are computationally powerful and efficient. The results in this paper still show a gap between the necessary and sufficient conditions for the existence of a stabilizing state feedback law. A number of suggestions is given to bridge this gap using a heuristic approach. Procedures for controller synthesis involve restricting the LF to be block diagonal, and using the LF from the centralized synthesis problem. Suggestions for reduction of the complexity of all problems are given, as to infer procedures that are computationally tractable even for systems of high state-space and input-space dimensions. This is shown in simulations on benchmark power systems. To the authors' knowledge the presented synthesis procedures are novel.

ACKNOWLEDGEMENTS

The authors would like to thank R.V. Bobiti MSc for providing the power system models and their system descriptions.

REFERENCES

Bakule, L. (2008). Decentralized control: An overview. *Annual Reviews in Control*, 32(1), 87–98.

Bobiti, R.V., Gielen, R.H., and Lazar, M. (2013). Non-conservative and tractable stability tests for general linear interconnected systems with an application to power systems. In *4th IFAC Workshop on Distributed Estimation and Control in Networked Systems, NecSys 2013*, 152–159.

Boyd, S., Ghaoul, L., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics. Society for Industrial and Applied Mathematics.

D'Andrea, R. and Dullerud, G. (2003). Distributed control design for spatially interconnected systems. *Automatic Control, IEEE Transactions on*, 48(9), 1478–1495.

Dullerud, G. and D'Andrea, R. (2004). Distributed control of heterogeneous systems. *Automatic Control, IEEE Transactions on*, 49(12), 2113–2128.

Hahn, W. and Baartz, A. (1967). *Stability of motion*. Grundlehren der mathematischen Wissenschaften. Springer.

Hermans, R.M. (2012). *Distributed control of deregulated electrical power networks*. Ph.D. thesis, Eindhoven University of Technology, The Netherlands.

Jokić, A. (2007). *Price–based Optimal Control of Electrical Power Systems*. Ph.D. thesis, Eindhoven University of Technology.

Jokic, A., Van der Els, T.F., and Weiland, S. (2012). Robust distributed h-infinity control of electrical power systems. In *American Control Conference (ACC), 2012*, 3637–3642.

Kundur, P. (1994). *Power system stability and control*. McGraw–Hill.

Langbort, C., Chandra, R.S., and D'Andrea, R. (2004). Distributed control design for systems interconnected over an arbitrary graph. *Automatic Control, IEEE Transactions on*, 49(9), 1502–1519.

Li, Y. and De Schutter, B. (2011). Stability and performance analysis of an irrigation channel with distributed control. *Control Engineering Practice*, 19(10), 1147 – 1156.

Lofberg, J. (2004). Yalmip : a toolbox for modeling and optimization in matlab. In *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, 284–289.

Napoli, M. and Bamieh, B. (2001). Modeling and observer design for an array of electrostatically actuated microcantilevers. In *Decision and Control, 2001. Proceedings of the 40th IEEE Conference on*, volume 5, 4274–4279 vol.5.

Pai, M.A. (1981). *Power System Stability*. North–Holland Publishing Company, Amsterdam, The Netherlands.

Raza, H. and Ioannou, P. (1996). Vehicle following control design for automated highway systems. *Control Systems, IEEE*, 16(6), 43–60.

Saadat, H. (1999). *Power system analysis*. The McGraw–Hill Companies, Inc.

Siljak, D. (1991). *Decentralized Control of Complex Systems*. Mathematics in science and engineering. Academic Press.

Sturm, J.F. (1999). Using sedumi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12, 625–653.

Van Horssen, E.P. and Weiland, S. (2013). Synthesis of distributed robust h-infinity controllers for interconnected discrete time systems. Unpublished: Submitted to IEEE Transactions on Control of Network Systems.

Venkat, A.N., Hiskens, I.A., Rawlings, J.B., and Wright, S.J. (2008). Distributed mpc strategies with application to power system automatic generation control. *IEEE Trans. Contr. Sys. Techn.*, 16(6), 1192–1206.

Yamin, H. and Shahidehpour, S.M. (2003). Transmission congestion and voltage profile management coordination in competitive electricity markets. *International Journal of Electrical Power & Energy Systems*, 25(10), 849–861.

Zecevic, A. and Siljak, D. (2010). *Control of Complex Systems: Structural Constraints and Uncertainty*. Communications and Control Engineering. Springer.

Zecevic, A. and Siljak, D. (2005). Global low-rank enhancement of decentralized control for large-scale systems. *Automatic Control, IEEE Transactions on*, 50(5), 740–744.