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Stochastic MPC for Systems with both Multiplicative and Additive Disturbances ⋆

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Abstract: A stochastic MPC strategy is proposed to handle systems with both multiplicative and additive random uncertainty. Through a dual mode strategy, the system can be divided into a nominal dynamics and an error dynamics. The errors are further decomposed into two parts: one for which it is possible to construct probabilistic tubes offline with the explicit use of the disturbance distribution information, and the other which can be handled through the use of a set of robust tubes with bounding facets of fixed orientation, whose distances from the origin are optimized online. The robust tubes can exhibit little conservativeness on account of the fact that the number of the bounding facets of tubes in the predictions can be varying through online optimization. A tailored terminal set is investigated to ensure the recursive feasibility and stability of the algorithm. The online computation is turned into a standard quadratic program, which is of comparable order of complexity as that of robust MPC. A numerical example is given to illustrate the effectiveness of the algorithm.

1. INTRODUCTION

Model predictive control (MPC) is a powerful control methodology that has been successfully applied in a very wide variety of fields [Qin and Badgwell, 2003, Kouvaritakis et al., 2004, Pasik-Duncan et al., 2004]. It employs an explicit model to optimize the predicted system performance subject to constraints on input, output, states in a receding horizon manner. In most cases, models can only approximate the physical systems and will inevitably introduce uncertainties. Researchers have developed two different methods to cope with uncertainties, i.e. robust MPC and stochastic MPC. In robust MPC, uncertainties are only considered in their worst case scenarios and no use is made of information about probabilistic distribution that may be available. Instead, stochastic MPC exploits the distribution of uncertainties and aims to achieve better performance compared with the corresponding robust MPC strategy.

To date, robust MPC has achieved a plethora of results and has reached a considerable state of maturity [Kothare et al., 1996, Mayne et al., 2000, Kouvaritakis et al., 2000, Löfberg, 2003, Mayne et al., 2005, Goulart et al., 2006, etc. and references therein]. Generally, robust MPC handles only hard constraints, which cannot be transgressed for any realization of the uncertainty. When constraints are allowed to be violated but with a frequency less than a given prescribed threshold, they are considered to be probabilistic and are addressed by stochastic MPC. One can always exploit the benefits afforded by allowable constraint violations. The challenge in stochastic MPC revolves around to two key issues. One is to reformulate, with the minimum degree of conservativeness, the probabilistic constraints as hard constraints which can be handled easily, and the other is to ensure the control theoretic properties of stochastic MPC such as recursive feasibility and stability.

Several stochastic MPC strategies have been proposed for the case where uncertainty takes the form of additive disturbances. For example Cannon et al. [2011] designs tubes with a series of stochastic ellipsoidal cross sections and then relaxes the probabilistic constraints so that the online optimization can be performed efficiently and such that the algorithm achieves recursive feasibility. Yet the employment of the ellipses incurs a degree of conservativeness. An alternative approach by Kouvaritakis et al. [2010] considers tubes with cross section of a polytopic nature and makes the explicit use of information on the probabilistic distribution of disturbances to compute offline the cross section parameters which allow for the tight satisfaction of constraints under the control structure adopted. In Korda et al. [2012] and Oldewurtel et al. [2013], the joint probabilistic constraints have been investigated, and the relaxed constraints are adjusted online according to their past violations, thereby further reducing the degree of conservativeness.

For systems with multiplicative uncertainty, the formulations of stochastic MPC raise more challenges. This arises from the multiplications between the uncertain parameters and future states, both of which are random variables. Tubes with cross sections consisting of polytopic sets are optimized online in Cannon et al. [2009a]. Although the recursive feasibility is guaranteed, a requirement that constraints are invoked at each vertex of a confidence polytope can be computationally demanding. A scenario

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approach to handle LPV systems is developed in Calafiore and Fagiano [2013], which imposes no restrictions on the distributions of the parameters. However, this approach may involve too many constraints if one aims at a high confidence level, and it does not address the problem of keeping, at each time instant, the probability of constraint violations below a prescribed level in closed loop and requires softening of constraints. In this paper, we decompose the uncertain part of the predictions into two components and construct two kinds of tubes separately. One deploys tubes with polytopic cross sections with facets whose distance from the origin is computed offline using the methodology developed in Kouvaritakis et al. [2010] and the other part is to be included by robust tubes with bounding facets of fixed orientation and variable scalings, which provide a low conservative estimation by allowing the number of the bounding facets (the corresponding active constraints) to change throughout the optimization [Evans et al., 2012, Cheng et al., 2013]. Our method is translated into a standard quadratic program to be solved online and thus can be implemented efficiently, in a time comparable to that of robust MPC. A tailored terminal set is devised to guarantee the recursive feasibility of the algorithm, which is shown to be quadratically stable.

Notation: The subscript k denotes the values at the instant k, while the subscript $i|k$, i is used to denote the future values of time $k+i$ which are predicted at the current instant k. $\mathbb{N}_{\geq 0} := \{0, 1, 2, \ldots\}$, $\mathbb{N}_{\geq i} := \{i, i+1, \ldots\}$, $\mathbb{N}_{[i,\,j]} := \{i, i+1, \ldots, j\}$, $i_\rho := i - \rho$. $\mathbb{E}$ is the expectation operator. $\Pr\{\diamondsuit\}$ represents the probability with which event $\diamondsuit$ happens.

2. PROBLEM FORMULATION

Consider the discrete-time system with the following uncertainty description

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \tag{1}$$

$$[A_k\ B_k] = [A^0\ B^0] + \sum_{j=1}^{L} [A^{(j)}\ B^{(j)}] q_{jk}, \tag{2}$$

$$\sum_{j=1}^{L} q_{jk} = 1, \quad q_{jk} \geq 0, \quad k \in \mathbb{N}_{\geq 0}, \tag{3}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $A_k \in \mathbb{R}^{n_x \times n_x}$, $B_k \in \mathbb{R}^{n_x \times n_u}$, A_k, B_k involves the multiplicative uncertainty, q_{jk} $(k = 0, 1, 2, \ldots)$ are temporally independent and identically distributed (i.i.d.) random variables such that $\mathbb{E}\big(\sum_{j=1}^{L}[A^{(j)}\ B^{(j)}]q_{jk}\big) = [0\ 0]$, w_k represents the i.i.d. random additive disturbance with known, compactly supported distributions and each element of w_k is independent. The probabilistic constraint set to be considered is

$$\Pr\{(x,\,u) \in \Xi\} \geq \mathbf{p}, \tag{4}$$

where Ξ is a compact convex polytope, $\mathbf{p}$ is a column vector and the vector inequality in (4) is interpreted elementwise with each element of $\mathbf{p}$ lying inside $(0, 1]$. For ease of discussion, one of the constraints defining (4) is explicitly expressed as

$$\Pr\{g^T x_k + f^T u_{k-1} \leq h\} \geq p, \quad 0 < p \leq 1, \tag{5}$$

where g and f are column vectors with appropriate dimensions and h is a constant. We will focus on (5) in the paper, as the other constraints can be handled in a similar way and hard constraints can be treated as a special case of (5) by setting p equal to 1.

The aim of the MPC strategy is to minimize a cost over the sequence of predicted inputs, $u_{i|k}$, subject to the satisfaction of constraint (5), and the cost is expressed as

$$J_k = \sum_{i=0}^{\infty} \mathbb{E}_k\big(x_{i|k}^T Q x_{i|k} + u_{i|k}^T R u_{i|k}\big). \tag{6}$$

A quasi-closed loop dual mode strategy will be employed according to which:

$$u_{i|k} = K x_{i|k} + c_{i|k}, \quad c_{N+i|k} = 0, \quad i \geq 0, \tag{7}$$

where K is designed offline, $u = Kx$ is assumed to be stabilising and optimal for (6) in the absence of constraints, and the perturbations $c_{i|k}$, $i = 0, 1, \ldots, N-1$ provide degrees of freedom to optimize system performance when constraints are present. The original system (1) will be decomposed into nominal dynamics and error dynamics:

$$x_{i+1|k} = \Phi_{i|k} x_{i|k} + B_{i|k} c_{i|k} + w_{i|k}, \tag{8}$$

$$x_{i|k} = z_{i|k} + e_{i|k}, \tag{9}$$

$$z_{i+1|k} = \Phi^0 z_{i|k} + B^0 c_{i|k}, \tag{10}$$

$$e_{i+1|k} = \Phi_{i|k} e_{i|k} + w_{i|k} + \delta\Phi_{i|k} z_{i|k} + \delta B_{i|k} c_{i|k}, \tag{11}$$

with $\Phi_{i|k} = A_{i|k} + B_{i|k}K$, $\Phi^0 = A^0 + B^0 K$ and $\delta\Phi_{i|k} = \Phi_{i|k} - \Phi^0$, $\delta B_{i|k} = B_{i|k} - B^0$. Then constraint (5) can be rewritten as

$$\Pr\{\eta^T z_{i|k} + \eta^T e_{i|k} + f^T c_{i-1|k} \leq h\} \geq p, \tag{12}$$

with $\eta^T = g^T + f^T K$ for $i \in \mathbb{N}_{\geq 1}$.

3. PREDICTIONS AND CONSTRAINT HANDLING

To handle the probabilistic constraint (12), we need to characterize the distribution of the error e. According to (11), the error dynamics involves not only the multiplicative and additive uncertainties, but also future control parameters $c_{i|k}$. Therefore, the computation of the probabilistic distributions of predictions on the basis of information on the distribution of the uncertainty is likely to be intractable. Instead this problem is handled here by decomposing the error e into a part, ε, that can be handled offline and a part, ζ, that is to be optimized online:

$$e_{i|k} = \varepsilon_{i|k} + \zeta_{i|k}, \tag{13}$$

$$\varepsilon_{i+1|k} = \Phi^0 \varepsilon_{i|k} + w_{i|k}, \tag{14}$$

$$\zeta_{i+1|k} = \Phi_{i|k}\zeta_{i|k} + \delta\Phi_{i|k} z_{i|k} + \delta B_{i|k} c_{i|k} + \delta\Phi_{i|k}\varepsilon_{i|k}. \tag{15}$$

The benefit of this formulation is that ε and ζ can be treated separately. It is assumed that state $x_{0|k}$ at time k is measurable, and *the initial conditions* $\varepsilon_{0|k} = 0$ and $z_{0|k} + \zeta_{0|k} = x_{0|k}$ will be deployed. Then the probabilistic distribution of ε, under the dynamics of (14), can be computed offline with the explicit use of the distribution information of $w_{i|k}$ [Kouvaritakis et al., 2010]. For the other part, ζ can be described by a series of tubes with *bounding facets of fixed orientation.* The tubes will be constructed to be robust (i.e. cater for all realization of the uncertainty), they can be arbitrarily complex in terms of the facet numbers and thus will reduce the degree of conservativeness, while the distance of the facets from the origin will be optimized efficiently online. This leads to the suboptimal receding horizon control strategy which is described in the sequel.

3.1 Recursively Feasible Probabilistic Tubes for ε

In this section, the probabilistic distribution of ε is analyzed and the recursively feasible probabilistic tubes for ε are constructed. This follows the methodology of [Kouvaritakis et al., 2010]. The evolution of the other random variable ζ will be treated in next section.

At instant k, for a given prediction horizon N and vector of perturbations $\mathbf{c}_k^T = \begin{bmatrix} c_{0|k}^T & c_{1|k}^T & \cdots & c_{N-1|k}^T \end{bmatrix}$, constraint (12) is equivalent to

$$\Pr\{\eta^T F_i \mathbf{c}_k + \eta^T (\Phi^0)^i z_k + \eta^T \varepsilon_{i|k} + \eta^T \zeta_{i|k} + f^T E M^{i-1} \mathbf{c}_k \leq h\} \geq p, \tag{16}$$

where $F_i = \begin{bmatrix} (\Phi^0)^{i-1} B & \cdots & B & 0 & \cdots & 0 \end{bmatrix}$, and

$$E = \begin{bmatrix} I \; 0 \cdots \; 0 \end{bmatrix}, \quad E\mathbf{c}_k = c_{0|k},$$

$$M = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad M\mathbf{c}_k = \begin{bmatrix} c_{1|k} \\ c_{2|k} \\ \vdots \\ c_{N-1|k} \\ 0 \end{bmatrix}.$$

Theorem 1. Recursive feasibility of constraint (12)/(16) is ensured if and only if, for each $i = 1, 2, \ldots,$

$$\eta^T F_i \mathbf{c}_k + \eta^T (\Phi^0)^i z_k + \eta^T \zeta_{i|k} + f^T E M^{i-1} \mathbf{c}_k \leq h - \beta_i, \tag{17}$$

where β_i is the maximum element of the ith column of the matrix

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots \\ \text{null} & \gamma_1 + a_1 & \gamma_2 + a_2 & \gamma_3 + a_3 & \cdots \\ \vdots & \text{null} & \gamma_1 + a_1 + a_2 & \gamma_2 + a_2 + a_3 & \cdots \\ \vdots & \vdots & \text{null} & \gamma_1 + a_1 + a_2 + \alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{18}$$

$$a_i = \max_w \eta^T (\Phi^0)^i w, \tag{19}$$

and 'null' means no value, γ_i is defined as the minimum value such that

$$\Pr\{\eta^T ((\Phi^0)^{i-1} w_k + \ldots + w_{k+i-1}) \leq \gamma_i\} = p. \tag{20}$$

Proof. According to (14), we perform predictions at the instant k,

$$\varepsilon_{i|k} = (\Phi^0)^{i-1} w_k + \ldots + w_{k+i-1}. \tag{21}$$

Then, in combination with the definition of γ_i in (20), it follows that if the term $\eta^T \varepsilon_{i|k}$ in (16) is replaced by the elements, γ_i, of the first row of the matrix (18), then the probabilistic constraint (16) is satisfied. However, since γ_i is computed by the predictions at instant k, this cannot guarantee the recursive feasibility of the perturbation vector $\mathbf{c}_k$. From the instant $k + j \, (j = 1, \ldots, i-1)$, predictions are made as follows

$$\varepsilon_{i-j|k+j} = (\Phi^0)^{i-j} \varepsilon_{0|k+j} + (\Phi^0)^{i-j-1} w_{0|k+j} + \ldots + w_{i-j-1|k+j}. \tag{22}$$

At $k + j$, $\varepsilon_{0|k+j}$ cannot be treated as a random variable because at that time a realization of $\varepsilon_{0|k+j}$ will have already occurred. As a consequence, one must consider the worst case for $\varepsilon_{0|k+j}$ when computing the distribution of $\varepsilon_{i-j|k+j}$ in (22):

$$\begin{aligned} &\max \eta^T (\Phi^0)^{i-j} \varepsilon_{0|k+j} \\ &= \max_w \eta^T (\Phi^0)^{i-j} ((\Phi^0)^{j-1} w_k + \ldots + w_{k+j-1}) \\ &= \sum_{t=i-j}^{i-1} a_t, \quad (i \geq 2 \text{ and } j = 1, \ldots, i-1) \end{aligned} \tag{23}$$

where a_t is defined in (19). It is then noted that the $(j+1)$th row in matrix (18) provides the prediction values of $\eta^T \varepsilon_{i-j|k+j}$ from instant $k+j$ when taking the worst case for $\varepsilon_{0|k+j}$ into account. Obviously, j should be no greater than i and 'null' is used to denote the case $j > i$. Therefore, the recursive feasibility of $\mathbf{c}_k$ can be guaranteed if and only if the largest element of the ith column of matrix (18), β_i, is utilized. $\square$

Constraint (17) defines the facets of a region in which state $x_{i|k}$ must lie. Therefore one can use (17) to cater for the the recursively feasible probabilistic tubes for ε, while the tube positions will be determined by the reference points, i.e. the nominal states $z_{i|k}$.

Assumption 2. For any finite $i \in \mathbb{N}_{\geq 1}$, $a_i > 0$.

This is a mild assumption, because an obvious sufficient condition to ensure $a_i > 0$ is that each element of w can range from negative to positive value.

Corollary 3. The sequence $\beta_1, \beta_2, \ldots$ is monotonically non-decreasing and converges to a limit, of which an arbitrarily precise upper bound, $\bar{\beta}$, can be calculated offline [Kouvaritakis et al., 2010].

Remark 4. The values of β_i and the upper bound, $\bar{\beta}$, can be computed offline. One can get a_i of (19) by solving a set of linear programs. Furthermore, the computation of γ_i can be implemented by discretizing the distributions of the elements of w and performing univariate convolutions, which can make the approximation error (due to discretization) as small as required with a reasonable computational cost. For further details, the interested reader is referred to Kouvaritakis et al. [2010].

3.2 Robust Tubes for ζ, with Bounding Facets of Fixed Orientation

The dynamics of $\zeta_{i|k}$ is described in (15). Since this involves the future nominal states $z_{i|k}$ and perturbations $c_{i|k}$, it is not possible to analyze offline the distribution of $\zeta_{i|k}$. Instead a series of robust tubes with *bounding facets of fixed orientation* is formulated here to characterize the evolution of $\zeta_{i|k}$ [Cheng et al., 2013]. These tubes deploy the following features:

1. the directions of bounding facets of the tubes are chosen offline and fixed online,
2. the distances from the origin to the facets are optimized online,
3. the facet number of the tubes may be less than the original design, since the corresponding constraints may become inactive for the optimized facet distances (from the origin).

The last feature provides a flexible mechanism to adjust the tube geometry, thus these tubes only incur a low degree of conservativeness. At each instant k, by assumption x_k is known and $z_{0|k} + \zeta_{0|k} = x_k$. For $i \geq 0$, a series of polytopic tubes are designed as

$$V\zeta_{i|k} \leq \theta_{i|k}, \tag{24}$$

with $V \in \mathbb{R}^{n_V \times n_x}$ to be chosen offline and parameters $\theta_{i|k} \in \mathbb{R}^{n_x}$ to be optimized online. Thus (24) prescribes n_V simultaneous linear inequalities:

$$V_j \zeta \leq \sigma_j, \quad j = 1, \ldots, n_V, \tag{25}$$

where V_j is the jth row of V and σ_j is the jth component of $\theta_{i|k}$. Each active constraint among the n_V simultaneous linear inequalities defines a bounding facet of the tube. If, at the instant $i|k$, some of inequalities in (25) become inactive, then the number of bounding facets will be less than n_V. However, the directions of the facets are determined by the vectors V_j, and these are to be designed offline and kept fixed during the online computations, the tubes are referred to as having *bounding facets of fixed orientation*.

The number of rows in the matrix V, n_V, reflects the complexity of the tubes. Generally, larger n_V results in less conservative tubes, however as discussed in Remark 10, the wish to keep computational complexity down will impose limits on how large n_V can be chosen to be.

The recursion equation can be obtained from (15):

$$V\zeta_{i+1|k} = V\Phi_{i|k}\zeta_{i|k} + V(\delta\Phi_{i|k}z_{i|k} + \delta B_{i|k}c_{i|k}) + V\delta\Phi_{i|k}\varepsilon_{i|k}. \tag{26}$$

Then the tube evolution can handled through a one-step-ahead scheme, following the methodology described in Evans et al. [2012].

Theorem 5. For any $\zeta_{i|k}$ in the polytope $\{\zeta : V\zeta \leq \theta_{i|k}\}$, a necessary and sufficient condition for $V\zeta_{i+1|k} \leq \theta_{i+1|k}$ is that there exist matrices $H^{(j)}$, $j = 1, \ldots, L$, with non-negative elements such that

$$H^{(j)}V = V\Phi^{(j)}, \tag{27a}$$

$$H^{(j)}\theta_{i|k} + V(\delta\Phi^{(j)}\, z_{i|k} + \delta B^{(j)}\, c_{i|k}) + d_i^{(j)} \leq \theta_{i+1|k}, \tag{27b}$$

with

$$d_i^{(j)} = \max_w V\delta\Phi^{(j)}\varepsilon_{i|k}, \tag{28}$$

where the maximisation is performed elementwise.

The proof of Theorem 5 is omitted here; the interested reader is referred to Evans et al. [2012] for more details. Recalling the definition of $\varepsilon_{i|k}$ in (21), it is noted that $d_i^{(j)}$ are not influenced by the information at k and can be obtained by implementing a set of linear programs offline.

Remark 6. For each vertex $j = 1, 2, \ldots, L$, the sequence $d_1^{(j)}, d_2^{(j)}, \ldots$ is monotonically non-decreasing (similar to sequence $\{\beta_i\}_{i \in \mathbb{N}_{\geq 1}}$), and an arbitrarily close upper bound, $\bar{d}^{(j)}$, can be computed offline.

Similar treatments can be used to handle constraint (17). The result is given below directly without proof.

Corollary 7. For any $\zeta_{i|k}$ in the polytope $\{\zeta : V\zeta \leq \theta_{i|k}\}$, constraint (17) is satisfied if there exists a matrix H_c with non-negative elements such that

$$H_c V = \eta^T, \tag{29a}$$

$$\eta^T F_i \mathbf{c}_k + \eta^T (\Phi^0)^i z_k + H_c \theta_{i|k} + f^T E M^{i-1} \mathbf{c}_k \leq h - \beta_i. \tag{29b}$$

The matrices $H^{(j)}, H_c, V,$ as well as the parameters $\theta_{i|k}$ should ideally be optimized online, but treating $H^{(j)}$, H_c, V and $\theta_{i|k}$ as free variables implies that the constraints of (27a), (27b), (29a) and (29b) are nonlinear, thereby leading to a difficult online optimization problem. Instead here $H^{(j)}$, H_c and V will be chosen offline subject to (27a), (29a) and therefore (27b), (29b) will in fact define linear inequalities in $\theta_{i|k}$, which thus leads to an online quadratic programming problem. To relax constraint (27b) and (29b), one way to calculate $H^{(j)}$, H_c offline is to minimise their respective row sums subject to (27a), (29a).

4. TERMINAL CONSTRAINTS

In order to construct a strategy based on finite-dimensional online and offline optimisation problems, we split the infinite prediction horizon into three stages, making different assumptions at each stage about the variables that parameterise the evolution of predicted states, inputs and tubes. For given N and $\hat{N}$, the three stages are defined as follows.

Stage 1: for $i \in \mathbb{N}_{[0,N-1]}$, $c_{i|k}$ and $\theta_{i|k}$ are decision variables,
Stage 2: for $i \in \mathbb{N}_{[N,N+\hat{N}]}$, $c_{i|k} = 0$, and $\theta_{i|k}$ are decision variables,
Stage 3: for $i \in \mathbb{N}_{\geq N+\hat{N}+1}$, $c_{i|k} = 0$ and $\theta_{i|k} = \bar{\theta}_k$ is a decision variable.

This 3-stage approach is a variant of the dual mode prediction strategy that is typically used to limit the number of decision variables in MPC [Mayne et al., 2000]. However our approach allows different horizons for the degrees of freedom parameterising predicted control trajectories and tube evolution. Hence by using $\hat{N} > 0$ the designer can choose, at the expense of introducing a greater number of variables in the online optimisation, to mitigate the effects of the finite parameterisation of predicted tubes.

Although the 3-stage scheme described above leads to a finite number of decision variables, the number of constraints implied by (27b) and (29b) remains infinite. However, using an extension of the method proposed in Gilbert and Tan [1991] for LTI systems, the constraints of (27b) and (29b) over the infinite horizon of Stage 3 are equivalent to a finite number of linear constraints, as we now show. In Stage 3, (27b), (29b) can be expressed

$$\begin{bmatrix} V\delta\Phi^{(j)} \\ \eta^T \end{bmatrix} (\Phi^0)^{i_N} z_{N|k} + \begin{bmatrix} H^{(j)} - I \\ H_c \end{bmatrix} \bar{\theta}_k \leq \begin{bmatrix} -d_i^{(j)} \\ h - \beta_i \end{bmatrix}. \tag{30}$$

The minimum number of constraints needed to ensure satisfaction of (30) for all $i \in \mathbb{N}_{\geq N+\hat{N}+1}$ can be found using the following result, which we state in terms of the set

$$\mathbb{S}_n = \{(z_{N|k}, \bar{\theta}_k) : (30) \text{ holds for all } i_{N+\hat{N}} \in \mathbb{N}_{[1,n]}\}. \tag{31}$$

Theorem 8. If

$$\max_{(z_{N|k},\bar{\theta}_k)\in\mathbb{S}_{n^*}} \left\{ \begin{bmatrix} V\delta\Phi^{(j)} \\ \eta^T \end{bmatrix} (\Phi^0)^{i_N} z_{N|k} + \begin{bmatrix} H^{(j)} - I \\ H_c \end{bmatrix} \bar{\theta} \right\} \leq \begin{bmatrix} -d_i^{(j)} \\ h - \beta_i \end{bmatrix}$$

for $i = N + \hat{N} + n^* + 1$, then:
(i). $(\Phi^0 z_{N|k} + (\Phi^0)^N w, \bar{\theta}_k) \in \mathbb{S}_{n^*}$ for all admissible values of w;
(ii). (30) holds for all $i \in \mathbb{N}_{\geq N+\hat{N}+1}$ if $(z_{N|k}, \bar{\theta}_k) \in \mathbb{S}_{n^*}$.

Proof. From the definitions of $d_i^{(j)}$ and β_i we have

$$\begin{bmatrix} d_{i+1}^{(j)} \\ \beta_{i+1} \end{bmatrix} = \begin{bmatrix} d_i^{(j)} \\ \beta_i \end{bmatrix} + \max_w \begin{bmatrix} V\delta\Phi^{(j)} \\ \eta^T \end{bmatrix} (\Phi^0)^i w, \tag{32}$$

and the condition of the theorem therefore implies that

$$\begin{bmatrix} V\delta\Phi^{(j)} \\ \eta^T \end{bmatrix}(\Phi^0)^{i_N}(\Phi^0 z_{N|k}+(\Phi^0)^N w)+\begin{bmatrix} H^{(j)}-I \\ H_c \end{bmatrix}\bar{\theta}_k \leq \begin{bmatrix} -d_i^{(j)} \\ h-\beta_i \end{bmatrix}$$

for $i = N + \hat{N} + n^*$. Hence property (i) holds, whereas property (ii) follows directly from (i) and (32). □

Theorem 8 shows that $(z_{N|k}, \bar{\theta}_k) \in \mathbb{S}_{n^*}$ constitutes a terminal constraint ensuring satisfaction of (27b) and (29b) in stage 3. Furthermore, for each $i \in \mathbb{N}_{[1,N+\hat{N}+n+1]}$, β_i and $d_i^{(j)}$ can be computed offline and the condition of Theorem 8 can be checked offline by solving a sequence of linear programs. Finally we note that feasibility of (30) when $d_i^{(j)}$ and β_i are replaced by the asymptotic bounds $\bar{d}^{(j)}$ and $\bar{\beta}$ is a sufficient condition for existence of n^* satisfying the condition of Theorem 8.

5. STOCHASTIC MPC ALGORITHM

Since the persistent non-zero additive uncertainty w_k exists, the stage cost $\mathbb{E}_k\big(x_{i|k}^T Q x_{i|k} + u_{i|k}^T R u_{i|k}\big)$ in (6) will tend asymptotically to a non-zero limit [Cannon et al., 2009b] and thus the objective of (6) will be infinite. Accordingly, we take the nominal cost to be the objective:

$$\tilde{J}_k = \sum_{i=0}^{\infty}\big(z_i^T Q z_i + (Kz_i + c_i)^T R (Kz_i + c_i)\big),$$

this can be further written in a lifted form:

$$\tilde{J}_k = \sum_{i=0}^{\infty}\Big(\begin{bmatrix} z_i \\ \mathbf{c}_i \end{bmatrix}^T \tilde{Q}\begin{bmatrix} z_i \\ \mathbf{c}_i \end{bmatrix}\Big),\quad \begin{bmatrix} z_{i+1} \\ \mathbf{c}_{i+1} \end{bmatrix} = \Psi^0\begin{bmatrix} z_i \\ \mathbf{c}_i \end{bmatrix}, \tag{33}$$

$$\Psi^0 = \begin{bmatrix} \Phi^0 & B^0 E \\ 0 & M \end{bmatrix},\ \mathbf{c}_i = M^i \mathbf{c}_k,\ \tilde{Q} = \begin{bmatrix} Q + K^T R K & K^T R E \\ E^T R K & E^T R E \end{bmatrix},$$

with $\mathbf{c}_k$, M, E defined in (16). Therefore, the nominal cost is equal to

$$\tilde{J}_k = \begin{bmatrix} z_k \\ \mathbf{c}_k \end{bmatrix}^T \tilde{W}\begin{bmatrix} z_k \\ \mathbf{c}_k \end{bmatrix}, \tag{34}$$

where $\tilde{W}$ is the solution of the Lyapunov equation $\tilde{W} - (\Psi^0)^T \tilde{W}\Psi^0 = \tilde{Q}$.

Algorithm 1.

Offline: Choose V in (24); compute β_i and $d_i^{(j)}$ and determine n^* satisfying the condition of Theorem 8; calculate $\tilde{W}$ in the cost function (34); set $d_0^{(j)} = 0$.

Online: At each time step $k = 0, 1, \ldots$ compute

$$\mathbf{c}_k = \arg\min_{\mathbf{c}_k,\, z_{0|k},\, \{\theta_{0|k}, \theta_{1|k}, \ldots, \theta_{N+\hat{N}|k}, \bar{\theta}_k\}} \tilde{J}_k \quad \text{subject to}$$

$$\begin{cases} V(x_k - z_{0|k}) \leq \theta_{0|k} \\ \text{constraints in Stages 1 and 2:} \\ \qquad (27\text{b}) \text{ for } i \in \mathbb{N}_{[0,N+\hat{N}-1]},\ (29\text{b}) \text{ for } i \in \mathbb{N}_{[1,N+\hat{N}]}, \\ \text{constraints in Stage 3: } \theta_{N+\hat{N}} \leq \bar{\theta}_k \text{ and } (z_{N|k}, \bar{\theta}_k) \in \mathbb{S}_{n^*}. \end{cases} \tag{35}$$

Then, using the first element, $c_{0|k}$, of the optimal value of $\mathbf{c}_k$, implement $u_k = Kx_k + c_{0|k}$ and repeat the online step at the next time instant.

Theorem 9. If the online optimisation (35) is feasible at $k = 0$, then it remains feasible for all $k > 0$. Under Algorithm 1, the probabilistic constraint (4) is satisfied for all $k \geq 0$ and the MPC law converges asymptotically to the unconstrained feedback law: $u_k \to Kx_k$ as $k \to \infty$.

Proof. Assume that the optimisation in (35) is feasible at instant k and has optimal solution

$$(\mathbf{c}_k,\ z_{0|k},\ \{\theta_{0|k},\ \theta_{1|k},\ \ldots,\ \theta_{N+\hat{N}|k},\ \bar{\theta}_k\}).$$

and denote the predicted sequence generated by (10) with this solution as $\{z_{0|k}, z_{1|k}, \ldots, z_{N|k}\}$. Then, at time $k+1$, the choice of parameters

$$(M\mathbf{c}_k,\ z_{1|k} + w_k,\ \{\theta_{1|k},\ \theta_{2|k},\ \ldots,\ \theta_{N+\hat{N}|k},\ \bar{\theta}_k,\ \bar{\theta}_k\})$$

necessarily satisfies $z_{0|k+1} + \zeta_{0|k+1} = x_{k+1}$ for some $\zeta_{0|k+1} \in \{\zeta : V\zeta \leq \theta_{0|k+1}\}$ and meets the constraints of (27b), (29b) for Stage 1 and 2. Furthermore (10) gives $z_{i|k+1} = z_{i+1|k} + (\Phi^0)^i w_k$, so property (i) of Theorem 8 ensures that the constraints of Stage 3 are also satisfied at time $k+1$. Hence this choice of parameters gives one feasible solution of the online optimisation (35) at time $k+1$. The implied recursive feasibility of (35) together with the property that $u_k = Kx_k$ is the unconstrained optimal control law ensures that the perturbation sequence $\{c_{0|0}, c_{0|1}, \ldots\}$ is in l^2 and hence $c_{0|k} \to 0$ as $k \to \infty$. Therefore x_k converges asymptotically to the set of states on which the unconstrained optimal control law $u_k = Kx_k$ satisfies constraints, and this unconstrained control law is stabilising by assumption. □

Remark 10. The online optimization (35) is a standard quadratic programming problem, and it involves $n_V(N + \hat{N} + 2) + n_u N + n_x$ scalar variables. Thus an increase in n_V results in more variables to be optimized online and more inequality constraints. On the other hand, as n_V represents the number of the tube facets, the larger n_V is, the less conservative the inclusion condition on ζ will be. Therefore, one needs to make a comprise between tightness of inclusion and the online computational burden when choosing n_V. However, with the combination of the offline design of probabilistic tubes and the online optimization of robust tubes, Algorithm 1 can be efficiently implemented online and employs guaranteed theoretical properties, i.e. recursive feasibility and stability with respect to probabilistic constraints.

6. NUMERICAL EXAMPLE

The system model parameters are given as

$$A^0 = \begin{bmatrix} 1.6 & 1.1 \\ -0.7 & 1.2 \end{bmatrix},\ A^{(1)} = \begin{bmatrix} 0.11 & -0.02 \\ -0.01 & 0.05 \end{bmatrix},\ A^{(2)} = -A^{(1)},$$

$$B^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B^{(1)} = \begin{bmatrix} -0.05 \\ 0.07 \end{bmatrix},\ B^{(2)} = -B^{(1)},$$

$$Q = [1\ 0.3]\ [1\ 0.3]^T,\quad R = 0.1,$$

and q_{1k} is a uniformly distributed random variable on $[0,\ 1]$ and $q_{2k} = 1 - q_{1k}$; each element of w_k is a truncated normal distribution with mean 0 and standard deviation 0.1 and satisfies: $\|w_k\|_\infty \leq 0.12$. The constraint set parameters in (4) are

$$\Xi := \left\{ x_k \left| \begin{pmatrix} 1 & 0.3 \\ 1 & -0.3 \\ -1 & -0.3 \\ -1 & 0.3 \end{pmatrix} x_k \leq \begin{pmatrix} 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \end{pmatrix} \right.\right\},\ \mathbf{p} = \begin{pmatrix} 0.8 \\ 0.8 \\ 0.8 \\ 0.8 \end{pmatrix}. \tag{36}$$

K is chosen to be unconstrained nominal control law $[-0.9798\ \ -1.0805]$; the lengths of Stage 1 and 2 are set to be $N = 4$, $\hat{N} = 6$ respectively; $n_V = 16$, and according to Theorem 8, $n^* = 7$.

For the same initial condition $x_0 = [1.65 \ -5]^T$, Figure 1 plots the state trajectories 1 and 2 of 100 realizations of uncertainty under Algorithm 1 and the unconstrained nominal control law respectively. In this figure, only constraints 2 and 3 of the set (36) are indicated since constraints 1 and 4 are inactive for both algorithms. It is found that at time instant $k = 1$, constraint 2 will be violated with a probability of 100% under the unconstrained control law. While under Algorithm 1, the violation probabilities of constraints 2 and 3 are 2.0% and 2.0% respectively (according to 1000 Monte Carlo simulations), which both satisfy the probabilistic constraint set (36).

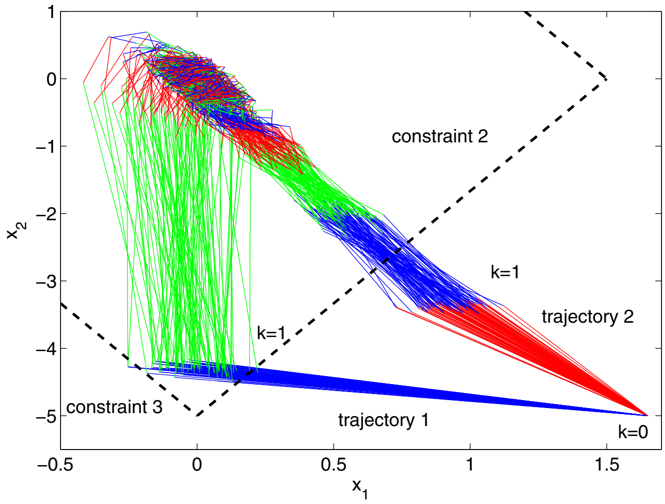

Fig. 1. Evolution of x_k under Algorithm 1 (trajectory 1) and unconstrained nominal control law (trajectory 2); state constraints (dashed line).

7. CONCLUSION

In this paper we have presented a stochastic MPC strategy in the presence of both multiplicative and additive disturbances. Constraint handling is achieved through a decomposition of the uncertain part of the predicted state into two parts: the first of which is constrained to lie in probabilistic tubes that are calculated explicitly offline, whereas the second part is constrained to lie in tubes with bounding facets of fixed orientation whose distances from the origin are optimized online. A tailored invariant terminal set is investigated to ensure the recursive feasibility and this in turn enables the proof of stability of the MPC algorithm. The online optimization is transformed into a quadratic programming problem and thus can be performed efficiently. The handling of the effects of multiplicative uncertainty through the use of robust tubes can incur a degree of conservativeness, the reduction of which forms a topic for future research.

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