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Stabilization of time-delay nonlinear discrete-time systems with saturating actuators through T-S models ⋆

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Abstract: In this paper the problem of local stabilization of nonlinear discrete-time systems with time-varying delay and saturating actuators is studied. Firstly, through a fuzzy Lyapunov-Krasovskii (L-K) function, we develop convex conditions to synthesize fuzzy state feedback gain controllers that stabilize the nonlinear system subject to saturating actuators. Next, we introduce a new approach to compute an estimate of the region of attraction where the initial condition sequence is split into two subsequences. The first one is composed of the state vector at the actual instant of sampling, i.e. for $k = 0$. The second one is composed of the state vectors at the delayed samplings. Then, we propose a convex optimization problem to maximize the estimated region of attraction of the closed loop control system. Finally, we give a numerical example to illustrate the obtained results.
Keywords: Discrete-time nonlinear systems, delayed states, saturating input, Lyapunov-Krasovskii fuzzy function, Takagi-Sugeno models, LMIs.

1. INTRODUCTION

In the last decades, the control system community has experienced the development and the application of fuzzy based control techniques. In particular, the fuzzy techniques based on the Takagi-Sugeno (T-S) modeling approach has received lots of attention, because it allows an exact or an approximate representation of nonlinear systems as a blend of linear models Feng [2009]. Some successful applications in controller synthesis can be found, for instance, in Tanaka and Wang [2001], Feng [2009], and Sung et al. [2012].

Dynamic systems with delay are often found in industrial processes especially when there is transfer of mass, energy, and/or information. The delay usually causes performance deterioration and even loss of stability Miranda and Leite [2011], Niculescu [2001]. In the last years, the academic community has given great attention to the control problem of systems with delayed states, as can be seen, for example, in Gassara et al. [2010], Liu et al. [2010], and Xu et al. [2012]. Also, actuator saturation is present in practical control systems. It is well known that it may cause loss of performance and, in some cases, even unstable behavior Hu and Lin [2001], Tarbouriech et al. [2011]. A fundamental issue in this context is to estimate the region of attraction, as the exact region of attraction is in general hard to determine or even impossible to find by analytical means Tarbouriech et al. [2011].

In the context of systems with state delay, the estimate of the region of attraction is even more challenging because a sequence of state vectors is required as the initial conditions. The actuator saturation problem for systems with time-delay has been studied, as can be seen in Bender et al. [2011], Dey et al. [2012], Gomes da Silva Jr. et al. [2009], Ting and Liu [2011], and Wang et al. [2013], where all theses papers deal with continuous-time systems with state delay. Moreover, the estimated region of attraction is characterized as one ball set, which may lead to conservative results. Similar research lines are pursued in Ting and Chang [2011] where continuous-time systems with actuator saturation and delay in the state are considered under the T-S modeling and the region of attraction is given as an ellipsoidal set.

The discrete-time version of T-S fuzzy systems with delayed states and saturating input has received much less attention, although it is practically appealing. In Xia-Na

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et al. [2012] the stabilization problem is studied by a non-convex design of a dynamic output fuzzy controller and an ellipsoidal set is employed to determine a region of local stability. However, it is not clear how the delayed states are handled. In Xu et al. [2012] a ball is used to characterize the region of local stability in the presence of some exogenous disturbances.

The objective in this paper is to propose some convex conditions in terms of linear matrix inequalities (LMIs) for the synthesis of fuzzy stabilizing feedback controllers of T-S discrete-time systems with time-varying delay and actuator saturation. The proposed conditions are based on an fuzzy Lyapunov-Krasovskii (L-K) function that assures the converging to the origin of the trajectories initiated in a so-called estimate of the region of attraction. The characterization and computation of such an estimate is a main result introduction in the present work for dealing with control saturations. Firstly the sequence of initial states is split into two subsequences: one composed of the state vector at the actual instant, i.e. at sampling time $k = 0$, and the other made up by the delayed state vectors present in the initial condition. Each of these subsequences are used to characterize two sets depending on the maximum value of the delay. The Cartesian product of these sets yields the estimated region of attraction. Additionally, a convex optimization problem is proposed to synthesize a fuzzy state feedback controller that maximizes the corresponding estimated region of attraction. A numerical example is given to illustrate the effectiveness of the proposed approach.

Notations: The r-th row of the matrix L is denoted as $L_{(r)}$. The symbol $\star$ represents the symmetric blocks in a symmetric matrix. Matrices $\mathbf{I}$ and $\mathbf{0}$ denote, respectively, identity and null matrices of appropriate dimensions. For $d \in \mathbb{N}^*$ and $k \in \mathbb{N}$, $\phi_{d,k} = \{x_{k-d}, x_{k-(d-1)}, \dots, x_{k-1}\}$ denotes a sequence of d vectors $x_j \in \mathbb{R}^n$, $j \in [-d, -1]$, where $[a, b]$ is the interval of the integer numbers starting in "a" and ending in "b". Consider that $\varphi_{d,k}$ defines a sequence of $d+1$ vectors $x_j \in \mathbb{R}^n$, $j \in [-d, 0]$, such that $\varphi_{d,k} = \{\phi_{d,k}, x_k\}$. The space of the vector sequence $\varphi_{d,k} = \{\phi_{d,k}, x_k\}$, which maps $[-d, 0]$ in $\mathbb{R}^n$, is $\mathcal{D}_d = \mathcal{D}([-d, 0], \mathbb{R}^n)$, with the norm $\|\phi_{d,k}\|_d = \sup_{-d \le j \le -1} \|x(k+j)\|$, where $\|\cdot\|$ is the Euclidean norm. The function $Y = \texttt{round}(X)$ rounds the elements of X to the nearest integers.

2. PROBLEM STATEMENT

Consider a time-varying delay discrete-time nonlinear system with saturating control inputs represented by:

$$x_{k+1} = f(x_k, x_{k-d_k}, \mathrm{sat}(u_k)), \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector with the initial condition given by a sequence $\varphi_{\bar{d},0}$, with $\varphi_{\bar{d},0} = \{\phi_{\bar{d},0}, x_0\}$ and $\varphi_{\bar{d},0} \in \mathcal{D}_{\bar{d}}$, $\phi_{\bar{d},0}(j) = x_j$, $j \in [-\bar{d}, -1]$ and $u_k \in \mathbb{R}^m$ is the control input vector, $\mathrm{sat}(u_k)$ is a classical vector-bounded saturating function, i.e, $\mathrm{sat}(u_k)_{(i)} = \mathrm{sign}(u_k)_{(i)} \min\{v_{0_{(i)}}, |u_{k_{(i)}}|\}$, $i = 1, \dots, m$, where $-v_{0_{(i)}}$ and $v_{0_{(i)}}$ are bounds on the i-th control input. The function $f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and it is always assumed that the origin is the equilibrium of the system, that is, $f(\mathbf{0}, \mathbf{0}, \mathbf{0}) = \mathbf{0}$. The time-varying delay is denoted by d_k with $1 \le \underline{d} \le d_k \le \bar{d}$, where $\underline{d}$ and $\bar{d}$ are the lower and upper bounds of the delay, respectively, and consequently d_k is subject to

$$|d_{k+1} - d_k| \le \delta, \tag{2}$$

where $\delta \in \mathbb{N}$ denotes the maximum modulus variation admissible by d_k between two samples with $\delta = \bar{d} - \underline{d}$.

The nonlinear system (1) can be represented by a T-S fuzzy model as:

$$\begin{array}{ll} \text{Rule } i : & \\ \quad \text{IF} & z_1(k) \text{ is } M_{i1} \text{ and } \cdots \text{ and } z_p(k) \text{ is } M_{ip}, \\ \text{THEN} & x_{k+1} = A_i x_k + A_{di} x_{k-d_k} + B_i \mathrm{sat}(u_k), \end{array} \tag{3}$$

where $z_j(k)$, $j = 1, \dots, p$, are the scalar premise variables supposed to be dependent only on the states, M_{ij} are the fuzzy sets, p is the number of premise variables. The matrices, $A_i \in \mathbb{R}^{n \times n}$, $A_{di} \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$, $i = 1, \dots, N$, are known.

From a standard fuzzy inference method, for example, a center-average defuzzifier, product fuzzy inference, and singleton fuzzifier, the dynamic fuzzy model (3) can be expressed by the following model Chen et al. [2009], Feng [2009]:

$$x_{k+1} = A(\alpha_k)x_k + A_d(\alpha_k)x_{k-d_k} + B(\alpha_k)\mathrm{sat}(u_k), \tag{4}$$

where the membership variables are $\alpha_{k(i)} = w_i(z(k)) / \sum_{j=1}^{N} w_j(z(k))$ with $w_i = \prod_{j=1}^{p} M_{ij}(z_j(k))$ and $z(k) = [z_1(k)\ z_2(k)\ \cdots\ z_p(k)]^T$. Note that α_k is a state-dependent time-varying parameter vector that is measurable or possible to be estimated in real time belonging to the unitary simplex:

$$\Xi = \left\{ \alpha_k \in \mathbb{R}^N;\ \sum_{i=1}^{N} \alpha_{k(i)} = 1,\ \alpha_{k(i)} \ge 0, i = 1, \dots, N \right\}. \tag{5}$$

Therefore, matrices in (4) can be rewritten as:

$$[A(\alpha_k)\ A_d(\alpha_k)\ B(\alpha_k)] = \sum_{i=1}^{N} \alpha_{k(i)} [A_i\ A_{di}\ B_i], \quad \alpha_k \in \Xi. \tag{6}$$

Consider the following two types of feedback control laws:

$$u_k = K(\alpha_k)x_k + K_d(\alpha_k)x_{k-d_k}, \tag{7}$$

$$u_k = K(\alpha_k)x_k. \tag{8}$$

The control law (7) can be used when the delay d_k is available on real time. Otherwise, only the control law (8) can be implemented.

Note that the matrices of the controllers are dependent of the membership variables and, similar to the matrices of the fuzzy system (4), they are defined as follows:

$$[K(\alpha_k)\ K_d(\alpha_k)] = \sum_{i=1}^{N} \alpha_{k(i)} [K_i\ K_{di}],\ \alpha_k \in \Xi, \tag{9}$$

where $K_i \in \mathbb{R}^{m \times n}$ and $K_{d_i} \in \mathbb{R}^{m \times n}$.

Because of the saturating actuators and presence of delay in the state vector, the local asymptotic stability of the nonlinear system (1) in closed loop with feedback control law (7)-(9) will be analyzed through two auxiliary sets

$$\mathcal{C}_x = \left\{ x_0 \in \mathcal{D}_{\bar{d}};\ V_1(x_0, \alpha_0) \le c(\phi_{\bar{d},0}) \right\} \tag{10}$$

and

$$\mathcal{B}(r) = \left\{ \phi_{\bar{d},0} \in \mathcal{D}_{\bar{d}};\ \|\phi_{\bar{d},0}\|_{\bar{d}} \le r \right\}, \tag{11}$$

where $V_1(x_0, \alpha_0)$ is a parameter dependent quadratic form, $c(\phi_{\bar{d},0})$ is a function on $\mathbb{R}^+$ with the sequence $\phi_{\bar{d},0}$ as argument and $0 \le r \in \mathbb{R}^+$. Then, the associated estimate region of attraction will be characterized by $\Upsilon_\varphi = \mathcal{B}(r) \times \mathcal{C}_x = \left\{ (\phi_{\bar{d},0}, x_0) \mid \phi_{\bar{d},0} \in \mathcal{B}(r) \text{ and } x_0 \in \mathcal{C}_x \right\}$.

Problem 1. Determine controller gains K_i and K_{di} and characterize the regions $\mathcal{C}_x$ and $\mathcal{B}(r)$, such that $\Upsilon_\varphi = \mathcal{B}(r) \times \mathcal{C}_x$ is a set of initial conditions such that the corresponding trajectories of the closed-loop system converge asymptotically to the origin.

3. PRELIMINARIES

Define $\Psi(u_k) = u_k - \text{sat}(u_k)$, i.e., $\Psi(u_k)$ corresponds to a decentralized dead zone nonlinearity. Using the fuzzy formulation (4)–(9), we have:

$$x_{k+1} = \hat{A}(\alpha_k)x_k + \hat{A}_d(\alpha_k)x_{k-d_k} - B(\alpha_k)\Psi(u_k), \quad (12)$$

where, by construction,

$$\hat{A}(\alpha_k) = \sum_{i=1}^{N}\sum_{j=i}^{N} \mu_{ij}\alpha_{k(i)}\alpha_{k(j)} 0.5\,(A_i + B_iK_j + A_j + B_jK_i), \quad (13)$$

$$\hat{A}_d(\alpha_k) = \sum_{i=1}^{N}\sum_{j=i}^{N} \mu_{ij}\alpha_{k(i)}\alpha_{k(j)} 0.5\,(A_{di} + B_iK_{dj} + A_{dj} + B_jK_{di}), \quad (14)$$

with

$$\mu_{ij} = \begin{cases} 2, & i \neq j, \\ 1, & \text{otherwise.} \end{cases} \quad (15)$$

Consider matrices $\mathcal{K}(\alpha_k) \in \mathbb{R}^{m\times 2n}$, $\mathcal{G}(\alpha_k) \in \mathbb{R}^{m\times 2n}$ and a vector $\xi_k \in \mathbb{R}^{2n}$, given respectively by $\mathcal{K}(\alpha_k) = [K(\alpha_k)\ K_d(\alpha_k)]$, $\mathcal{G}(\alpha_k) = [G(\alpha_k)\ G_d(\alpha_k)]$, and $\xi_k = \left[x_k^T\ x_{k-d_k}^T\right]^T$. Also, define the polyhedral set:

$$\mathcal{S} \equiv \{\xi_k \in \mathbb{R}^{2n} : |(\mathcal{K}(\alpha_k)_{(i)} - \mathcal{G}(\alpha_k)_{(i)})\xi_k| \leq v_{0_{(i)}},\ i = 1,\dots,m\}. \quad (16)$$

The following Lemma regarding nonlinearity $\Psi(u_k)$ was directly adapted from Jungers and Castelan [2011] and Tarbouriech et al. [2004].

Lemma 1. If $\xi_k \in \mathcal{S}$, then the relation

$$\Psi(u_k)^T T(\alpha_k)[\Psi(u_k) - G(\alpha_k)x_k - G_d(\alpha_k)x_{k-d_k}] \leq \mathbf{0} \quad (17)$$

is verified for all diagonal positive definite matrices $T(\alpha_k) \in \mathbb{R}^{m\times m}$.

Consider the following fuzzy L-K candidate function, $V(x_k,\alpha_k) : \mathbb{R}^n \times \Xi \to \mathbb{R}^+$:

$$V(x_k,\alpha_k) = \sum_{i=1}^{3} V_i(x_k,\alpha_k) > 0, \quad (18)$$

with $V_1(x_k,\alpha_k) = x_k^T Q^{-1}(\alpha_k)x_k$,
$V_2(x_k,\alpha_k) = \sum_{i=k-d_k}^{k-1} x_i^T R^{-1}(\alpha_i)x_i$,
$V_3(x_k,\alpha_k) = \sum_{\ell=2-\delta}^{1}\sum_{i=k+\ell-1}^{k-1} x_i^T R^{-1}(\alpha_i)x_i$,
where $Q(\alpha_k) = \sum_{i=1}^{N}\alpha_{k(i)}Q_i$, $\mathbf{0} < Q_i^T = Q_i \in \mathbb{R}^{n\times n}$ and $R(\alpha_k) = \sum_{i=1}^{N}\alpha_{k(i)}R_i$, $\mathbf{0} < R_i^T = R_i \in \mathbb{R}^{n\times n}$. We can associate to this L-K function some level sets defined as follows:

Definition 1. For all scalar $c > 0$ and the L-K function (18), we define the level set $\mathcal{L}_{V_1}(c)$ as the intersection of the ellipsoidal sets associated with matrices $Q_i > \mathbf{0}$, $i = 1,\dots,N$, as follows:

$$\mathcal{L}_{V_1}(c) = \left\{\mathcal{E}(Q_i^{-1},c),\ \forall\alpha_k \in \Xi\right\} = \bigcap_{\alpha_k\in\Xi}\mathcal{E}(Q^{-1}(\alpha_k),c) = \bigcap_{i\in\{1,\dots,N\}}\mathcal{E}(Q_i^{-1},c), \quad (19)$$

where $\mathcal{E}(Q_i^{-1},c)$, $i = 1,\dots,N$, denote the ellipsoidal sets

$$\mathcal{E}(Q_i^{-1},c) = \left\{x_k \in \mathbb{R}^n;\ x_k^T Q_i^{-1}x_k \leq c\right\}. \quad (20)$$

The equality given in (19) was proved in [Jungers and Castelan, 2011, Lemma 4]. If we assume $c = 1$ in this definition, then the simplified notation $\mathcal{L}_{V_1} \equiv \mathcal{L}_{V_1}(1)$ and $\mathcal{E}(Q_i^{-1}) \equiv \mathcal{E}(Q_i^{-1},1)$ is used.

The following lemmas are also used in the next section to establish the main result of this paper.

Lemma 2. Let $R(\alpha_k) = \sum_{i=1}^{N}\alpha_{k(i)}R_i$ and $R_i^T = R_i > \mathbf{0}$, then

$$\lambda_{\max}\left(R^{-1}(\alpha_k)\right) \leq \max_i\left(\lambda_{\max}\left(R_i^{-1}\right)\right), \quad (21)$$

for $i = 1,\dots,N$ and $\forall\alpha_k \in \Xi$.

Proof 1. Using the positive definiteness of R_i, we have

$$0 < \lambda_{\max}\left(R_i^{-1}\right) \leq \max_i\left(\lambda_{\max}\left(R_i^{-1}\right)\right). \quad (22)$$

Assuming $\tilde{\lambda} = \max_i\left(\lambda_{\max}\left(R_i^{-1}\right)\right)$, where $\tilde{\lambda}$ is a positive scalar, we have $\lambda_{\max}\left(R_i^{-1}\right) \leq \tilde{\lambda} \Leftrightarrow R_i^{-1} \leq \tilde{\lambda}\mathbf{I}$. By Schur's complement we get:

$$\begin{bmatrix}\tilde{\lambda}\mathbf{I} & \mathbf{I} \\ \mathbf{I} & R_i\end{bmatrix} \geq \mathbf{0}. \quad (23)$$

Multiplying the inequality (23) by $\alpha_{k(i)}$, summing up for $i = 1,\dots,N$, and knowing that $\sum_{i=1}^{N}\alpha_{k(i)}R_i = R(\alpha_k)$, we obtain:

$$\begin{bmatrix}\tilde{\lambda}\mathbf{I} & \mathbf{I} \\ \mathbf{I} & R(\alpha_k)\end{bmatrix} \geq \mathbf{0}. \quad (24)$$

By Schur's complement, (24) yields $R^{-1}(\alpha_k) \leq \tilde{\lambda}\mathbf{I} \Leftrightarrow \lambda_{\max}\left(R^{-1}(\alpha_k)\right) \leq \tilde{\lambda}$. Therefore, we prove the Lemma 2.

From the initial condition $\phi_{\bar{d},0}$, terms $V_2(x_k,\alpha_k)$ and $V_3(x_k,\alpha_k)$ of the L-K function (18), and using Lemma 2, we get:

$$\sum_{i=k-d_k}^{k-1} x_i^T R^{-1}(\alpha_i)x_i + \sum_{\ell=2-\delta}^{1}\sum_{i=k+\ell-1}^{k-1} x_i^T R^{-1}(\alpha_i)x_i \leq \sum_{i=-d_k}^{-1}\phi_{\bar{d},0}(i)^T R^{-1}(\alpha_i)\phi_{\bar{d},0}(i) + \sum_{\ell=2-\delta}^{1}\sum_{i=\ell-1}^{1}\phi_{\bar{d},0}(i)^T \times R^{-1}(\alpha_i)\phi_{\bar{d},0}(i) \leq \rho\|\phi_{\bar{d},0}\|_{\bar{d}}^2, \quad (25)$$

where

$$\rho = \max_{i=1,\dots,N}\left(\lambda_{\max}\left(R_i^{-1}\right)\right)\left(\bar{d} + \frac{\delta^2-\delta}{2}\right). \quad (26)$$

Thus, set $\mathcal{C}_x$ can be calculated as

$$\mathcal{C}_x = \mathcal{L}_{V_1}(1 - \rho\|\phi_{\bar{d},0}\|_{\bar{d}}^2) = \left\{x_0 \in \mathbb{R}^n; x_0^T Q^{-1}(\alpha_0)x_0 \leq 1 - \rho\|\phi_{\bar{d},0}\|_{\bar{d}}^2\right\}. \quad (27)$$

To keep $1 - \rho\|\phi_{\bar{d},0}\|_{\bar{d}}^2$ non negative it follows from (11) that one needs to bound the radius of the ball $\mathcal{B}(r)$ as $0 \leq r \leq \rho^{-\frac{1}{2}}$.

The connection between the sets $\mathcal{C}_x$ and $\mathcal{B}(r)$ in terms of the confinement of trajectories in $\mathcal{L}_{V_1}$ and local asymptotic stability is shown in the following Lemma.

Lemma 3. Let (18) be fuzzy L-K candidate function. If $\forall k \geq 0$ and $\forall \alpha_k \in \Xi$ it is verified

$$\Delta V(x_k, \alpha_k) = V(x_{k+1}, \alpha_{k+1}) - V(x_k, \alpha_k) < 0, \quad (28)$$

then

$$V(x_k, \alpha_k) < V(x_0, \alpha_0) \leq x_0^T Q^{-1}(\alpha_0) x_0 + \rho \|\phi_{\bar{d},0}\|_{\bar{d}}^2. \quad (29)$$

Therefore, $\forall\ x_0 \in \mathcal{C}_x = \mathcal{L}_{V_1}(1 - \rho\|\phi_{\bar{d},0}\|_{\bar{d}}^2)$ and $\forall\ \phi_{\bar{d},0} \in \mathcal{B}(r)$, inequality (29) implies that $x_k \in \mathcal{L}_{V_1}$ and $\lim_{k\to\infty} x_k = 0$.

Proof 2. Once (28) is verified with (18) then we have

$$x_k^T Q^{-1}(\alpha_k) x_k \leq V(x_k, \alpha_k) < V(x_0, \alpha_0). \quad (30)$$

Moreover,

$$V(x_0, \alpha_0) = x_0^T Q^{-1}(\alpha_0) x_0 + \sum_{i=-d_k}^{-1} \phi_0^T(i) R^{-1}(\alpha_i) \phi_0(i) + \sum_{\ell=2-\bar{d}}^{1-\underline{d}} \sum_{i=\ell-1}^{-1} \phi_0^T(i) R^{-1}(\alpha_i) \phi_0(i).$$

By using (25)–(27), we have:

$$V(x_0, \alpha_0) \leq x_0^T Q^{-1}(\alpha_0) x_0 + \rho \|\phi_{\bar{d},0}\|_{\bar{d}}^2. \quad (31)$$

From (30) and (31), it can be verified that if $x_0^T Q^{-1}(\alpha_0) x_0 \leq 1 - \rho\|\phi_{\bar{d},0}\|_{\bar{d}}^2$, then $x_k^T Q^{-1}(\alpha_k) x_k \leq 1$. Therefore we can conclude that trajectories emanating from $\Upsilon = \mathcal{B}(r) \times \mathcal{C}_x$ with $0 \leq r \leq \rho^{-\frac{1}{2}}$ and ρ given by (26) remain inside $\mathcal{L}_{V_1}$. Besides, due to the negative definiteness of (28) we can assure the local asymptotic stability.

4. MAIN RESULTS

Based on the results shown in the previous section, we present some convex conditions for the synthesis of T-S fuzzy controllers for local stabilization of nonlinear discrete-time systems with time-varying delay and saturating actuators.

Theorem 1. Suppose that there exist symmetric definite positive matrices $Q_i \in \mathbb{R}^{n\times n}$ and $R_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, N$, diagonal and positive definite matrices $S_i \in \mathbb{R}^{m\times m}$ and matrices $U \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times n}$, $Y_i \in \mathbb{R}^{m\times n}$, $Y_{di} \in \mathbb{R}^{m\times n}$, $Z_i \in \mathbb{R}^{m\times n}$, and $Z_{di} \in \mathbb{R}^{m\times n}$ satisfying the following LMIs:

$$\begin{bmatrix} -Q_q & 0.5(A_iU + B_iY_j + A_jU + B_jY_i) & \begin{pmatrix} 0.5\left(A_{di}H + B_iY_{dj}\right. \\ \left.+A_{dj}H + B_jY_{di}\right) \end{pmatrix} & -0.5(B_iS_j + B_jS_i) & \mathbf{0} \\ \star & 0.5(Q_i + Q_j) - U^T - U & \mathbf{0} & 0.5(Z_i^T + Z_j^T) & U^T \\ \star & \star & R_\ell - H^T - H & 0.5(Z_{di}^T + Z_{dj}^T) & \mathbf{0} \\ \star & \star & \star & -(S_i + S_j) & \mathbf{0} \\ \star & \star & \star & \star & -0.5\dfrac{R_i + R_j}{1+\delta} \end{bmatrix} < \mathbf{0},$$

$$\forall i, \ell, q = 1, \ldots, N,\ j = i, \ldots, N \quad (32)$$

and

$$\begin{bmatrix} \mathcal{Q}_i - \mathcal{U}^T - \mathcal{U} & \mathcal{Y}_{i_{(\ell)}}^T - \mathcal{Z}_{i_{(\ell)}}^T \\ \star & -v_{0_{(\ell)}}^2 \end{bmatrix} \leq \mathbf{0},$$

$$\forall i = 1, \ldots, N,\ \forall \ell = 1, \ldots, m, \quad (33)$$

where $\mathcal{Q}_i = \text{diag}\{Q_i, R_i\}$, $\mathcal{U} = \text{diag}\{U, H\}$, $\mathcal{Y}_i = [Y_i\ Y_{di}]$, and $\mathcal{Z}_i = [Z_i\ Z_{di}]$. Then, the controller matrices in (9) obtained through

$$K_i = Y_i U^{-1} \quad \text{and} \quad K_{di} = Y_{di} H^{-1} \quad (34)$$

are such that the origin of the nonlinear system (1) in closed-loop with control law (7)–(9) is asymptotically stable for any set of initial conditions starting in $\Upsilon_\varphi = \mathcal{B}(r) \times \mathcal{C}_x$, with $0 \leq r \leq \rho^{-\frac{1}{2}}$ and ρ given by (26), and the corresponding trajectories remain in $\mathcal{L}_{V_1}$.

Proof 3. Suppose that (32) is satisfied, then we have assured the positivity of R_i, Q_i and S_i, $i = 1, \ldots, N$, by consequence $V(x_k, \alpha_k)$ verifies (18) and the Lemma 1 is verified too. Then, we show the relationship between the diagonal matrix $T(\alpha_k)$ and the diagonal matrix S_i. Besides, the regularity of U and H is assured by blocks $(2, 2)$ and $(3, 3)$. By replacing Y_i, Y_{di}, Z_i and Z_{di} by K_iU, $K_{di}H$, G_iU, and $G_{di}H$, respectively, multiplying the resulting inequality successively by $\alpha_{k(i)}$, $\alpha_{k(j)}$, $\alpha_{k+1(q)}$, and $\alpha_{k-d_k(\ell)}$, and summing up on $i = 1, \ldots, N$, $j = i, \ldots, N$, $q = 1, \ldots, N$, $\ell = 1, \ldots, N$, we get

$$\begin{bmatrix} -Q(\alpha_k^+) & \hat{A}(\alpha_k)U & \hat{A}_d(\alpha_k)H & -B(\alpha_k)S(\alpha_k) & \mathbf{0} \\ \star & Q(\alpha_k) - U - U^T & \mathbf{0} & U^TG^T(\alpha_k) & U^T \\ \star & \star & R(\alpha_k^-) - H - H^T & H^TG_d^T(\alpha_k) & \mathbf{0} \\ \star & \star & \star & -2S(\alpha_k) & \mathbf{0} \\ \star & \star & \star & \star & -\dfrac{1}{1+\delta}R(\alpha_k) \end{bmatrix} < \mathbf{0}. \quad (35)$$

where $\hat{A}(\alpha_k)$ and $\hat{A}_d(\alpha_k)$ are given in (13) and (14), respectively. Note that the positive definite matrices $Q(\alpha_k)$, $R(\alpha_k)$ and $S(\alpha_k)$ can be written as

$$F(\alpha_k) = \left(\sum_{j=1}^N \alpha_j\right) F(\alpha_k) = \sum_{i=1}^N \sum_{j=i}^N \mu_{ij} \alpha_{k(i)} \alpha_{k(j)} \times 0.5\,(F_i + F_j)\,,$$

where F stands for Q, R and S, and μ_{ij} is given in (15), $Q(\alpha_k^+) = \sum_{q=1}^N \alpha_{k(q)}^+ Q_q$, $R(\alpha_k^-) = \sum_{\ell=1}^N \alpha_{k(\ell)}^- R_\ell$, and the shorthands $\alpha_k^+ \equiv \alpha_{k+1}$ and $\alpha_k^- \equiv \alpha_{k-d_k}$.

From Geromel et al. [2007] we know that $-M^TE^{-1}M \leq E - M^T - M$, for all square matrices M and $E = E^T > \mathbf{0}$. Applying this inequality on the blocks $(2, 2)$ and $(3, 3)$ of Θ_k we obtain, respectively, $-U^TQ^{-1}(\alpha_k)U$ and $-H^TR^{-1}(\alpha_k^-)H$. Then, applying Schur's complement on the resulting expression, we get

$$\tilde{\Pi}_k = \begin{bmatrix} -Q(\alpha_k^+) & \hat{A}(\alpha_k)U & \hat{A}_d(\alpha_k)H & -B(\alpha_k)S(\alpha_k) \\ \star & \begin{pmatrix} U^T(1+\delta)R^{-1}(\alpha_k)U \\ -U^TQ^{-1}(\alpha_k)U \end{pmatrix} & \mathbf{0} & U^TG^T(\alpha_k) \\ \star & \star & -H^TR^{-1}(\alpha_k^-)H & H^TG_d^T(\alpha_k) \\ \star & \star & \star & -2S(\alpha_k) \end{bmatrix} < \mathbf{0}. \quad (36)$$

Taking into account the regularity of U and H, due to blocks $(2,2)$ and $(3,3)$ in (32) we consider the congruence transformation $\Pi_k = \mathcal{T}^T\tilde{\Pi}_k\mathcal{T}$ with $\mathcal{T} = \text{diag}\{\mathbf{I}, U^{-1}, H^{-1}, S^{-1}(\alpha_k)\}$ and by using Schur's complement, we obtain

$$\begin{bmatrix} (1+\delta)R^{-1}(\alpha_k) \\ -Q^{-1}(\alpha_k) & \mathbf{0} & G^T(\alpha_k)S^{-1}(\alpha_k) \\ \star & -R^{-1}(\alpha_k^-) & G_d^T(\alpha_k)S^{-1}(\alpha_k) \\ \star & \star & -2S^{-1}(\alpha_k) \end{bmatrix}$$
$$+ \begin{bmatrix} \hat{A}^T(\alpha_k) \\ \hat{A}_d^T(\alpha_k) \\ -B^T(\alpha_k) \end{bmatrix} Q^{-1}(\alpha_k^+) \left[\hat{A}(\alpha_k) \ \hat{A}_d(\alpha_k) \ -B(\alpha)\right] < \mathbf{0}. \tag{37}$$

Pre- and post-multiplying (37) by $\tilde{X}_k^T = \left[x_k^T \ x_{k-d_k}^T \ \Psi^T(u)\right]$ and its transpose, respectively, and from (12)–(14), we can replace $\hat{A}(\alpha_k)x_k + \hat{A}_d(\alpha_k)x_{k-d_k} - B(\alpha_k)\Psi(u_k)$ by x_{k+1}, getting

$$\Omega_k \equiv x_{k+1}^T Q^{-1}(\alpha_k^+)x_{k+1} + x_k^T\left[(1+\delta)R^{-1}(\alpha_k) -Q^{-1}(\alpha_k)\right]x_k - x_{k-d_k}^T R^{-1}(\alpha_k^-)x_{k-d_k} - 2\Psi^T(u_k)S^{-1}(\alpha_k) \times (\Psi(u_k) - G(\alpha_k)x_k - G_d(\alpha_k)x_{k-d_k}) < 0. \tag{38}$$

Choosing $S^{-1}(\alpha_k) = T(\alpha_k)$, we can obtain from (18):

$$\Delta V(x_k, \alpha_k) - 2\Psi^T(u_k)T(\alpha_k)\left(\Psi(u_k) - G(\alpha_k)x_k -G_d(\alpha_k)x_{k-d_k}\right) \le \Omega_k < 0. \tag{39}$$

Thus, we can conclude that the feasibility of (32) assures the negativity of $\Delta V(x_k, \alpha_k)$ and verifies the sector generalized condition (17) which with the positivity of $V(x_k, \alpha_k)$ and the Lyapunov-Krasovskii's theorem (see Leite and Miranda [2008] for details) assure the stability of T-S fuzzy model (3) in closed-loop with saturating actuators by control law (7)–(9).

Now we assume that (32) is verified and additionally (33) is satisfied. Then, we multiply (33) by $\alpha_{k(i)}$ and sum up on $i = 1, \ldots, N$, getting:

$$\Lambda = \begin{bmatrix} \mathcal{Q}(\alpha_k) - \mathcal{U}^T - \mathcal{U} & \mathcal{Y}^T(\alpha_k)_{(\ell)} - \mathcal{Z}^T(\alpha_k)_{(\ell)} \\ \star & -v_{0_{(\ell)}}^2 \end{bmatrix} \le \mathbf{0}. \tag{40}$$

Following what we did above, in the block $(1,1)$ we have $-\mathcal{U}^T\mathcal{Q}^{-1}(\alpha_k)\mathcal{U} \le \mathcal{Q}(\alpha_k) - \mathcal{U}^T - \mathcal{U}$, that is,

$$\Lambda \ge \begin{bmatrix} -\mathcal{U}^T\mathcal{Q}^{-1}(\alpha_k)\mathcal{U} & \mathcal{Y}^T(\alpha_k)_{(\ell)} - \mathcal{Z}^T(\alpha_k)_{(\ell)} \\ \star & -v_{0_{(\ell)}}^2 \end{bmatrix} \le \mathbf{0}. \tag{41}$$

Pre- and post-multiplying (41) by $\mathcal{F} = \text{diag}\{\mathcal{U}^{-T}, 1\}$ and its transpose, respectively, and then applying the Schur's complement, it follows that:

$$-\xi_k^T\mathcal{Q}(\alpha_k)\xi_k + (v_{0_{(\ell)}})^{-2}\xi_k^T(\mathcal{K}^T(\alpha_k)_{(\ell)} - \mathcal{G}^T(\alpha_k)_{(\ell)}) \times (\mathcal{K}(\alpha_k)_{(\ell)} - \mathcal{G}(\alpha_k)_{(\ell)})\xi_k \le \mathbf{0}, \ \forall \ell = 1, \ldots, m. \tag{42}$$

This implies that the intersection ellipsoidal set $\mathcal{E}(\mathcal{Q}(\alpha_k)) \equiv \{\xi_k \in \mathbb{R}^{2n}; \xi_k^T\mathcal{Q}^{-1}(\alpha_k)\xi_k \le 1\}$ is contained in $\mathcal{S}$.

Remark 1. In the case where d_k is not available in real time, we consider control law (8) and solve Theorem 1 with $Z_{di} = \mathbf{0}$ and $\mathcal{Q}_i = Q_i$, $\mathcal{U} = U$, $\mathcal{Y}_i = Y_i$, and $\mathcal{Z}_i = Z_i$.

4.1 Convex optimization problem

The objective here is to solve Problem 1 using Theorem 1 by computing the set Υ_φ as big as possible. In this sense a fundamental issue is to maximize the size of $\mathcal{L}_{V_1} \subseteq \mathcal{S}$. Such an optimization can be achieved by considering the maximization of an ellipsoidal set included in the level set $\mathcal{L}_{V_1}$ as follows

$$D(W) = \left\{x \in \mathbb{R}^n; x^TWx \le 1\right\} \subseteq \mathcal{L}_{V_1}. \tag{43}$$

This inclusion is equivalent to:

$$\begin{bmatrix} W & \mathbf{I} \\ \mathbf{I} & Q_i \end{bmatrix} \ge \mathbf{0}, \ i = 1, \ldots, N. \tag{44}$$

Therewith, a convex optimization problem can be proposed as follows:

$$\mathcal{P}_\Upsilon \ : \ \begin{cases} \min \ \text{trace}(W) \\ \text{s. t.} \ \ (32), (33), \text{ and } (44). \end{cases} \tag{45}$$

5. A NUMERICAL EXAMPLE

Consider the T-S fuzzy model with saturating input signal described by (3) with $i = 1, 2$,

$$A_1 = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}, \ A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0.2 & -0.5 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \ A_{d2} = \begin{bmatrix} 0.1 & -0.2 \\ 0 & -0.5 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$d_k \in [1, 5]$, $v_0 = [5 \ 15]^T$ is the control signal bound, and the fuzzy sets are defined as $M_1(x_{2,k}) = \sin^2(x_{2,k})$ and $M_2(x_{2,k}) = \cos^2(x_{2,k})$. Then we have the following membership function

$$\alpha_k = \left[M_1(x_{2,k}) \ M_2(x_{2,k})\right]. \tag{46}$$

By solving the optimization problem $\mathcal{P}_\Upsilon$ given in (45) we obtain the law (7) with control gain matrices:

$$K_1 = \begin{bmatrix} 2.4637 & -0.809 \\ -5.1263 & 1.1064 \end{bmatrix}, \ K_2 = \begin{bmatrix} 1.882 & -0.9636 \\ -2.8734 & 2.2807 \end{bmatrix}, \tag{47}$$

$$K_{d1} = \begin{bmatrix} -0.0324 & -0.1066 \\ -0.0505 & 0.464 \end{bmatrix}, \ K_{d2} = \begin{bmatrix} -0.0879 & 0.0793 \\ 0.0757 & 0.3613 \end{bmatrix}, \tag{48}$$

and $\rho = 0.0323$.

Considering $\|\phi_{10,0}\|_{10}^2 = 0$, we have the sets $\mathcal{C}_x = \mathcal{L}_{V_1}$ and $\mathcal{B}(r) = \{0\}$. To illustrate the closed-loop behavior, two simulations with different initial conditions indicated by $\times$ marks in Figure 1. The set $\mathcal{C}_x$ is shown in Figure 1 and also the stable trajectories for two initial conditions $\varphi_{5,0} = \{\phi_{5,0}, x_0\} \in \mathcal{D}_5$, $\|\phi_{5,0}\|_{10}^2 = 0$ and the initial condition 1 takes $x_0 = [2.476 \ -0.8619]^T$, and the initial condition 2 takes $x_0 = [-0.7438 \ 4.902]^T$. The respective control signals are shown in Figure 2 (see $\circ$ marks for initial condition 1 and $\square$ for the initial condition 2). Both control signals stabilize the closed loop system even with the saturation of these signals. For these simulations, the time-varying delay was assumed as $d_k = \texttt{round}(3 + 2\cos(k))$.

Additionally, we used [Gao et al., 2009, Theorem 2] to synthesize the fuzzy gains for the control law (7), considering $\epsilon = 1$. We obtain the following gains:

$$K_1 = \begin{bmatrix} 3.8728 & -1 \\ -2 & 1.7456 \end{bmatrix}, \ K_2 = \begin{bmatrix} 2.8728 & -1 \\ -2 & 3.7456 \end{bmatrix}, \tag{49}$$

$$K_{d1} = \begin{bmatrix} -0.1003 & 0 \\ -0.4 & 0.9993 \end{bmatrix}, \ K_{d2} = \begin{bmatrix} -0.1003 & 0.2 \\ 0 & 0.9993 \end{bmatrix}. \tag{50}$$

Since the condition proposed in [5] does not take into account the saturation, the designer cannot a priori know about the guaranteed convergence of the closed-loop trajectories for given initial conditions. For example, by

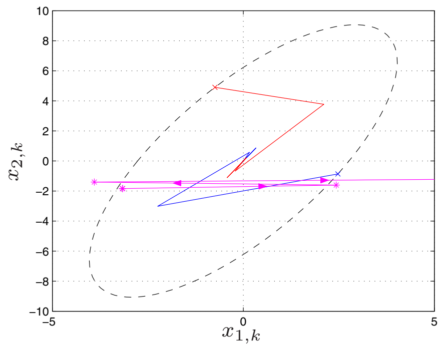

Fig. 1. The set $\mathcal{C}_x = \mathcal{L}_{V_1}$ and trajectories stable.

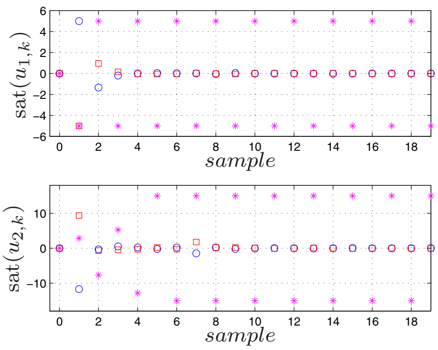

Fig. 2. Saturating control signals.

choosing initial condition $\varphi_{5,0} = \{\phi_{5,0}, x_0\} \in \mathcal{D}_5$ with $\|\phi_{5,0}\|_{10}^2 = 0$ and $x_0 = [-3.163\ -1.837]^T$, see $*$ mark in Figure 1, we obtain an unstable trajectory starting from a region where our controller is assured to be stabilizing. Also observe in Figure 2 the respective saturating control signals generated by this controller ($*$ marks). Thus, this clearly demonstrate the relevance of taking into account the saturating control signals and the region of stability for the initial conditions.

6. CONCLUSIONS

We have proposed convex conditions to synthesize fuzzy control gains stabilizing a nonlinear discrete-time systems with time-varying delay in the states and saturating actuators. These conditions were developed based on a fuzzy Lyapunov-Krasovskii function and described as LMIs. The proposed design is based on a Takagi-Sugeno representation of nonlinear systems where each local subsystem has saturating control inputs. To handle the input saturation, we used a generalized sector condition. To this end, we introduced a new characterization of the estimated region of attraction that is based on splitting the initial conditions sequence into two convex sets.

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