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Formulation of MPC for Multiplicative Stochastic Uncertainty by Multi-step Probabilistic Sets ⋆

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Abstract: This paper designs multi-step probabilistic sets for linear, discrete-time, stochastic systems with unbounded multiplicative noise and probabilistic constraints. Multi-step probabilistic sets strengthen IWPp by bringing more degrees of freedom to optimize the applicable region of finite-step probabilistic constraints, and extending the prediction horizon of IWPp to infinity for infinite-horizon probabilistic constraints. Conditions for multi-step probabilistic sets are then incorporated into a stochastic model predictive control algorithm to satisfy probabilistic constraints. Closed-loop mean-square stability is guaranteed by the algorithm. A numerical example shows the performance of the proposed algorithm.

Keywords: probabilistic constraints, multi-step feedback, multiplicative uncertainty

1. INTRODUCTION

Model predictive control (MPC) has attracted much interest from theoretical research and practical application (e.g., Mayne et al. (2000); Ding (2011); Li et al. (2013); Zheng et al. (2013)). It optimizes performance and takes account of system constraints, and control move is implemented in a receding horizon fashion. As a relatively recent development, stochastic MPC (SMPC) has been proposed in some papers (e.g. Primbs (2007); Kouvaritakis et al. (2010); Chatterjee et al. (2011); Hokayem et al. (2012); Zou and Niu (2013)) in order to satisfy soft constraints or to achieve better control performance.

The difficulty of handling probabilistic constraints for systems with unbounded multiplicative uncertainty is concerned with the propagation of the uncertainty over the prediction horizon (Cannon et al., 2009). IWPp (invariance with probability p) in Cannon et al. (2009); Su et al. (2011) is effective to handle this problem, since it ensures that state within IWPp remains inside with probability no less than p. It uses one-step ahead prediction for the uncertainty propagation, and can be extended to satisfy finite-step ahead probabilistic constraints.

This paper proposes multi-step probabilistic sets (MSP sets) to strengthen IWPp. MSP sets consist of a sequence of sets, where state in one set enters the following set with given transition probability. Compared with the single feedback law for IWPp, the multi-step feedback policy for MSP sets admits more degrees of freedom, so that the finite-step probabilistic constraints are ensured with a larger region of applicability. For the infinite-horizon probabilistic constraints, an extra mild condition concerning the convergence of the controlled state is introduced. The supermartingale inequality is then used to give a lower bound on the probability of any predicted state belonging to MSP sets.

The SMPC algorithm incorporates conditions of MSP sets to satisfy the probabilistic constraints. At each time instance it online solves a convex optimization problem to minimize the upper bound on control performance. A feedback sequence is obtained by the optimization, while the first feedback law is implemented to control the system. The SMPC algorithm guarantees the recursive feasibility and mean-square stability.

The paper is organized as follows. Section 2 introduces the probabilistic constraints and some preliminary results. Section 3 includes the design and theoretical results for MSP sets. Section 4 presents the MPC algorithm. The numerical example in Section 5 verifies the effectiveness of the proposed algorithm.

Notation. For brevity, we use $E_k(X)$ to mean the expectation of random variable X with x_k known. One of the two symmetric blocks in a symmetric matrix is replaced by symbol $*$. $\{\Omega_i\}_{i=0}^N$ means a sequence of sets $\Omega_0, \Omega_1, \cdots, \Omega_N$. $\mathrm{diag}(M_1, \ldots, M_n)$ represents a block diagonal matrix in which $M_1, \ldots, M_n$ are its diagonal blocks and the off-diagonal matrices are zero.

2. SYSTEM DESCRIPTION

Consider the system with multiplicative uncertainty

$$x_{k+1} = A(k)x_k + B(k)u_k, \tag{1}$$

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where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, and $A(k)$, $B(k)$ are random matrices of proper dimensions

$$A(k) = \bar{A} + \sum_{j=1}^{m} \tilde{A}_j q_j(k),\ B(k) = \bar{B} + \sum_{j=1}^{m} \tilde{B}_j q_j(k), \quad (2)$$

where $q_j(k)$, $k = 0, 1, \ldots, j = 1, \ldots, m$, follow independent and identical standard normal distribution.

The constraints for (1) are assumed to be soft and probabilistic, which take the form

$$\Pr\{Fx_{k+i} + Gu_{k+i} \leq h\} \geq p, \quad i = 1, \ldots, N_p, \quad (3)$$
$$\Pr\{Fx_{k+i} + Gu_{k+i} \leq h\} \geq p_\infty, \quad i = N_p + 1, \ldots, \infty, \quad (4)$$

where $h \in \mathbb{R}^{n_h}$, and the inequality sign $\leq$ apply element-wise to vectors. Note that p, N_p, and p_∞ in (3) and (4) are parameters that can be specified by designers for control specifications. For instance, when $N_p = 1$ and $F = 0$, (3) has the form of probabilistic constraints in Cannon et al. (2009); Su et al. (2011). If $p_\infty = p$, then (3) and (4) together define $\Pr\{\ Fx_{k+i} + Gu_{k+i} \leq h\} \geq p$, $i = 1, \ldots,$ which are widely studied.

IWPp $\hat{E}_z$ in Su et al. (2011) ensures that $\Pr\{z_{k+1} \in \hat{E}_z\} \geq p$ if $z_k \in \hat{E}_z$. It is proposed to formulate (3) with $N_p = 1$. To ensure both (3) and (4), we design MSP sets under the control of multi-step feedback policy (Li et al., 2009). This control policy enables more degrees of freedom than the single feedback law for RMPC. Here MSP sets are expected to have a larger applicable region than IWPp for (3). The detailed proof is offered in later sections.

3. MULTI-STEP PROBABILISTIC SETS

The probabilistic transition of state is guaranteed by MSP sets over control inputs specified by following multi-step feedback policy (Li et al., 2009; Li and Xi, 2009)

$$u_{k+i} = \begin{cases} K_i x_{k+i}, & i = 0, \ldots, N-1, \\ K_N x_{k+i}, & i = N, N+1, \ldots. \end{cases} \quad (5)$$

Here K_N is the terminal feedback law for the prediction of $\{u_{k+i}\}_{i=N}^{\infty}$. Thus at each time k, there are totally $N+1$ steps of feedback laws to be optimized. In the following, conditions involving these feedback laws are expressed in terms of linear matrix inequalities to form a convex optimization problem.

3.1 Definition and design of MSP sets

In order to predict the probabilistic distribution region of state for the satisfaction of probabilistic constraints, we give the definition of MSP sets.

Definition 1. A sequence of sets $\{\Omega_i\}_{i=0}^{N}$, where $\Omega_i = \{x | x^T Q_i^{-1} x \leq 1\}$, is said to be multi-step probabilistic sets (MSP sets) if the following conditions are satisfied

1. State in Ω_i is steered into Ω_{i+1} with probability no less than $p_1 = p^{1/N_p}$ under feedback K_i. Particularly, Ω_N (including the case N =0) is an IWPp under feedback K_N.
2. For state in the sets $\{\Omega_i\}_{i=0}^{N}$, the following deterministic joint constraints are satisfied
$$Fx_{k+i} + Gu_{k+i} \leq h. \quad (6)$$
3. The quadratic expression of MSP sets is a supermartingale, i.e., there exists $\alpha \in (0, 1]$ satisfying
$$\alpha x_{k+i}^T Q_i^{-1} x_{k+i} \geq E_{k+i}\{x_{k+i+1}^T Q_{i+1}^{-1} x_{k+i+1}\}, \quad (7)$$
with the definition $Q_i = Q_N$ for $i > N$.

The setting of p_1 in Condition 1 ensures that predicted state belongs to $\{\Omega_i\}_{i=0}^{N}$ with probability no less than p. To see this, we write Condition 1 as

$$\Pr\{x_{k+i+1} \in \Omega_{i+1} | x_{k+i} \in \Omega_i\} \geq p_1,\ i = 0, \ldots, N, \quad (8)$$

where $\Omega_{N+1} = \Omega_N$. If $x_k \in \Omega_0$, then (8) can be used to obtain $\Pr\{x_{k+i} \in \Omega_i\} \geq p$ for $i = 1, \ldots, N_p$. Combining this result with Condition 2, we deduce that (6) is satisfied in probability p for time $k+1$, …, $k + N_p$. Hence multi-step probabilistic constraints (3) is guaranteed if the two conditions of MSP sets are ensured and $x_k \in \Omega_0$.

Theorem 2. A sequence of sets $\{\Omega_i\}_{i=0}^{N}$, where $\Omega_i = \{x | x^T Q_i^{-1} x \leq 1\}$, are MSP sets meeting Definition 1 if there exist matrices Q_i, Y_i, and W_i of proper dimensions satisfying the LMIs below

$$\begin{bmatrix} Q_{i+1} & * & * \\ Q_i \bar{A}^T + Y_i^T \bar{B}^T & (1-\lambda) Q_i & * \\ \begin{bmatrix} Q_i \tilde{A}_1^T + Y_i^T \tilde{B}_1^T \\ \vdots \\ Q_i \tilde{A}_m^T + Y_i^T \tilde{B}_m^T \end{bmatrix} & 0 & \dfrac{\lambda}{r^2} \begin{bmatrix} Q_i & & \\ & \ddots & \\ & & Q_i \end{bmatrix} \end{bmatrix} \succeq 0, \quad (9)$$

$$\begin{bmatrix} W_i & FQ_i + GY_i \\ * & Q_i \end{bmatrix} \succeq 0,\ e_l^T W_i e_l \leq h_l^2,\ l = 1, \ldots, n_h, \quad (10)$$

$$\begin{bmatrix} \alpha Q_i & * & * \\ \bar{A} Q_i + \bar{B} Y_i & Q_{i+1} & * \\ \begin{bmatrix} \tilde{A}_1 Q_i + \tilde{B}_1 Y_i \\ \vdots \\ \tilde{A}_m Q_i + \tilde{B}_m Y_i \end{bmatrix} & 0 & \begin{bmatrix} Q_{i+1} & & \\ & \ddots & \\ & & Q_{i+1} \end{bmatrix} \end{bmatrix} \succeq 0, \quad (11)$$

$$i = 0, \ldots, N,\ Q_{N+1} = Q_N, Y_{N+1} = Y_N,$$

where e_l is the l-th column of the identity matrix, r is the confidence radius for a χ^2 distribution $\Pr\{\ \chi^2(m) \leq r^2\} \geq p_1$, and λ is a scalar in the interval $(0, 1)$.

The feedback law (5) that regulates the probabilistic transition of state between $\{\Omega_i\}_{i=0}^{N}$ is determined by Y_i and Q_i in (9) and (10), i.e., $K_i = Y_i Q_i^{-1}$. One of the alternatives to choose λ in (9) is to search over $(0, 1)$ for the largest volume of Q_N, similar to the method introduced in Cannon et al. (2009).

Proof. Conditions in Definition 1 are guaranteed by (9), (11) and (10) in Theorem 2 respectively. Firstly, it can be verified that (9) is sufficient for (8). The main idea is to prove that, for $x_{k+i} \in \Omega_i$ and the confidence ellipsoid $\mathcal{O}_{i+1}$ with probability p_1 (i.e., $\Pr\{x_{k+i+1} \in \mathcal{O}_{i+1}\} \geq p_1$), the relation

$$\mathcal{O}_{i+1} \subseteq \Omega_{i+1}$$

always holds. This procedure follows from the one about IWPp in Cannon et al. (2009), hence the proof is omitted for brevity. Therefore, Condition 1 is sufficed by (9). Meanwhile, referring to Kothare et al. (1996), (10) guarantees (6) for $x_{k+i} \in \Omega_i$, verifying the sufficing of Condition 2. Finally, (11) is equivalent to

$$\alpha Q_i^{-1} - (\bar{A} + \bar{B}K_i)^T Q_{i+1}^{-1}(\bar{A} + \bar{B}K_i) - \sum_{j=1}^{m}(\tilde{A}_j + \tilde{B}_j K_i)^T Q_{i+1}^{-1}(\tilde{A}_j + \tilde{B}_j K_i) \succeq 0. \quad (12)$$

For x_{k+i} controlled by K_i, it can be easily verified that (12) ensures (7). □

3.2 Some characteristics of MSP sets

First we compare the applicable region of finite-step probabilistic constraints (3) ensured by MSP sets with that by IWPp in Su et al. (2011). For IWPp, its applicable region is the elliptic projection from Z space (augmented state space in Su et al. (2011)) to X space (system state space in this paper). We give the following lemma about the applicable region of IWPp.

Lemma 3. For system (1), the applicable region of any given IWPp in Su et al. (2011) can be covered by a X-space IWPp resulted by a single feedback control law.

Proof. It suffices to show that the X-space projection region $\hat{E}_x$ of any given IWPp $\hat{E}_z$ in Su et al. (2011) can be realized by an IWPp $\bar{E}_x$ designed in X space.

First of all, referring to Su et al. (2011), conditions for IWPp $\hat{E}_z := \{z|z^T\hat{P}z \le 1\}$ to ensure probabilistic constraints are briefly listed below

$$\begin{bmatrix} \hat{P}^{-1} & \bar{\Psi}\hat{P}^{-1} & [\tilde{\Psi}_1\hat{P}^{-1}, \cdots, \tilde{\Psi}_m\hat{P}^{-1}] \\ * & (1-\lambda)\hat{P}^{-1} & 0 \\ * & * & \frac{\lambda}{r^2}\begin{bmatrix} \hat{P}^{-1} & & \\ & \ddots & \\ & & \hat{P}^{-1} \end{bmatrix} \end{bmatrix} \succeq 0, \quad (13)$$

$$[K\ \Gamma_u^T]\hat{P}^{-1}\begin{bmatrix} K^T \\ \Gamma_u \end{bmatrix} \preceq W,\ W_{ii} \le \bar{u}_i^2,\ i = 1, \ldots, n_u, \quad (14)$$

where matrix variables are $\hat{P}^{-1}$ and W, and the other constant matrices are identical to those in Su et al. (2011) and Cannon et al. (2009). Define $\hat{P}^{-1} = \left[\begin{smallmatrix} X & V \\ V^T & U \end{smallmatrix}\right]$, according to Cannon et al. (2009), projection of $\hat{E}_z$ from Z space to X space is given by $\hat{E}_x = \{x|x^TX^{-1}x \le 1\}$, where $X = \Gamma_x^T\ \hat{P}^{-1}\ \Gamma_x$.

An IWPp $\bar{E}_x = \{x|x^T\bar{X}^{-1}x \le 1\}$ of X space under linear feedback law $u = \bar{K}x$ is ensured in Cannon et al. (2009) by

$$\begin{bmatrix} \bar{X} & * & * \\ \bar{X}\bar{A}^T + \bar{Y}^T\bar{B}^T & (1-\lambda)\bar{X} & * \\ \begin{bmatrix} \bar{X}\tilde{A}_1^T + \bar{Y}^T\tilde{B}_1^T \\ \vdots \\ \bar{X}\tilde{A}_m^T + \bar{Y}^T\tilde{B}_m^T \end{bmatrix} & 0 & \frac{\lambda}{r^2}\begin{bmatrix} \bar{X} & & \\ & \ddots & \\ & & \bar{X} \end{bmatrix} \end{bmatrix} \succeq 0, \quad (15)$$

$$\begin{bmatrix} W & \bar{Y} \\ * & \bar{X} \end{bmatrix} \succeq 0, \quad W_{ii} \le \bar{u}_i^2, \quad i = 1, \ldots, n_u. \quad (16)$$

where $\bar{X}$, $\bar{Y}$ and W are matrix variables and the feedback law $\bar{K} = \bar{Y}\bar{X}^{-1}$.

The proof below shows that, there exists $\hat{P}$ satisfying (13), (14) only if there exists $\bar{X} = X$, $\bar{Y}$ satisfying (15), (16). In this way, every projection region $\hat{E}_x$ can be realized by $\bar{E}_x$, so that the lemma is proved.

First we show that (13) is sufficient for (15). Pre- and postmultiplying both sides of (13) respectively by Π and Π^T, with $\Pi = [\Gamma_x^T, \ldots, \Gamma_x^T]$, gives

$$\begin{bmatrix} X & * & * \\ \Gamma_x^T\hat{P}^{-1}\bar{\Psi}^T\Gamma_x & (1-\lambda)X & 0 \\ \begin{bmatrix} \Gamma_x^T\hat{P}^{-1}\tilde{\Psi}_1^T\Gamma_x \\ \vdots \\ \Gamma_x^T\hat{P}^{-1}\tilde{\Psi}_m^T\Gamma_x \end{bmatrix} & 0 & \frac{\lambda}{r^2}\begin{bmatrix} X & & \\ & \ddots & \\ & & X \end{bmatrix} \end{bmatrix} \succeq 0. \quad (17)$$

Referring to definitions of $\Gamma_x, \bar{\Psi}, \tilde{\Psi}_1, \ldots, \tilde{\Psi}_m$ in Su et al. (2011), and noting that $\hat{P}^{-1} = \left[\begin{smallmatrix} X & V \\ V^T & U \end{smallmatrix}\right]$, it is easy to verify that (17) is equivalent to (15) if $X = \bar{X}$ and $KX + \Gamma_u^TV^T = \bar{Y}$.

Second we prove that (14) implies (16). With partitioned $\hat{P}^{-1}$, (14) is equivalent to

$$KXK^T + \Gamma_u^TV^TK^T + KV\Gamma_u + \Gamma_u^TU\Gamma_u \preceq W. \quad (18)$$

By Schur complement, $\hat{P}^{-1} \succeq 0$ is equivalent to $U \succeq V^TX^{-1}V$. Hence $\Gamma_u^TU\Gamma_u \succeq \Gamma_u^TV^TX^{-1}V\Gamma_u$. Combining this with (18), it is sufficient to get

$$KXK^T + \Gamma_u^TV^TK^T + KV\Gamma_u + \Gamma_u^TV^TX^{-1}V\Gamma_u \preceq W,$$

which is (16) when $X = \bar{X}$ and $KX + \Gamma_u^TV^T = \bar{Y}$. □

For MSP sets, (3) is guaranteed by (9) and (10) in Theorem 2. From the definition, it is easy to find that an IWPp designed in X space (state space in this paper) is a special form of MSP sets $\{\Omega_i\}_{i=0}^N$ by setting $\Omega_i = \Omega$, $i = 0, \ldots, N$. Hence we have the following remark.

Remark 4. With extra degrees of freedom provided by Ω_i, $i = 0, \ldots, N$, or more specifically provided by Q_i and Y_i in (9) (10) in Theorem 2, the voulme of MSP sets $\{\Omega_i\}_{i=0}^N$ can be larger than the X-space IWPp.

Next we combine this property with Lemma 3 to obtain a further result.

Remark 5. Lemma 3 already shows that there exists a X-space IWPp to cover the applicable region of any given IWPp in Su et al. (2011), and we also know that the volume of MSP sets can be larger than the X-space IWPp. It follows from these two results that MSP sets can ensure a larger applicable region than IWPp in Su et al. (2011).

In the following, we analyze the satisfaction of (4) by MSP sets. Using Condition 2 and 3 in Definition 1, we can give the following lemma to ensure (4).

Lemma 6. For MSP sets $\{\Omega_i\}_{i=0}^N$ defined in Theorem 2, if (7) holds with $\alpha \in (0, 1]$, and state x_k satisfies

$$x_k^TQ_0^{-1}x_k \le \frac{1 - p_\infty}{\alpha^{N_p+1}}, \quad (19)$$

then (4) is sufficed under the control of feedback $\{K_i\}_{i=0}^N$ in Theorem 2.

Proof. By (7), the sequence $\{x_{k+i}Q_i^{-1}x_{k+i}\}_{i=0}^\infty$, where $Q_i = Q_N$ for $i \ge N$, is a supermartingale. Then we use the supermartingale inequality in Kushner (1971) and (19) to get

$$\Pr\left\{\max_{\infty > i \ge N_p+1} x_{k+i}^TQ_i^{-1}x_{k+i} \ge 1\right\} \le 1 - p_\infty.$$

An equivalent form of the equation above is

$$\Pr\left\{\max_{\infty > i \ge N_p+1} x_{k+i}^TQ_i^{-1}x_{k+i} < 1\right\} \ge p_\infty,$$

which implies $\Pr\{x_{k+i} \in \Omega_i\} \geq p_\infty$ for $i \geq N_p + 1$. Meanwhile Condition 2 of MSP sets ensures (6) for $x_{k+i} \in \Omega_i$. These two terms in combination ensure (4). ☐

Theorem 7. The applicable region of (3) and (4) that can be ensured under feedback $\{K_i\}_{i=0}^N$ satisfying (9), (10), (11) and (22) is determined by

$$x_k^T Q_0^{-1} x_k \leq \rho, \quad \rho = \min\{1, \frac{1-p_\infty}{\alpha^{N_p+1}}\}. \tag{20}$$

The proof of Theorem 7 is straightforward by using previous results. Since (20) is the intersection of the region Ω_0 (from which (3) is ensured by Definition 1) and (19) (from which (4) is ensured by Lemma 6), both (3) and (4) can be satisfied if state belongs to (20).

4. MPC BASED ON MULTI-STEP PROBABILISTIC SETS

In this section we develop a SMPC algorithm, in which MSP sets are used to ensure probabilistic constraints. The expected cost function to be optimized by the algorithm at time k is

$$J_k = E_k \left\{ \sum_{i=0}^{\infty} x_{k+i}^T \mathcal{Q} x_{k+i} + u_{k+i}^T \mathcal{R} u_{k+i} \right\} \tag{21}$$

where x_{k+i} and u_{k+i} denote predicted values of the state and input, and $\mathcal{Q}$ and $\mathcal{R}$ are positive definite weighting matrices. An upper bound of (21) under the multi-step feedback policy (5) is deduced by the following theorem.

Theorem 8. An upper bound on J_k defined by (21) is given by $\gamma x_k^T Q_0^{-1} x_k$ if the following LMIs hold

$$\begin{bmatrix} Q_i & * & * & * \; * \\ \bar{A}Q_i + \bar{B}Y_i & Q_{i+1} & * & * \; * \\ \begin{bmatrix} \tilde{A}_1 Q_i + \tilde{B}_1 Y_i \\ \vdots \\ \tilde{A}_m Q_i + \tilde{B}_m Y_i \end{bmatrix} & 0 & \begin{bmatrix} Q_{i+1} & & \\ & \ddots & \\ & & Q_{i+1} \end{bmatrix} & * \; * \\ \mathcal{Q}^{1/2} Q_i & 0 & 0 & \gamma \; * \\ \mathcal{R}^{1/2} Y_i & 0 & 0 & 0 \; \gamma \end{bmatrix} \succeq 0,$$

$$i = 0, \ldots, N, \quad Q_{N+1} = Q_N, \quad Y_{N+1} = Y_N, \tag{22}$$

where the matrix variables are Q_i and Y_i, and the feedback law is $K_i = Y_i Q_i^{-1}$.

Proof. By Schur complement, (22) is equivalent to

$$\gamma Q_i^{-1} - \sum_{j=1}^{m} (\tilde{A}_j + \tilde{B}_j K_i)^T \gamma Q_{i+1}^{-1} (\tilde{A}_j + \tilde{B}_j K_i)$$
$$- (\bar{A} + \bar{B} K_i)^T \gamma Q_{i+1}^{-1} (\bar{A} + \bar{B} K_i) \succeq \mathcal{Q} + K_i^T \mathcal{R} K_i,$$

which is sufficient to

$$x_{k+i}^T \gamma Q_i^{-1} x_{k+i} - E_{k+i}\{x_{k+i+1}^T \gamma Q_{i+1}^{-1} x_{k+i+1}\} \geq x_{k+i}^T (\mathcal{Q} + K_i^T \mathcal{R} K_i) x_{k+i}, \tag{23}$$

where $Q_i = Q_N$, $K_i = K_N$ when $i \geq N$. Therefore the sequence $\{x_{k+i}^T \gamma Q_i^{-1} x_{k+i}\}_{i=0}^{\infty}$ is a nonnegative supermartingale. According to Williams (1991), $x_{k+i}^T \gamma Q_i^{-1} x_{k+i}$ is finite and converges almost surely. Taking conditional expectation on time k and summing both sides of (23) for $i = 0, 1, \ldots,$ gives

$$x_k^T \gamma Q_0^{-1} x_k - \lim_{i \to \infty} E_k\{x_{k+i}^T \gamma Q_N^{-1} x_{k+i}\} \geq J_k, \tag{24}$$

implying that J_k is bounded from above. Since $\mathcal{Q} + K_i^T \mathcal{R} K_i \succ 0$, it can be concluded that

$$\lim_{i \to \infty} E_k\{x_{k+i}^T (\mathcal{Q} + K_i^T \mathcal{R} K_i) x_{k+i}\} = 0.$$

This is sufficient for $E_k(x_{k+i} x_{k+i}^T) \to 0$ as $i \to \infty$, which together with (24) ensures $x_k^T \gamma Q_0^{-1} x_k \geq J_k$. ☐

In the following, we propose a SMPC algorithm to optimize control performance (23) and satisfy probabilistic constraints (3) and (4). Based on Theorem 7, the calculation of MSP sets in this algorithm ensures probabilistic constraints. The multi-step feedback laws are optimized in each online computation to satisfy conditions on MSP sets and cost function with the updating measure of system state.

Algorithm 1. Given system state x_k, at time $k = 0,1,\ldots,$ do the following steps:

Step 1 Calculate ρ_k by

$$\rho_k = \begin{cases} x_k^T Q_{1|k-1}^{-1} x_k, & \text{if } k \geq 1, \text{ and } x_k \notin \hat{\Omega}_{1|k-1}, \\ \rho \text{ in (20)}, & \text{if } k = 0, \text{ or } x_k \in \hat{\Omega}_{1|k-1}, \end{cases} \tag{25}$$

where $\hat{\Omega}_{1|k-1}$ is defined by Step 3 at time $k-1$.

Step 2 Solve $\mathcal{P}(x_k, \rho_k)$, which is defined by

$$\underset{\{Q_i\}_{i=0}^N, \{Y_i\}_{i=0}^N, \{W_i\}_{i=0}^N, \gamma}{\text{minimize}} \quad \gamma$$
$$\text{subject to } (9), (10), (11), (22),$$
$$\begin{bmatrix} \rho_k & x_k^T \\ x_k & Q_0 \end{bmatrix} \succeq 0, \tag{26}$$

and let $Y_{i|k} = Y_i$, $Q_{i|k} = Q_i$, $K_{i|k} = Y_i Q_i^{-1}$, $\gamma_k^* = \gamma$.

Step 3 Implement control move $u_k = K_{0|k} x_k$, let $\hat{\Omega}_{1|k} = \{x | x^T \; Q_{1|k}^{-1} \; x \leq \rho\}$, where ρ is given in (20), and go to Step 1 at time $k+1$.

Remark 9. For $k = 1, \ldots, \hat{\Omega}_{1|k-1}$ computed at time $k-1$ in Step 3 approximates applicable region of (3) and (4) at time k. This follows from Theorem 7, which proves that state starting from $\hat{\Omega}_{1|k-1}$ satisfies (3) and (4) under feedback $\{K_{i|k-1}\}_{i=1}^N$.

Remark 10. In Step 1, ρ_k adjusts volume of $\hat{\Omega}_{1|k-1}$ to capture x_k, for the unbounded system parameter allows state to escape any finite region. If $x_k \notin \hat{\Omega}_{1|k-1}$, then ρ_k is set by (25), such that (3) and (4) are relaxed to ensure feasibility of the optimization. If $x_k \in \hat{\Omega}_{1|k-1}$, then ρ_k is set to ρ by Theorem 7.

Theorem 11. Algorithm 1 is recursively feasible. Moreover, the closed-loop system is mean-square stable, so that both state and input converge to zero almost surely.

Proof. A feasible solution for $\mathcal{P}(x_{k+1}, \rho_{k+1})$ is given by

$$\begin{aligned} &\gamma_{k+1} = \beta_{k+1} \gamma_k^*, \quad Q_{i|k+1} = \beta_{k+1} Q_{i+1|k}, \\ &Y_{i|k+1} = \beta_{k+1} Y_{i+1|k}, \quad W_{i|k+1} = \beta_{k+1} W_{i+1|k}. \end{aligned} \tag{27}$$

where $\beta_{k+1} = \frac{1}{\rho} \; x_{k+1}^T \; Q_{1|k}^{-1} \; x_{k+1}$ if $x_{k+1} \in \hat{\Omega}_{1|k}$, otherwise $\beta_{k+1} = 1$. It can be verified that constraints of $\mathcal{P}(x_{k+1}, \rho_{k+1})$ hold under the solution in (27), hence recursive feasibility is proved.

According to Theorem 8, J_k has an upper bound $\rho_k \gamma_k^*$. At time $k+1$, solving $\mathcal{P}(x_{k+1}, \rho_{k+1})$ yields a smaller γ_{k+1}^*

than the feasible γ_{k+1} in (27), hence $\gamma_{k+1}^* \leq \beta_{k+1}\gamma_k^*$. Then the expectation of $\rho_{k+1}\gamma_{k+1}^*$ satisfies

$$E_k\{\rho_{k+1}\gamma_{k+1}^*\} \leq E_k\{\rho_{k+1}\beta_{k+1}\gamma_k^*\} \leq E_k\{x_{k+1}^T Q_{1|k}^{-1} x_{k+1}\gamma_k^*\}.$$

Noting that $Q_{1|k} = Q_1$ for Step 2 in $\mathcal{P}_1(x_k, \rho_k)$, and $E_k\{x_{k+1}^T Q_1^{-1} x_{k+1}\gamma_k^*\}$ satisfies (23), it is easy to see

$$x_k^T Q_0^{-1} x_k \gamma_k^* \geq E_k\{\rho_{k+1}\gamma_{k+1}^*\} + x_k^T(\mathcal{Q} + K_k^T \mathcal{R} K_k)x_k.$$

Since $x_k^T Q_0^{-1} x_k \leq \rho_k$ by (26), we have $x_k^T Q_0^{-1} x_k \gamma_k^* \leq \rho_k \gamma_k^*$, then the equation above is sufficient to

$$\rho_k \gamma_k^* \geq E_k\{\rho_{k+1}\gamma_{k+1}^*\} + x_k^T(\mathcal{Q} + K_k^T \mathcal{R} K_k)x_k, \qquad (28)$$

which verifies that $\{\rho_k \gamma_k^*\}_{k=0}^{\infty}$ is a nonnegative supermartingale. By Williams (1991), $\rho_k \gamma_k^*$ converges almost surely, hence x_k converges almost surely. In addition (28) implies that $E_0\{x_k^T(\mathcal{Q} + K_k^T \mathcal{R} K_k)x_k\} \to 0$ as $k \to \infty$, which is sufficient for $E_0\{x_k^T x_k\} \to 0$ since $\mathcal{Q} \succ 0$ in (21). Thus the closed-loop system is mean-square stable. □

5. NUMERICAL EXAMPLE

Consider system (1) with $m = 1$ and

$$\bar{A} = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.5 \end{bmatrix}, \tilde{A}_1 = \begin{bmatrix} 0.004 & -0.031 \\ -0.002 & 0.142 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \tilde{B}_1 = \begin{bmatrix} 0.003 \\ -0.010 \end{bmatrix}.$$

Performance weighting matrices in (21) are

$$\mathcal{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0.0001 \end{bmatrix}, \quad \mathcal{R} = 0.0001,$$

so that the control performance mainly concerns the first entry of the state x_k.

We denote Algorithm 1 as A1, and denote the algorithm in Su et al. (2011) with $N = 5$ as S. The one-step ahead probabilistic constraints take the form

$$\Pr\{|u_{k+i}| \leq 2\} \geq 0.9, \quad i = 1, \qquad (29)$$

which is satisfied by S in Su et al. (2011). (29) can also be ensured by A1 by setting $N_p = 1$, $p = 0.9$, $p_\infty = 0$, and α in (11) to a sufficient large number. λ (appears in (9)) for both algorithms is set to 0.23, which corresponds to the largest volume of IWPp under S.

The regions of applicability of A1 with $N = 1$ and $N = 2$, and S are drawn in Fig. 1. It is shown that A1 has a larger region than S, owing to MSP sets employed. With the increasing of N in A1, the region of applicability is enlarged, since more degrees of freedom are provided.

For each of 30 initial points evenly distributed around the boundary of the applicable region of A1 with $N = 2$, 500 random simulations are carried out to count the frequency of x_1 escaping $\hat{\Omega}_1$ in (25). The total violation times are 1247, showing that $x_1 \in \Omega_1$ in frequency around 0.92, which satisfies the given probability 0.9.

Among these simulations, input trajectories under A1 from $x_0 = [2.495, -13.172]^T$, when the violation of MSP sets ever happens (i.e., $x_k \notin \hat{\Omega}_k$ in (25) for $k = 0, \ldots, 10$), are shown in Fig. 2. As expected, input converges to zero, and the violation time of $|u_1| \leq 2$ takes value of 44, which is within the allowable limit.

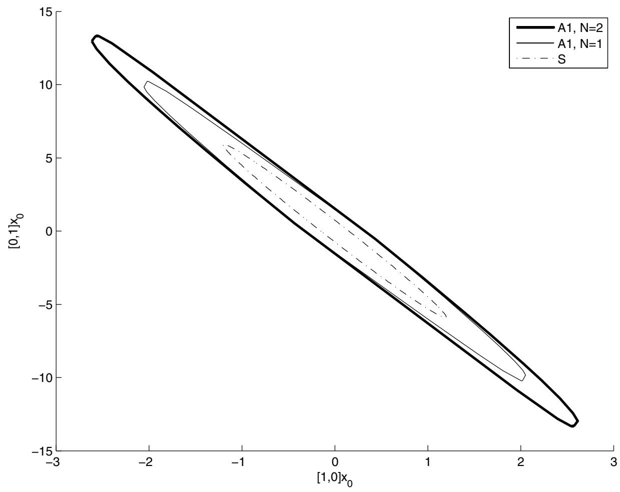

Fig. 1. Initial applicable regions of A1 and S

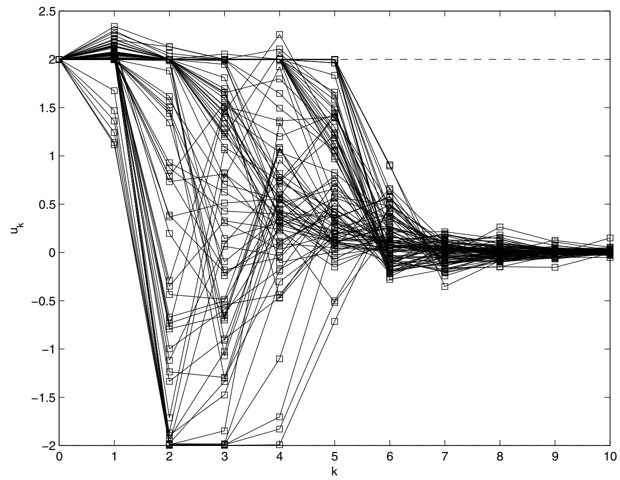

Fig. 2. Input trajectories under A1

For $x_0 = [-0.024, 0.862]^T$ inside the region of applicability of S, its average control performances of 500 random simulations under A1 and S are 0.025 and 0.035 respectively. The average state trajectories are shown in Fig. 3, demonstrating that the first entry of x_k under A1 converges to the origin faster than that under S.

Then we consider to satisfy both multi-step and infinite-horizon probabilistic constraints

$$\begin{aligned} \Pr\{|u_{k+i}| \leq 2\} \geq 0.75, \quad i = 1, 2, 3, 4, 5, \\ \Pr\{|u_{k+i}| \leq 2\} \geq 0.87, \quad i = 6, \ldots, \infty, \end{aligned} \qquad (30)$$

which are ensured by setting $N_p = 5$, $p = 0.75$, and $p_\infty = 0.87$ for A1. λ in (9) is set to 0.4, α is set to 0.709, so ρ in (20) takes value of 1. The initial applicable regions of A1 under $N = 1$ and $N = 2$ are shown in Fig. 4.

For $x_0 = [1.521, -8.221]^T$ in the initial applicable region of A1, we run 500 times of random simulations under the control of A1. In each simulation, the state trajectory lasts for 10 steps, i.e., x_k is computed for $k = 1, \ldots, 10$. As expected, the finite-length input and state trajectories all converge to the vicinity of the origin. The violations of $|u_k| \leq 2$ occur from time 1 to time 3, as shown in Fig. 5, and the violation times clearly suffice constraints (30).

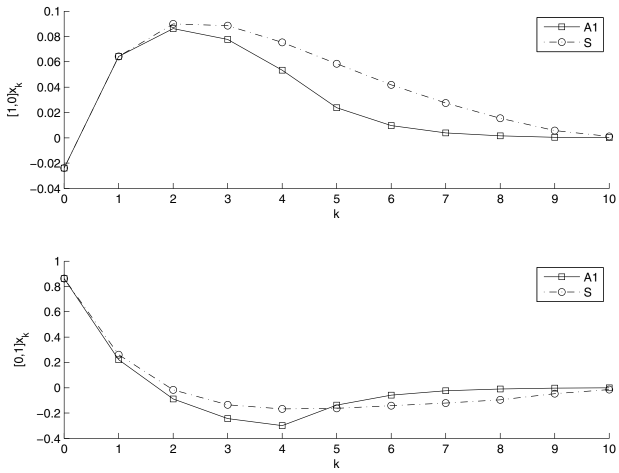

Fig. 3. Average state trajectories under A1 and S

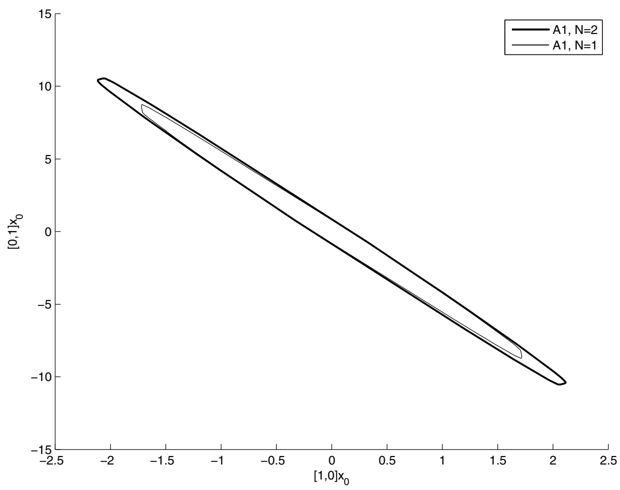

Fig. 4. Initial applicable regions of A1

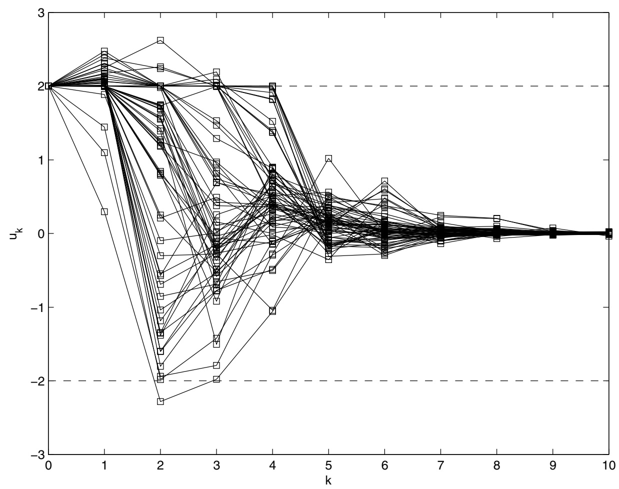

Fig. 5. Control trajectories under A1

6. CONCLUSION

A stochastic MPC algorithm is developed for systems with multiplicative uncertainty. Both multi-step and infinite-horizon probabilistic constraints are ensured by MSP sets proposed in this paper, which can also enlarge region of applicability of the MPC algorithm. The results in the numerical example verifies these properties.

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