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On the stabilizability of continuous-time systems over a packet based communication system with loss and delay ⋆

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Abstract: We raise the question when a continuous-time system can be stabilized over a packet based communication system with loss and delay. Thereby, packet loss is assumed to be an iid random process; packet delay follows a known delay distribution and packets that arrive after a maximal waiting time are ignored, i.e., dropped. Moreover, due to the packet based communication system, the continuous-time system must be sampled. By setting the maximal waiting time for a packet identical to the sampling period, we are able to derive conditions on the sampling period for the existence of a stabilizing controller.

Keywords: Networked Control System, stabilizability, packet delay, packet loss.

1. INTRODUCTION

Since the raise of the Internet more and more control engineers work towards closing the loop by a packet based communication system. This approach leads to a higher flexibility, a reduced wiring harness and thereby a reduction of the cost; but there is one challenge to overcome: packet loss and delay. Thus, such Networked Control Systems (NCS) with packet loss and delay have been studied thoroughly, see, e.g., Tipsuwan and Chow (2003); Hespanha et al. (2007); Bemporad et al. (2010); Lunze (2014) for an overview.

Thereby, it is often assumed that the system is already given as a discrete-time system. However, most physical systems are indeed continuous-time systems. Hence, we consider the control of a continuous-time system and show how the choice of the sampling period affects the properties of the corresponding discrete-time system as well as the arrival probability and thereby the stabilizability.

Obviously, there already exist works that consider continuous time systems with loss and delay. E.g., in Hetel et al. (2006), the delay is assumed to be random but bounded and the control input is immediately applied when the control packet arrives. In doing so, the dynamic of the system depends on the delay. To derive LMIs to find a stabilizing controller, the authors formulate the resulting closed loop system as a time-varying system with parametric uncertainties. Similarly, Donkers et al. (2011) consider a NCS with loss and delay and formulate it as a discrete-time switched uncertain system.

In contrast to these works, the control input is not applied immediately but buffered until the next sampling instance within this work. Moreover, the delay follows a probability distribution with possibly infinite support. Finally, we do not assume that the sampling period is given. Instead, we interpret the sampling period as a degree of freedom that can be chosen and affects the arrival probability as well as the properties of the corresponding discrete-time system.

Parts of this work build on the conditions for the existence of an optimal Kalman filter or optimal controller from Sinopoli et al. (2004); Schenato et al. (2007), which follow from the convergence of a Modified Algebraic Riccati Equation (MARE). However, since we are interested in conditions for the existence of a stabilizing controller, we use a more natural definition of stability than the convergence of a modified algebraic Riccati equation. Hence, we formulate the system with packet loss as a Markov Jump Linear System (MJLS) and rederive the conditions of Sinopoli et al. (2004); Schenato et al. (2007) within this framework. Finally, note that a similar problem setup as the one considered in the present paper, i.e., packets can get lost and the delay follows a probability distribution, can also be found in Schenato (2008), where the optimal estimation for such a system is studied. However, in Schenato (2008), the system is already given as a discrete-time system. Thus, the properties of the discrete-time system can not be affected by a proper choice of the sampling period.

The remainder of this work is outlined as follows. We start with a concrete problem statement in Section 2 and the preliminaries in Section 3. Then, conditions for the stabilizability of a discrete-time system over a packet based communication system with loss and delay are given in Section 4. Based on these results, we derive conditions for the stabilizability of a continuous-time system over a packet based communication system with loss and delay in Section 5. In Section 6, the usefulness of these conditions is demonstrated with the help of an example. Finally, we draw our conclusions in Section 7.

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2. PROBLEM STATEMENT

We consider the control of a continuous-time system

$$\dot{\mathbf{x}}(t) = A_\mathrm{c}\mathbf{x}(t) + B_\mathrm{c}\mathbf{u}(t) \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system and $\mathbf{u} \in \mathbb{R}^m$ the control input.

Note that there is no loss and delay within the physical system (1). This is due to the fact that loss and delay is caused solely by the communication system. Moreover, due to the usage of a packet based communication system, it is not possible to close the loop continuously, i.e., with a controller $\mathbf{u}(t) = K\mathbf{x}(t)$. Instead, the system is sampled at discrete times. The new control input is transmitted within a packet to the actuator. The transmission of each packet takes some time, i.e., they are delayed, and can even get lost.

The model of the communication system is taken from Carabelli et al. (2014). The delay d is a random variable with known probability density function (PDF) $f_d(\tau)$. The corresponding cumulative distribution function (CDF) $F_d(\tau)$ gives the probability that the delay d is less than a certain value:

$$\Pr\{d \le \tau\} = F_d(\tau) = \int_0^\tau f_d(t)\mathrm{d}t. \tag{2}$$

Figure 1 shows an example of such a delay distribution. First of all, note that this distribution is parameterized such that there is a minimal delay for each packet. Moreover, most packets arrive with a delay between 0.05 and 0.3. Nevertheless, there does not exist an upper bound for the delay, i.e., no matter how large the maximal waiting time is chosen, there will always be packets that will not arrive on time.

Moreover, packet loss is an iid process with a packet arrival probability p_a, which is independent of the delay. Thus, for a given maximal waiting time T_max, the effective arrival probability p_eff becomes

$$p_\mathrm{eff}(T_\mathrm{max}) = \Pr\{d \le T_\mathrm{max}\}p_\mathrm{a} = F_d(T_\mathrm{max})p_\mathrm{a}. \tag{3}$$

Obviously, lost packets can also be interpreted as packets with an infinite delay. Nevertheless, in doing so, (3) will in general be not a CDF because $\lim_{\tau\to\infty} p_\mathrm{eff}(\tau) = 1$ holds only for $p_\mathrm{a} = 1$.

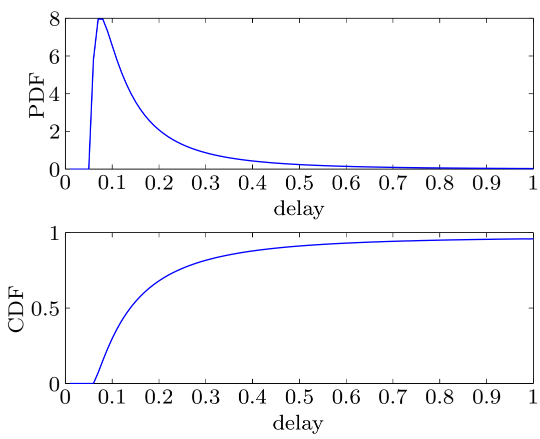

Fig. 1. Exemplary delay distribution.

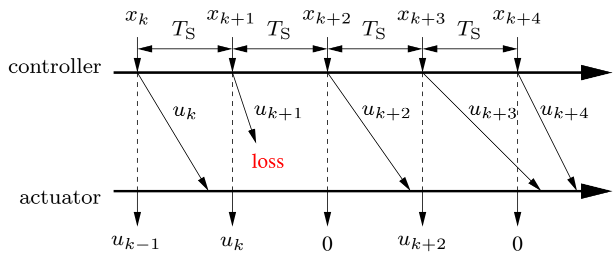

Fig. 2. Timing.

Since a packet based communication system is used to close the loop, packets are sent at discrete times. Thus, system (1) is sampled with a constant sampling period T_S.

Figure 2 depicts how the system with packet loss and delay is sampled. As can be seen, the sampling period T_S is chosen such that most control packets arrive before the next sampling instance. The actuator waits for the end of the sampling interval and then applies the corresponding control input. Nevertheless, due to the delay distribution, it is possible that a packet arrives after the next sampling instance. Within this work, we assume that such packets are ignored, i.e., dropped, and the input of the plant is set to zero. In doing so, the maximal waiting time is given by the sampling period, i.e., $T_\mathrm{max} = T_\mathrm{S}$. Thus, the choice of the sampling period T_S affects the effective packet loss probability, as given in (3).

Obviously, there exist other approaches to sample such a system. First, the system can be sampled much faster such that the delay will be multiples of the sampling period. Since the delay distribution is known, this approach results in a discrete-time system with known delay distribution, see, e.g., Blind et al. (2008); Xiao et al. (2000); Zhang et al. (2005). Nevertheless, at each time, there will be more than one packet on the fly. Depending on the arrival probability, this might not be TCP-fair and we thus have to worry that the communication system does not become overloaded. On the other hand, it would also be possible to apply the new control input as soon as the control packet arrives, as done in e.g., Hetel et al. (2006).

To discretize the continuous-time system (1), we define $\mathbf{x}_k := \mathbf{x}(kT_\mathrm{S})$ as the state of the corresponding discrete-time system. The input $\mathbf{u}$ is kept constant during each sampling interval, i.e.,

$$\mathbf{u}(t) = \mathbf{u}_k \qquad \text{for } kT_\mathrm{S} \le t < (k+1)T_\mathrm{S}.$$

When ignoring packet loss and delay, the corresponding discrete-time system becomes

$$\mathbf{x}_{k+1} = A_\mathrm{d}(T_\mathrm{S})\mathbf{x}_k + B_\mathrm{d}(T_\mathrm{S})\mathbf{u}_k, \tag{4}$$

where the matrices A_d and B_d are

$$A_\mathrm{d}(T_\mathrm{S}) = e^{A_\mathrm{c}T_\mathrm{S}} \text{ and } B_\mathrm{d}(T_\mathrm{S}) = \int_0^{T_\mathrm{S}} e^{A_\mathrm{c}\tau}B_\mathrm{c}\mathrm{d}\tau,$$

see, e.g., Franklin et al. (1997) for the details of the discretization.

However, in our setup, packets are lost and delayed due to the packet based communication system. Thus, we use $\beta_k \in \{0,1\}$ with $\mathrm{E}[\beta_k] = p$ to describe packet loss. Moreover, note that the control input can not be applied immediately. Instead it is applied during the next sampling period. Thus, the delay is exactly one sampling interval.

Consequently, the discrete-time system with packet loss and delay becomes

$$\mathbf{x}_{k+1} = A_{\mathrm{d}}(T_{\mathrm{S}})\mathbf{x}_k + B_{\mathrm{d}}(T_{\mathrm{S}})\beta_{k-1}\mathbf{u}_{k-1}. \tag{5}$$

Now, the interesting questions are: For which delay distributions exists a stabilizing controller? For which choice of the sampling period exists a stabilizing controller?

3. PRELIMINARIES

The stabilizability analysis of this work is based on the well studied Markov Jump Linear System (MJLS) approach, see Costa et al. (2005) for an introduction to MJLS. Basically, a MJLS is a linear system, which jumps between different dynamics, i.e., the state evolves as

$$\mathbf{z}_{k+1} = \mathcal{A}_{\sigma_k}\mathbf{z}_k, \tag{6}$$

where $\sigma_k \in \{1, \ldots, N\}$ is a Markovian process, described by a Markov chain with transition probabilities p_{ij}.

Definition 1. [Costa et al. (2005), Definition 3.8] System (6) *is* Mean Square Stable (MSS) *if for any initial condition* $\mathbf{z}_0$, σ_0*:*

$$\mathrm{E}[\mathbf{z}_k] \to 0 \text{ as } k \to \infty,$$
$$\mathrm{E}[\mathbf{z}_k\mathbf{z}_k^{\mathsf{T}}] \to 0 \text{ as } k \to \infty.$$

Theorem 2. [Costa et al. (2005), Corollary 3.26] Suppose, the transition probabilities are iid, i.e., $p_{ij} = p_j$*, then the system* (6) *is MSS if and only if there exists a* $\mathcal{V} > 0$, $\mathcal{V}^{\mathsf{T}} = \mathcal{V}$ *such that*

$$\mathcal{V} - \sum_{j=1}^{N} p_j \mathcal{A}_j^{\mathsf{T}} \mathcal{V} \mathcal{A}_j > 0. \tag{7}$$

4. STABILIZABILITY OF THE DISCRETE-TIME SYSTEM

4.1 Ideal System

The controllability and stabilizability of a discretized system are already studied in Kalman et al. (1963) and Hautus (1970, 1969). It turns out that for certain sampling periods the controllability or stabilizability can get lost. As in Hautus (1970), we thus call system (4) *properly sampled* if for λ, $\mu \in \mathrm{eig}(A_{\mathrm{c}})$ and k a positive integer, the condition

$$\lambda - \mu \neq 2\pi i k/T_{\mathrm{S}} \tag{8}$$

is satisfied.

Theorem 3. [Hautus (1970, 1969)] Let the continuous-time system (1) *be properly sampled, then the corresponding discrete-time system* (4) *is controllable (stabilizable) if and only if the continuous-time system* (1) *is controllable (stabilizable).*

4.2 No Delay, Only Packet Loss

Before we analyze the stabilizability of the NCS with loss and delay (5), we first have a look at the delay free version

$$\mathbf{x}_{k+1} = A_{\mathrm{d}}\mathbf{x}_k + B_{\mathrm{d}}\beta_k\mathbf{u}_k, \quad \mathrm{E}[\beta_k] = p. \tag{9}$$

By using the controller $\mathbf{u}_k = K\mathbf{x}_k$, we get a jump linear system with $\mathcal{A}_1 = (A_{\mathrm{d}} + B_{\mathrm{d}}K)$ and $\mathcal{A}_2 = A_{\mathrm{d}}$. Thus, we define the stabilizability of (9) as follows.

Definition 4. We say that system (9) *is* stabilizable *if there exists a* K *such that with* $\mathbf{u}_k = K\mathbf{x}_k$*, system* (9) *is MSS.*

From Theorem 2, we get the following stabilizability test.

Corollary 5. System (9) *is stabilizable if and only if there exists a* K *and* $V > 0$, $V^{\mathsf{T}} = V$ *such that*

$$V - p(A_{\mathrm{d}}+B_{\mathrm{d}}K)^{\mathsf{T}}V(A_{\mathrm{d}}+B_{\mathrm{d}}K) - (1-p)A_{\mathrm{d}}^{\mathsf{T}}VA_{\mathrm{d}} > 0. \tag{10}$$

Obviously, when all control packets arrive, i.e., $p = 1$, there exists a V and K if and only if $(A_{\mathrm{d}}, B_{\mathrm{d}})$ is stabilizable. On the other hand, when all control packets get lost, i.e., $p = 0$, then there exists a V if and only if A_{d} is stable. Moreover, the following quite intuitive theorem can be derived.

Theorem 6.

- *(i) Suppose,* (9) *is stabilizable for some* $p = p_{\mathrm{s}}$*, then* (9) *is stabilizable for all* $p_{\mathrm{s}} \leq p \leq 1$.
- *(ii) Suppose,* (9) *is not stabilizable for some* $p = p_{\mathrm{u}}$*, then* (9) *is not stabilizable for all* $0 \leq p \leq p_{\mathrm{u}}$.

The proof is given in the appendix.

Theorem 6 states that system (9) is stabilizable when p is large enough but not stabilizable when p is too small. Thus, we are interested in the smallest p such that system (9) is stabilizable. We define this critical arrival probability as follows.

Definition 7. For system (9)*, the critical arrival probability is*

$$p_{\mathrm{c}} = \inf_p\{0 \leq p \leq 1 | \textit{system } (9) \textit{ is stabilizable}\}.$$

Numerically, a tight upper bound for the critical arrival probability can be found by searching for the smallest probability p where system (9) is stabilizable, i.e.,

$$\bar{p}_{\mathrm{c}} = \min_p\{0 \leq p \leq 1 | \text{system (9) is stabilizable}\}.$$

Remark 8. Since the stabilizability test (10) is a strict inequality, the minimum does not achieve the infimum. Consequently, $p_{\mathrm{c}} < \bar{p}_{\mathrm{c}}$. Moreover, it follows that (9) is stabilizable if and only if $p > p_{\mathrm{c}}$. Obviously, (9) is stabilizable if $p \geq \bar{p}_{\mathrm{c}}$.

Now, the interesting question is: For which arrival probabilities does there exist a stabilizing controller? This question has also been studied in Sinopoli et al. (2004); Elia (2005); Schenato et al. (2007) and can be summarized with the following theorem.

Theorem 9. Suppose that A_{d} *is unstable and* $(A_{\mathrm{d}}, B_{\mathrm{d}})$ *stabilizable. Then*

- *(i) there exists a stabilizing controller if and only if* $p > p_{\mathrm{c}}$*, where* p_{c} *is the critical arrival probability.*
- *(ii) the critical arrival probability* p_{c} *satisfies the following analytical bounds:*

$$p_{\min} \leq p_{\mathrm{c}} \leq p_{\max}, \tag{11}$$

where

$$p_{\min} = 1 - \frac{1}{\max_i |\mathrm{eig}_i^u(A_{\mathrm{d}})|^2},$$
$$p_{\max} = 1 - \frac{1}{\prod_i |\mathrm{eig}_i^u(A_{\mathrm{d}})|^2},$$

$\mathrm{eig}^u(A_\mathrm{d})$ are the unstable eigenvalues of A_d, i.e., all eigenvalues whose absolute value is larger than one.

(iii) $p_\mathrm{c} = p_{\min}$ when B_d is square and invertible.

(iv) $p_\mathrm{c} = p_{\max}$ when B_d has rank one.

(v) $p_\mathrm{c} = 1 - 1/|\lambda_u^2|$ when A_d has only one unstable eigenvalue λ_u.

(vi) a tight upper bound $\bar{p}_\mathrm{c}$ of the critical arrival probability p_c can be computed via the solution of the following LMI

$$\bar{p}_\mathrm{c} = \arg\min_p \Psi_p(Y,Z) > 0, \qquad 0 \le Y \le I, \tag{12}$$

where

$$\Psi_p(Y,Z) = \begin{bmatrix} Y & \sqrt{p}(YA_\mathrm{d}^\intercal + ZB_\mathrm{d}^\intercal) & \sqrt{1-p}YA_\mathrm{d}^\intercal \\ * & Y & 0 \\ * & * & Y \end{bmatrix}.$$

See Remark 8 for a discussion on the tightness of $\bar{p}_\mathrm{c}$.

Proof. (i) follows directly from Definition 7.
(ii), (iii), and (iv): The lower bound $p_{\min}$ follows from studying (10) with $A_\mathrm{d} + B_\mathrm{d}K = 0$, see also Katayama (1976). The upper bound $p_{\max}$ was derived in Elia (2005).
(v) follows form the fact that the upper and lower bound is identical when A_d has only one unstable eigenvalue.
(vi): This LMI follows by applying the Schur complement twice on (10). □

Note that Theorem 9 is similar to Theorem 4 of Sinopoli et al. (2004) and Lemma 5.4 of Schenato et al. (2007). However, in contrast to the present work, where the stability of the NCS is based on its formulation as a jump linear system, the stability analysis in Sinopoli et al. (2004); Schenato et al. (2007) is based on the convergence of a modified algebraic Riccati equation.

4.3 With Loss and Delay

Now, we are ready to consider the NCS with loss and delay as given by (5), i.e.,

$$\mathbf{x}_{k+1} = A_\mathrm{d}(T_\mathrm{S})\mathbf{x}_k + B_\mathrm{d}(T_\mathrm{S})\beta_{k-1}\mathbf{u}_{k-1}.$$

To compensate the delay, we use a predictive controller:

$$\mathbf{u}_k = K\hat{\mathbf{x}}_{k+1},$$
$$\hat{\mathbf{x}}_{k+1} = A_\mathrm{d}\mathbf{x}_k + B_\mathrm{d}\beta_{k-1}\mathbf{u}_{k-1},$$

i.e.,

$$\mathbf{u}_k = KA_\mathrm{d}\mathbf{x}_k + KB_\mathrm{d}\beta_{k-1}\mathbf{u}_{k-1}. \tag{13}$$

By augmenting the state, (5) can be written as

$$\begin{bmatrix}\mathbf{x}_{k+1} \\ \mathbf{u}_k\end{bmatrix} = \begin{bmatrix} A_\mathrm{d} & \beta_{k-1}B_\mathrm{d} \\ 0 & 0 \end{bmatrix}\begin{bmatrix}\mathbf{x}_k \\ \mathbf{u}_{k-1}\end{bmatrix} + \begin{bmatrix}0 \\ I\end{bmatrix} u_k. \tag{14}$$

In doing so, the controller (13) becomes

$$\mathbf{u}_k = [KA_\mathrm{d} \quad KB_\mathrm{d}\beta_{k-1}]\begin{bmatrix}\mathbf{x}_k \\ \mathbf{u}_{k-1}\end{bmatrix}. \tag{15}$$

Moreover, the closed loop becomes

$$\begin{bmatrix}\mathbf{x}_{k+1} \\ \mathbf{u}_k\end{bmatrix} = \begin{bmatrix} A_\mathrm{d} & \beta_{k-1}B_\mathrm{d} \\ KA_\mathrm{d} & \beta_{k-1}KB_\mathrm{d} \end{bmatrix}\begin{bmatrix}\mathbf{x}_k \\ \mathbf{u}_{k-1}\end{bmatrix}. \tag{16}$$

Note that (16) can be written as a MJLS with

$$\mathbf{z}_k = \begin{bmatrix}\mathbf{x}_k \\ \mathbf{u}_{k-1}\end{bmatrix}, \quad \mathcal{A}_1 = \begin{bmatrix} A_\mathrm{d} & B_\mathrm{d} \\ KA_\mathrm{d} & KB_\mathrm{d}\end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} A_\mathrm{d} & 0 \\ KA_\mathrm{d} & 0\end{bmatrix}.$$

Thereby, $\mathcal{A}_1$ represents the case that the control packet arrives and $\mathcal{A}_2$ the case that the control packet is lost.

Theorem 10. For the system with loss and delay, i.e., system (5), there exists a stabilizing controller of the form (13) if and only if there exists a stabilizing controller $\mathbf{u}_k = K\mathbf{x}_k$ for the system without delay, i.e., system (9).

The proof is given in the appendix.

Note that Theorem 10 is crucial for studying the NCS with packet loss and delay since it shows that Theorem 9 also holds when we also take the packet delay into account.

5. STABILIZABILITY OF THE CONTINUOUS-TIME SYSTEM

In this section, we bring the different observations of the previous sections together and derive conditions on the sampling period, delay distribution, and continuous-time system that determine the existence of a stabilizing controller for system (1) with packet loss and delay.

As already stated, the maximal waiting time is chosen to be identical to the sampling period, i.e., the maximal waiting time follows from the choice of the sampling period and thus the sampling period affects the effective arrival probability. Moreover, the sampling period also affects the matrices A_d and B_d of the corresponding discrete-time system and thereby the eigenvalues of A_d.

Now, remember that Theorem 9 states how the critical arrival probability depends on the discrete-time system. Theorem 9 also gives an upper and a lower bound for the critical arrival probability that both depend on the eigenvalues of A_d. Based on this theorem, we can derive the following conditions on the sampling period, delay distribution, and continuous-time system that determine the existence of a stabilizing controller.

Theorem 11. Suppose that B_c has rank one, the continuous-time system (1) is stabilizable, and properly sampled. Then there exists a stabilizing controller if and only if the sampling period T_S satisfies

$$p_\mathrm{c} = 1 - e^{-\lambda_M T_\mathrm{S}} < F_d(T_\mathrm{S})p_\mathrm{a} = p_{\mathit{eff}}, \tag{17}$$

where $\lambda_M = \sum_i 2\,\mathrm{Re}(\mathrm{eig}_i^u(A_\mathrm{c}))$. Thereby, $\mathrm{eig}^u(A_\mathrm{c})$ are the unstable eigenvalues of A_c, i.e., all eigenvalues whose real part is greater than zero.

Theorem 12. Suppose that B_c is square and invertible and the continuous-time system (1) is properly sampled. Then, there exists a stabilizing controller if and only if the sampling period T_S satisfies

$$p_\mathrm{c} = 1 - e^{-\lambda_m T_\mathrm{S}} < F_d(T_\mathrm{S})p_\mathrm{a} = p_{\mathit{eff}}, \tag{18}$$

where $\lambda_m = \max_i 2\,\mathrm{Re}(\mathrm{eig}_i^u(A_\mathrm{c}))$. Thereby, $\mathrm{eig}^u(A_\mathrm{c})$ are the unstable eigenvalues of A_c, i.e., all eigenvalues whose real part is greater than zero.

Theorem 13. Suppose that A_c has only one unstable eigenvalue λ_u, the continuous-time system (1) is stabilizable, and properly sampled. Then there exists a stabilizing controller if and only if the sampling period T_S satisfies

$$p_\mathrm{c} = 1 - e^{-2\lambda_u T_\mathrm{S}} < F_d(T_\mathrm{S})p_\mathrm{a} = p_{\mathit{eff}}. \tag{19}$$

Theorem 14. Suppose that the continuous-time system (1) is stabilizable and properly sampled. Then, a stabilizing controller exists

(i) *only if there exists a sampling period* T_{S} *that satisfies* (18)
(ii) *if there exists a sampling period* T_{S} *that satisfies* (17).
(iii) *if there exists a sampling period* T_{S} *such that*

$$F_d(T_{\mathrm{S}})p \geq \bar{p}_{\mathrm{c}} = \arg\min_{p} \Psi_p(Y, Z, T_{\mathrm{S}}) > 0, \qquad 0 \leq Y \leq I,$$

where

$$\Psi_p(Y, Z, T_{\mathrm{S}}) =$$
$$\begin{bmatrix} Y & \sqrt{p}(YA_{\mathrm{d}}(T_{\mathrm{S}})^{\mathsf{T}} + ZB_{\mathrm{d}}(T_{\mathrm{S}})^{\mathsf{T}}) & \sqrt{1-p}YA_{\mathrm{d}}(T_{\mathrm{S}})^{\mathsf{T}} \\ * & Y & 0 \\ * & * & Y \end{bmatrix}.$$

Proof. These theorems follow from the observation how the effective arrival probability depends on the choice of the sampling period, i.e., eq. (3), Theorem 9, and the fact that the multispectrum of the matrix exponential is

$$\mathrm{mspec}\left(e^{A_{\mathrm{c}}T_{\mathrm{S}}}\right) = \left\{e^{\lambda T_{\mathrm{S}}} : \lambda \in \mathrm{mspec}(A_{\mathrm{c}})\right\}_m,$$

see, e.g., Proposition 11.2.3 in Bernstein (2009). □

Note that the left hand side of (17), (18), and (19) is the CDF of a negative exponential distribution with coefficient λ_m, λ_M, and λ_u, respectively. Thus, (17), (18), and (19) can be easily checked. Loosely speaking, these conditions can be interpreted as follows. There exists a stabilizing controller if and only if the communication system is faster than the control system. Moreover, since negative exponential distributions are very well known in communication theory, these theorems might be very interesting and helpful when the controller and the communication system are designed together, so called control and communication co-design.

6. EXAMPLE

Finally, we demonstrate the usefulness of Theorem 11 - 14 with the help of an example where the stabilization of an inverted pendulum on a cart is considered. Therefore, we assume that the mass of the cart is 5 kg. The mass of the pendulum is 2 kg, the length to its center of mass is 3 m, and its mass moment of inertia is 0.06 kgm^2.

With these parameters, we get

$$A_{\mathrm{c}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 3.9018 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 4.5521 & 0 \end{bmatrix}, \qquad B_{\mathrm{c}} = \begin{bmatrix} 0 \\ 0.1997 \\ 0 \\ 0.0664 \end{bmatrix}.$$

The delay distribution is the one already depicted in Figure 1, which is a shifted log normal distribution as given in (20) with the parameters $\mu = -2.5$, $\sigma = 1.1$ and $d_0 = 0.05$. Note that the transmission delay is at least d_0 but there does not exist an upper bound for the delay.

$$f_d(\tau|\mu, \sigma, d_0) = \begin{cases} 0 & \tau \leq d_0 \\ \frac{1}{(\tau - d_0)\sigma\sqrt{2\pi}} e^{-\frac{(\ln(\tau-d_0)-\mu)^2}{2\sigma^2}} & \tau > d_0 \end{cases} \tag{20}$$

Moreover, for simplicity we assume that all packets arrive, i.e., $p_{\mathrm{a}} = 1$ within this example.

Since B_{c} is of rank one and A_{c} has an unstable eigenvalue at 2.1336, we can use Theorem 11 or 13 to conclude that there exists a stabilizing controller if and only if $p_{\mathrm{c}} = 1 -$

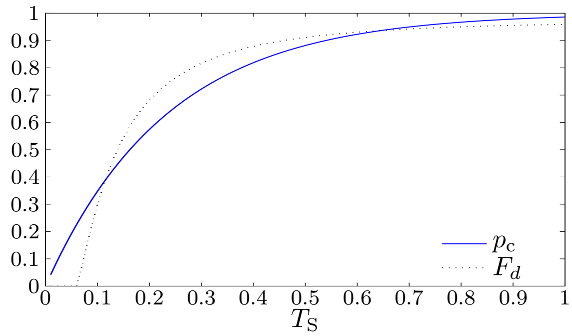

Fig. 3. The critical arrival probability together with the CDF of the delay.

$e^{-4.2672T_{\mathrm{S}}} < F_d(T_{\mathrm{S}}) = p_{\mathrm{eff}}$. Figure 3 depicts this condition. From this figure, we see that there exists a stabilizing controller if and only if the sampling period T_{S} is chosen between 0.1149 and 0.6456.

7. CONCLUSIONS

Within this work, we considered the stabilizability of continuous-time systems over a packet based communication system with loss and delay. Thereby, we used a slightly more realistic and complex model for the loss and delay than previous works and assumed that the delay follows a known delay distribution with possibly infinite support. Based on results from recent literature, we derived conditions on the sampling period for the existence of a stabilizing controller. For the special cases B_{c} square, B_{c} rank one, or A_{c} has only one unstable eigenvalue, it turns out that the condition is necessary and sufficient and simply requires a comparison between the CDF of the delay and a negative exponential distribution whose parameter follows from the unstable eigenvalues of the control system.

Since the necessary and sufficient conditions give a relationship between the CDF of the delay and the unstable eigenvalues of the control system this result might be very interesting and helpful when the controller and the communication system are designed together, so called controller and communication co-design.

Appendix A. PROOF OF THEOREM 6

The poof of Theorem 6 uses some ideas of Sinopoli et al. (2004).

First, note that (10) can also be written as

$$p(A_{\mathrm{d}} + B_{\mathrm{d}}K)^{\mathsf{T}}V(A_{\mathrm{d}} + B_{\mathrm{d}}K) + (1-p)A_{\mathrm{d}}^{\mathsf{T}}VA_{\mathrm{d}} < V. \tag{A.1}$$

For simplicity of notation, we use $\Phi_p(K, V)$ as a shortcut for the left hand side of (A.1), i.e.,

$$\Phi_p(K, V) := p(A_{\mathrm{d}} + B_{\mathrm{d}}K)^{\mathsf{T}}V(A_{\mathrm{d}} + B_{\mathrm{d}}K) + (1-p)A_{\mathrm{d}}^{\mathsf{T}}VA_{\mathrm{d}}$$

and define

$$g_p(V) := \min_K \Phi_p(K, V) \quad \text{and} \quad K_V := \arg\min_K \Phi_p(K, V).$$

The minimizer K_V can be found by solving $\partial\Phi_p(K, V)/\partial K = 0$. In doing so, we obtain

$$K_V = -(B_{\mathrm{d}}^{\mathsf{T}}VB_{\mathrm{d}})^{-1}B_{\mathrm{d}}^{\mathsf{T}}VA_{\mathrm{d}}.$$

By noting that $B_{\mathrm{d}}^{\mathsf{T}}V(A_{\mathrm{d}} + B_{\mathrm{d}}K_V) = 0$, we get

$$g_p(V) = A_{\mathrm{d}}^{\mathsf{T}} V A_{\mathrm{d}} - p A_{\mathrm{d}}^{\mathsf{T}} V B_{\mathrm{d}} (B_{\mathrm{d}}^{\mathsf{T}} V B_{\mathrm{d}})^{-1} B_{\mathrm{d}}^{\mathsf{T}} V A_{\mathrm{d}}. \quad \text{(A.2)}$$

Since $A_{\mathrm{d}}^{\mathsf{T}} V B_{\mathrm{d}} (B_{\mathrm{d}}^{\mathsf{T}} V B_{\mathrm{d}})^{-1} B_{\mathrm{d}}^{\mathsf{T}} V A_{\mathrm{d}} \geq 0$, we get the following lemma.

Lemma 15. If $p_1 \leq p_2$, then $g_{p_1}(V) \geq g_{p_2}(V)$.

Now, we are ready to prove Theorem 6 (i). Since (9) is stabilizable for p_{s}, there exists a V such that $g_{p_{\mathrm{s}}}(V) < V$. Using Lemma 15, we see that $g_p(V) \leq g_{p_{\mathrm{s}}}(V) < V$ holds for $p \geq p_{\mathrm{s}}$. Consequently (9) is stabilizable for $p \geq p_{\mathrm{s}}$.

Similarly, to prove Theorem 6 (ii) we use Lemma 15 to see that $g_{p_{\mathrm{u}}}(V) \leq g_p(V)$ holds for $p \leq p_{\mathrm{u}}$. Since there does not exist a V such that $g_{p_{\mathrm{u}}}(V) < V$, there does not exist a V such that $g_p(V) < V$ for $p \leq p_{\mathrm{u}}$. Consequently (9) is not stabilizable for $p \leq p_{\mathrm{u}}$.

Appendix B. PROOF OF THEOREM 10

First, we show that if there exists a controller that stabilizes the system with delay, then there exists also a controller that stabilizes the system without delay.

From Theorem 2, we know that there exists a $\mathcal{V}$ such that

$$\mathcal{V} - p\mathcal{A}_1^{\mathsf{T}} \mathcal{V} \mathcal{A}_1 - (1-p)\mathcal{A}_2^{\mathsf{T}} \mathcal{V} \mathcal{A}_2 > 0. \quad \text{(B.1)}$$

Now, note that $\mathcal{A}_1$ and $\mathcal{A}_2$ can be written as

$$\mathcal{A}_1 = \begin{bmatrix} I \\ K \end{bmatrix} [A_{\mathrm{d}}\ B_{\mathrm{d}}], \qquad \mathcal{A}_2 = \begin{bmatrix} I \\ K \end{bmatrix} [A_{\mathrm{d}}\ 0].$$

Using this observation and multiplying (B.1) with $\left[I\ K^{\mathsf{T}}\right]$ and $\begin{bmatrix} I \\ K \end{bmatrix}$ from left and right, respectively, we get

$$\begin{aligned} &\left[I\ K^{\mathsf{T}}\right] \mathcal{V} \begin{bmatrix} I \\ K \end{bmatrix} \\ &\quad - p\left[I\ K^{\mathsf{T}}\right] \begin{bmatrix} A_{\mathrm{d}}^{\mathsf{T}} \\ B_{\mathrm{d}}^{\mathsf{T}} \end{bmatrix} \left[I\ K^{\mathsf{T}}\right] \mathcal{V} \begin{bmatrix} I \\ K \end{bmatrix} [A_{\mathrm{d}}\ B_{\mathrm{d}}] \begin{bmatrix} I \\ K \end{bmatrix} \\ &- (1-p)\left[I\ K^{\mathsf{T}}\right] \begin{bmatrix} A_{\mathrm{d}}^{\mathsf{T}} \\ 0 \end{bmatrix} \left[I\ K^{\mathsf{T}}\right] \mathcal{V} \begin{bmatrix} I \\ K \end{bmatrix} [A_{\mathrm{d}}\ 0] \begin{bmatrix} I \\ K \end{bmatrix} > 0. \end{aligned}$$

Now, let $V = \left[I\ K^{\mathsf{T}}\right] \mathcal{V} \begin{bmatrix} I \\ K \end{bmatrix}$, note that $[A_{\mathrm{d}}\ B_{\mathrm{d}}] \begin{bmatrix} I \\ K \end{bmatrix} = A_{\mathrm{d}} + B_{\mathrm{d}} K$, and $[A_{\mathrm{d}}\ 0] \begin{bmatrix} I \\ K \end{bmatrix} = A_{\mathrm{d}}$, to rediscover (10).

To show that (B.1) holds when there exits a V and K such that (10) holds, we simply use $\mathcal{V} = \left[I\ K^{\mathsf{T}}\right]^{\dagger} V \begin{bmatrix} I \\ K \end{bmatrix}^{\dagger}$, where $M^{\dagger}$ is the Moore-Penrose pseudoinverse of the matrix M.

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