Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

On Computing Control Inputs to Achieve State Transition for a Class of Nearly Controllable Discrete-time Bilinear Systems ⋆

Lin Tie *

* *The author is with the School of Automation Science and Electrical Engineering, Beihang University (Beijing University of Aeronautics and Astronautics), 100191, Beijing, P. R. China. (E-mail: tielinllc@gmail.com).*

Abstract: Near-controllability is defined for those systems that are uncontrollable but have a large controllable region. It is a property of nonlinear control systems introduced recently, and it has been well demonstrated on discrete-time bilinear systems. The purpose of this paper is to propose a useful algorithm to compute the control inputs, which achieve the transition of a given pair of states, for a class of discrete-time bilinear systems that are nearly controllable. Accordingly, for such class of bilinear systems, not only near-controllability is proved, but also the computability of control inputs for near-controllability is shown. An example is provided to demonstrate the effectiveness of the proposed algorithm.

1. INTRODUCTION

Bilinear systems comprise an important class as well as a special class of nonlinear systems, which have received considerable attention over decades (Elliott [2009]). Many real world processes, ranging from engineering to sciences, can be modeled or approximated by bilinear systems (Bruni, Pillo, and Koch [1974], Mohler and Kolodziej [1980], Fliess [1981], and Mohler [1991]). Furthermore, such systems are thought to be simpler and better understood than most other nonlinear systems. Indeed, bilinear systems have been a hot research topic in the literature of nonlinear systems.

Controllability is clearly one of the most important issues in control theory. The concept of controllability was identified in the early 1960s (Kalman, Ho, and Narendra [1963]) and the theory for controllability of linear systems and nonlinear systems was well established (Wonham [1985], Sussmann and Jurdjevic [1972], Hermann and Krener [1977], Isidori [1995], Jurdjevic [1997], Fliess and Normand-Cyrot [1981], Jakubczyk and Sontag [1990], Albertini and Sontag [1994], and Wirth [1998]). Today, controllability has become one of the fundamental concepts in mathematical control theory (Klamka [1991] and Sontag [1998]). There are many different kinds of definitions of controllability, such as local controllability, global controllability, null controllability, approximate controllability, and positive controllability. Roughly speaking, controllability is defined as the ability of a system that the system can be steered from an arbitrary initial state to an arbitrary terminal state under the action of admissible control inputs. For linear systems, controllability can be proved by directly deriving the control inputs to achieve the state transition. For nonlinear systems, however, such control inputs are, in general, difficult to obtain even if controllability has been proved.

For bilinear systems, controllability in continuous-time case has been extensively investigated profiting from the Lie algebra methods. Various Lie-algebraic criteria on controllability of continuous-time bilinear systems were provided in the literature, which have been summarized and updated in the recent monograph Elliott [2009] on bilinear systems. The controllability results for discrete-time bilinear systems, however, are sparse compared with their continuous-time counterparts. Most of the work on controllability of discrete-time bilinear systems was done in the 1970s (Tarn, Elliott, and Goka [1973], Goka, Tarn, and Zaborszky [1973], and Evans and Murthy [1977]), which considered bilinear systems of the form

$$x\left(k+1\right)=\left(A+u\left(k\right)B\right)x\left(k\right) \tag{1}$$

where $x(k)\in\mathbb{R}^n$, $A,B\in\mathbb{R}^{n\times n}$, and $u(k)\in\mathbb{R}$. For controllability of system (1), Tarn et al. [1973] gave a sufficient condition, which requires A to be similar to an orthogonal matrix. Goka et al. [1973] studied controllability of system (1) under the assumption of $\mathrm{rank}B=1$ and presented necessary as well as sufficient conditions; based on the work in Goka et al. [1973], Evans and Murthy [1977] improved these conditions by raising necessary and sufficient ones. Since then, few work on controllability of discrete-time bilinear systems has been reported until the 2000s (Tie and Cai [2012], Tie and Lin [2013b], and Tie [2013b]), where controllability criteria were obtained for system (1) (mainly in dimension two). In summary, for controllability of system (1), only specific subclasses are considered, while most cases remain unsolved. The reasons

⋆ This work was supported by the China Postdoctoral Science Foundation funded project under Grant 2012T50035 and the National Natural Science Foundation of China under Grants 61203231, 61203022.

are basically due to the nonlinearity and poor algebraic structure of the systems.

For controllability of discrete-time bilinear systems, it is of interest to notice the recent result obtained in Tie, Cai, and Lin [2010]. If a system is uncontrollable, it is natural to consider its controllable regions [1]. Tie et al. [2010] fulfilled this thought by investigating system (1) with $A = I$:

$$x(k+1) = (I + u(k) B) x(k). \tag{2}$$

System (2) is uncontrollable if the system dimension $n \geq 3$ (Tie and Cai [2012] and Tie and Lin [2013b]). Nevertheless, it was shown in Tie et al. [2010] that if B has only real eigenvalues that are nonzero and pairwise distinct, then the system (2) has a large controllable region nearly covering the whole space, i.e. the system is nearly controllable. Near-controllability is thus introduced to describe those systems that are uncontrollable but own a very large controllable region. If we only use "uncontrollable" to describe a system that is not controllable according to the general controllability definition, we may miss some valuable properties of it. Near-controllability was first defined and was demonstrated on system (2) in Tie et al. [2010], and it was later investigated for another type of system (1):

$$x(k+1) = (A + u(k) I) x(k) \tag{3}$$

in Tie and Cai [2011], where a necessary and sufficient condition for near-controllability of system (3) was given provided that A has only real eigenvalues.

However, Tie et al. [2010] and Tie and Cai [2011] only gave the criteria for determining near-controllability, while the problem of computing control inputs to achieve state transition has not been discussed. Recently, a root loucs approach is proposed in Tie [2014b] to study near-controllability of system (2), where not only an improved result is obtained, but also the control inputs that achieve the transition of any given pair of states are computed. The similar idea was also used in Tie [2013a] and Tie [2014a] to prove near-controllability and obtain the computable control inputs. The difference of the use of the root loucs technique between Tie [2014b] and Tie [2013b], Tie [2014a] is that, most of the poles of the corresponding closed loop transfer function in Tie [2014b] are double poles, while the corresponding closed loop transfer functions in Tie [2013a], Tie [2014a] only contain single poles or at most one pair of double poles, so that the root loci first move on the real axis no matter what the zeros are and the Implicit Function Theorem is not needed in Tie [2013a], Tie [2014a]. In addition, nearly-controllable subspaces are derived in Tie [2014b] and Tie [2013a], respectively, for systems (2) and (3).

In this paper, by applying the similar root locus technique, a useful algorithm is proposed to compute the required control inputs that achieve the transition of a given pair of states for the nearly controllable system (3). In order to formulate the algorithm, a new sufficiency proof of near-controllability of the system (3) is presented. Accordingly, for such class of bilinear systems, the problems of near-controllability and computability of control inputs for near-controllability are both solved. Finally, an example is provided to show the effectiveness of the proposed algorithm.

[1] A controllable region is a region in $\mathbb{R}^n$ on which the system is controllable, namely, for any ξ, η in this region, there exist control inputs that steer the system from ξ to η.

2. NEAR-CONTROLLABILITY WITH AN ALGORITHM

We first introduce the near-controllability definition.

Definition 1. *A continuous-time system $\dot{x}(t) = f(x(t), u(t))$ (discrete-time system $x(k+1) = f(x(k), u(k))$) defined on $\mathbb{R}^n$ is said to be nearly controllable if, for any $\xi \in \mathbb{R}^n \setminus \mathcal{E}$ and any $\eta \in \mathbb{R}^n \setminus \mathcal{F}$, there exist piecewise continuous control $u(t)$ and $\mathrm{T} > 0$ (a finite control sequence $u(k)$, $k = 0, 1, \ldots, l-1$, where l is a positive integer) such that ξ can be transferred to η at some $t \in (0, \mathrm{T})$ ($k = l$), where $\mathcal{E}$ and $\mathcal{F}$ are two sets of Lebesgue measure zero in $\mathbb{R}^n$.*

In this section, we give the algorithm for computing the required control inputs to achieve the state transition of the nearly controllable system (3). We first present a new sufficiency proof of near-controllability by applying the root locus theory and root locus technique. To this end, the following lemma is needed.

Lemma 1. *Let*

$$C_1 = \begin{bmatrix} \lambda_1^{2m} & \lambda_1^{2m-1} & \cdots & \lambda_1 \\ 2m\lambda_1^{2m-1} & (2m-1)\lambda_1^{2m-2} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_m^{2m} & \lambda_m^{2m-1} & \cdots & \lambda_m \\ 2m\lambda_m^{2m-1} & (2m-1)\lambda_m^{2m-2} & \cdots & 1 \end{bmatrix},$$

$$d_1 = \begin{bmatrix} -\lambda_1^{2m+1} \\ -(2m+1)\lambda_1^{2m} \\ \vdots \\ -\lambda_m^{2m+1} \\ -(2m+1)\lambda_m^{2m} \end{bmatrix}, \; d_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix},$$

where $\lambda_1, \ldots, \lambda_m$ are nonzero real and pairwise distinct. Then, C_1 is nonsingular and

$$C_1^{-1}\left(d_1 - (-1)^{2m+1} a \prod_{i=1}^{m} \lambda_i^2 d_2\right) = \begin{bmatrix} (-1)\left(2\sum_{i=1}^{m}\lambda_i + a\right) \\ \vdots \\ (-1)^{2m}\left(\prod_{i=1}^{m}\lambda_i^2 + a\sum_{i=1}^{m}\frac{\prod_{j=1}^{m}\lambda_j^2}{\lambda_i}\right) \end{bmatrix}, \tag{4}$$

where a is a nonzero real number and is distinct with λ_i for $i = 1, \ldots, m$.

Proof. Let $C_2 = \begin{bmatrix} a^{2m} & a^{2m-1} & \cdots & a & 1 \end{bmatrix}$. Consider the linear equation

$$Cz = \begin{bmatrix} d_1 \\ -a^{2m+1} \end{bmatrix}, \tag{5}$$

where

$$C = \begin{bmatrix} C_1 & d_2 \\ & C_2 \end{bmatrix}$$

and $z \in \mathbb{R}^{2m+1}$. Since the transpose of C is a general Vandermonde matrix and $\lambda_1, \ldots, \lambda_m, a$ are nonzero and pairwise distinct, C is nonsingular, and C_1 is as well for the similar reason. Therefore, linear equation (5) is solvable and has a unique solution. Assume that $z = \left[\, z_1 \ \cdots \ z_{2m+1} \,\right]^T$ is the solution. From (5), we have

$$\lambda_i^{2m+1} + z_1 \lambda_i^{2m} + \cdots + z_{2m} \lambda_i + z_{2m+1} = 0,$$
$$(2m+1)\,\lambda_i^{2m} + z_1 2m \lambda_i^{2m-1} + \cdots + z_{2m} = 0$$

for $i = 1, \ldots, m$ and

$$a^{2m+1} + z_1 a^{2m} + \cdots + z_{2m} a + z_{2m+1} = 0.$$

Thus, $\lambda_1, \ldots, \lambda_m, a$ are all the roots of the following $(2m+1)$th-degree equation

$$s^{2m+1} + z_1 s^{2m} + \cdots + z_{2m} s + z_{2m+1} = 0,$$

where $\lambda_1, \ldots, \lambda_m$ are double roots. According to the Viète's formulas,

$$z_1 = (-1) \left(2 \sum_{i=1}^{m} \lambda_i + a \right),$$
$$\vdots$$
$$z_{2m} = (-1)^{2m} \left(\prod_{i=1}^{m} \lambda_i^2 + a \sum_{i=1}^{m} \frac{\prod_{j=1}^{m} \lambda_j^2}{\lambda_i} \right),$$
$$z_{2m+1} = (-1)^{2m+1} a \prod_{i=1}^{m} \lambda_i^2.$$

As a result, by eq. (5) and noting $z_{2m+1} = (-1)^{2m+1} a \prod_{i=1}^{m} \lambda_i^2$, we have

$$C_1 \left[\, z_1 \ \cdots \ z_{2m} \,\right]^T + z_{2m+1} d_2 = d_1$$
$$\Rightarrow \left[\, z_1 \ \cdots \ z_{2m} \,\right]^T = C_1^{-1} \left(d_1 - (-1)^{2m+1} a \prod_{i=1}^{m} \lambda_i^2 d_2 \right).$$

Eq. (4) is proved. ■

Recall the necessary and sufficient condition for near-controllability obtained in Tie and Cai [2011].

Theorem 1. *The system (3), where A has only real eigenvalues, is nearly controllable if and only if A is cyclic* [2] *and the dimension of the largest Jordan block in the Jordan canonical form of A is no greater than two.*

A New Sufficiency Proof of Near-controllability of the System (3) with Computable Control Inputs. For the system (3), we write

$$A = \begin{bmatrix} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_r & 1 & & & \\ & & & & \lambda_r & & & \\ & & & & & \lambda_{r+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_m \end{bmatrix} \tag{6}$$

as in Tie and Cai [2011] without lose of generality, where $\lambda_1, \ldots, \lambda_m$ are real and pairwise distinct and $m + r = n$. Moreover, it is assumed that $\lambda_1, \ldots, \lambda_m$ are nonzero since A can be replaced by $(A - bI)$ with b unequal to λ_i for $i = 1, \ldots, m$ in view of the structure of system (3). In the following, we continue from the second step in the proof of Theorem 1 in Tie and Cai [2011]. It has been shown that (quoted from Tie and Cai [2011]), for any ξ in

$$\mathbb{R}^n \setminus \left\{ \xi \,\middle|\, \left| \xi \ A\xi \ \cdots \ A^{n-2}\xi \ A^{n-1}\xi \right| = 0 \right\}, \tag{7}$$

there exists an attainable open neighborhood of itself, i.e.

$$(A + u(2m) I) \cdots (A + u(n) I)\,(A + t_{n-1}(y) I) \cdots (A + t_0(y) I)\,\xi = \xi + y, \tag{8}$$

where y belongs to an open neighborhood of the origin in $\mathbb{R}^n$, $t_0(y), \ldots, t_{n-1}(y) \in \mathbb{R}$ are continuous functions with respect to y, and $u(n), \ldots, u(2m) \in \mathbb{R}$. Furthermore,

$$\left\{ \xi \,\middle|\, \left| \xi \ A\xi \ \cdots \ A^{n-2}\xi \ A^{n-1}\xi \right| = 0 \right\} = \left\{ \xi = \left[\, \xi_1 \ \cdots \ \xi_n \,\right]^T \mid \xi_2 \xi_4 \cdots \xi_{2r} \xi_{2r+1} \cdots \xi_n = 0 \right\}$$

that separates $\mathbb{R}^n$ into 2^m open orthants. [3] Therefore, ξ can be transferred to any state that is close enough to ξ. Next, we prove that the system (3) is controllable on each of the 2^m open orthants. For any two states ξ, η in one orthant, establish the *transition matrix*

$$\Phi_{\xi \to \eta} = \begin{bmatrix} \frac{\eta_2}{\xi_2} & \alpha_1 & & & & & & \\ & \frac{\eta_2}{\xi_2} & & & & & & \\ & & \ddots & & & & & \\ & & & \frac{\eta_{2r}}{\xi_{2r}} & \alpha_r & & & \\ & & & & \frac{\eta_{2r}}{\xi_{2r}} & & & \\ & & & & & \frac{\eta_{2r+1}}{\xi_{2r+1}} & & \\ & & & & & & \ddots & \\ & & & & & & & \frac{\eta_n}{\xi_n} \end{bmatrix} \tag{9}$$

where

$$\alpha_1 = \frac{\eta_1}{\xi_2} - \frac{\xi_1 \eta_2}{\xi_2^2}, \ \ldots, \ \alpha_r = \frac{\eta_{2r-1}}{\xi_{2r}} - \frac{\xi_{2r-1}\eta_{2r}}{\xi_{2r}^2}.$$

We can see that $\eta = \Phi_{\xi \to \eta} \xi$ and all eigenvalues of $\Phi_{\xi \to \eta}$ are positive since ξ, η belong to the same orthant. Then,

[2] A matrix is said to be cyclic if its characteristic polynomial is equal to its minimal polynomial, namely only one Jordan block exists for each eigenvalue in the Jordan canonical form of the matrix.

[3] The orthants are

$$C_m^0 : \{\xi \,|\xi_2 > 0, \xi_4 > 0, \ldots, \xi_{2r} > 0, \xi_{2r+1} > 0, \ldots, \xi_n > 0\};$$
$$C_m^1 : \begin{cases} \{\xi \,|\xi_2 < 0, \xi_4 > 0, \ldots, \xi_{2r} > 0, \xi_{2r+1} > 0, \ldots, \xi_n > 0\}, \\ \{\xi \,|\xi_2 > 0, \xi_4 < 0, \ldots, \xi_{2r} > 0, \xi_{2r+1} > 0, \ldots, \xi_n > 0\}, \\ \qquad \ldots, \\ \{\xi \,|\xi_2 > 0, \xi_4 > 0, \ldots, \xi_{2r} > 0, \xi_{2r+1} > 0, \ldots, \xi_n < 0\}; \end{cases}$$
$$C_m^2 : \{\ldots; \quad \ldots; \ C_m^{m-2} : \{\ldots;$$
$$C_m^{m-1} : \begin{cases} \{\xi \,|\xi_2 < 0, \ldots, \xi_{2r} < 0, \xi_{2r+1} < 0, \ldots, \xi_{n+1} < 0, \xi_n > 0\}, \\ \{\xi \,|\xi_2 < 0, \ldots, \xi_{2r} < 0, \xi_{2r+1} < 0, \ldots, \xi_{n+1} > 0, \xi_n < 0\}, \\ \qquad \ldots, \\ \{\xi \,|\xi_2 > 0, \xi_4 < 0, \ldots, \xi_{2r} < 0, \xi_{2r+1} < 0, \ldots, \xi_n < 0\}; \end{cases}$$
$$C_m^m : \{\xi \,|\xi_2 < 0, \xi_4 < 0, \ldots, \xi_{2r} < 0, \xi_{2r+1} < 0, \ldots, \xi_n < 0\}.$$

$$\lim_{q\to+\infty}\Phi_{\xi\to\eta}^{\frac{1}{q}} =$$

$$\begin{bmatrix} \left(\frac{\eta_2}{\xi_2}\right)^{\frac{1}{q}} & \beta_1 & & & & & & \\ & \left(\frac{\eta_2}{\xi_2}\right)^{\frac{1}{q}} & & & & & & \\ & & \ddots & & & & & \\ & & & \left(\frac{\eta_{2r}}{\xi_{2r}}\right)^{\frac{1}{q}} & \beta_r & & & \\ & & & & \left(\frac{\eta_{2r}}{\xi_{2r}}\right)^{\frac{1}{q}} & & & \\ & & & & & \left(\frac{\eta_{2r+1}}{\xi_{2r+1}}\right)^{\frac{1}{q}} & & \\ & & & & & & \ddots & \\ & & & & & & & \left(\frac{\eta_n}{\xi_n}\right)^{\frac{1}{q}} \end{bmatrix} = I$$

where

$$\beta_1 = \frac{\frac{\eta_1}{\xi_2}-\frac{\xi_1\eta_2}{\xi_2^2}}{q\left(\frac{\eta_2}{\xi_2}\right)^{\frac{q-1}{q}}},\ \ldots,\ \beta_r = \frac{\frac{\eta_{2r-1}}{\xi_{2r}}-\frac{\xi_{2r-1}\eta_{2r}}{\xi_{2r}^2}}{q\left(\frac{\eta_{2r}}{\xi_{2r}}\right)^{\frac{q-1}{q}}}.$$

As a result, we can always choose a positive integer q such that $\Phi_{\xi\to\eta}^{\frac{1}{q}}\xi$ is sufficiently close to ξ, and hence it can be reached from ξ according to eq. (8). That is, there exist $(2m+1)$ control inputs $u_0, u_1, \ldots, u_{2m-1}, u_{2m}$ such that

$$(A+u_{2m}I)(A+u_{2m-1}I)\cdots(A+u_0I)\,\xi = \Phi_{\xi\to\eta}^{\frac{1}{q}}\xi. \quad (10)$$

We now show that $u_0, u_1, \ldots, u_{2m-1}, u_{2m}$ are on the root loci of the characteristic equation of a closed loop transfer function that is related to A's eigenvalues. Let

$$\Pi \triangleq (A+u_{2m}I)(A+u_{2m-1}I)\cdots(A+u_1I)(A+u_0I).$$

$\Phi_{\xi\to\eta}^{\frac{1}{q}}$ and A are of the same structure, and $\Phi_{\xi\to\eta}^{\frac{1}{q}}$ and Π are as well. Denote by $\Phi_{\xi\to\eta}^{\frac{1}{q}}(i,j)$ and $\Pi(i,j)$ the (i,j)th entries of $\Phi_{\xi\to\eta}^{\frac{1}{q}}$ and Π, respectively. From eq. (10)

$$\Pi(2i-1,2i-1)\,\xi_{2i-1} + \Pi(2i-1,2i)\,\xi_{2i} = \Phi_{\xi\to\eta}^{\frac{1}{q}}(2i-1,2i-1)\,\xi_{2i-1} + \Phi_{\xi\to\eta}^{\frac{1}{q}}(2i-1,2i)\,\xi_{2i},$$

$$\Pi(2i,2i)\,\xi_{2i} = \Phi_{\xi\to\eta}^{\frac{1}{q}}(2i,2i)\,\xi_{2i}$$

for $i=1,\ldots,r$ and

$$\Pi(i,i)\,\xi_i = \Phi_{\xi\to\eta}^{\frac{1}{q}}(i,i)\,\xi_i$$

for $i=2r+1,\ldots,n$. Since $\xi_2,\xi_4,\ldots,\xi_{2r},\xi_{2r+1},\ldots,\xi_n$ are nonzero, we deduce

$$\Pi(2i-1,2i-1) = \Pi(2i,2i) = \Phi_{\xi\to\eta}^{\frac{1}{q}}(2i,2i) = \Phi_{\xi\to\eta}^{\frac{1}{q}}(2i-1,2i-1),$$

$$\Pi(2i-1,2i) = \Phi_{\xi\to\eta}^{\frac{1}{q}}(2i-1,2i)$$

for $i=1,\ldots,r$ and

$$\Pi(i,i) = \Phi_{\xi\to\eta}^{\frac{1}{q}}(i,i)$$

for $i=2r+1,\ldots,n$. Therefore, we have $\Pi = \Phi_{\xi\to\eta}^{\frac{1}{q}}$, i.e.

$$(A+u_{2m}I)(A+u_{2m-1}I)\cdots(A+u_1I)(A+u_0I) = \Phi_{\xi\to\eta}^{\frac{1}{q}},$$

which is equivalent to

$$A^{2m+1} + \sum_{k=0}^{2m}u_kA^{2m} + \cdots + \sum u_{k_0}\cdots u_{k_{2m-1}}A + \prod_{k=0}^{2m}u_kI = \Phi_{\xi\to\eta}^{\frac{1}{q}}. \quad (11)$$

Writing (11) as two groups of equations yields

$$\begin{cases} \lambda_1^{2m+1} + \sum_{k=0}^{2m}u_k\lambda_1^{2m} + \cdots + \sum u_{k_0}\cdots u_{k_{2m-1}}\lambda_1 + \prod_{k=0}^{2m}u_k = \Phi_{\xi\to\eta}^{\frac{1}{q}}(2,2) \\ (2m+1)\,\lambda_1^{2m} + 2m\sum_{k=0}^{2m}u_k\lambda_1^{2m-1} + \cdots + \sum u_{k_0}\cdots u_{k_{2m-1}} = \Phi_{\xi\to\eta}^{\frac{1}{q}}(1,2) \\ \vdots \\ \lambda_r^{2m+1} + \sum_{k=0}^{2m}u_k\lambda_r^{2m} + \cdots + \sum u_{k_0}\cdots u_{k_{2m-1}}\lambda_r + \prod_{k=0}^{2m}u_k = \Phi_{\xi\to\eta}^{\frac{1}{q}}(2r,2r) \\ (2m+1)\,\lambda_r^{2m} + 2m\sum_{k=0}^{2m}u_k\lambda_r^{2m-1} + \cdots + \sum u_{k_0}\cdots u_{k_{2m-1}} = \Phi_{\xi\to\eta}^{\frac{1}{q}}(2r-1,2r) \end{cases}$$

$$\begin{cases} \lambda_{r+1}^{2m+1} + \sum_{k=0}^{2m}u_k\lambda_{r+1}^{2m} + \cdots + \sum u_{k_0}\cdots u_{k_{2m-1}}\lambda_{r+1} + \prod_{k=0}^{2m}u_k = \Phi_{\xi\to\eta}^{\frac{1}{q}}(2r+1,2r+1) \\ \vdots \\ \lambda_m^{2m+1} + \sum_{k=0}^{2m}u_k\lambda_m^{2m} + \cdots + \sum u_{k_0}\cdots u_{k_{2m-1}}\lambda_m + \prod_{k=0}^{2m}u_k = \Phi_{\xi\to\eta}^{\frac{1}{q}}(n,n) \end{cases}, \quad (12)$$

in which the first group contains $2r$ equations and the second one contains $(m-r)$ equations. Adding the following $(m-r)$ constraints

$$(2m+1)\,\lambda_{r+1}^{2m} + 2m\sum_{k=0}^{2m}u_k\lambda_{r+1}^{2m-1} + \cdots + \sum u_{k_0}\cdots u_{k_{2m-1}} = 0,$$

$$\vdots$$

$$(2m+1)\,\lambda_m^{2m} + 2m\sum_{k=0}^{2m}u_k\lambda_m^{2m-1} + \cdots + \sum u_{k_0}\cdots u_{k_{2m-1}} = 0$$

to the second group of equations in (12) and putting the above $2r+(m-r)+(m-r)=2m$ equations into matrix form yields

$$\begin{bmatrix} \lambda_1^{2m} & \cdots & \lambda_1 \\ 2m\lambda_1^{2m-1} & \cdots & 1 \\ \vdots & \vdots & \vdots \\ \lambda_m^{2m} & \cdots & \lambda_m \\ 2m\lambda_m^{2m-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{2m} u_k \\ \vdots \\ \sum u_{k_0}\cdots u_{k_{2m-1}} \end{bmatrix} + \left(\prod_{k=0}^{2m} u_k - 1\right) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda_1^{2m+1} \\ -(2m+1)\lambda_1^{2m} \\ \vdots \\ -\lambda_m^{2m+1} \\ -(2m+1)\lambda_m^{2m} \end{bmatrix} + \begin{bmatrix} \Phi_{\xi\to\eta}^{\frac{1}{q}}(2,2)-1 \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(1,2) \\ \vdots \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(2r,2r)-1 \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(2r-1,2r) \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(2r+1,2r+1)-1 \\ 0 \\ \vdots \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(n,n)-1 \\ 0 \end{bmatrix}.$$

Let

$$\prod_{k=0}^{2m} u_k = (-1)^{2m+1} a \prod_{i=1}^{m} \lambda_i^2 + 1 \tag{13}$$

where a is a positive number greater than λ_i for $i = 1,\ldots,m$. It follows

$$\begin{bmatrix} \sum_{k=0}^{2m} u_k \\ \vdots \\ \sum u_{k_0}\cdots u_{k_{2m-1}} \end{bmatrix} = \begin{bmatrix} (-1)\left(2\sum_{i=1}^{m}\lambda_i + a\right) \\ \vdots \\ (-1)^{2m}\left(\prod_{i=1}^{m}\lambda_i^2 + a\sum_{i=1}^{m}\frac{\prod_{j=1}^{m}\lambda_j^2}{\lambda_i}\right) \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{2m} \end{bmatrix} \tag{14}$$

by using Lemma 1, where

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{2m} \end{bmatrix} = \begin{bmatrix} \lambda_1^{2m} & \cdots & \lambda_1 \\ 2m\lambda_1^{2m-1} & \cdots & 1 \\ \vdots & \vdots & \vdots \\ \lambda_m^{2m} & \cdots & \lambda_m \\ 2m\lambda_m^{2m-1} & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{\xi\to\eta}^{\frac{1}{q}}(2,2)-1 \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(1,2) \\ \vdots \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(2r,2r)-1 \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(2r-1,2r) \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(2r+1,2r+1)-1 \\ 0 \\ \vdots \\ \Phi_{\xi\to\eta}^{\frac{1}{q}}(n,n)-1 \\ 0 \end{bmatrix}. \tag{15}$$

Then, by noting (13) and (14) and using the Viète's formulas, it can be seen that $u_0, u_1, \ldots, u_{2m-1}, u_{2m}$ are the roots of the following $(2m+1)$th-degree equation

$$s^{2m+1} + \left(2\sum_{i=1}^{m}\lambda_i + a - \mu_1\right)s^{2m} + \cdots + \left(\prod_{i=1}^{m}\lambda_i^2 + a\sum_{i=1}^{m}\frac{\prod_{j=1}^{m}\lambda_j^2}{\lambda_i} + \mu_{2m}\right)s + a\prod_{i=1}^{m}\lambda_i^2 - 1 = 0,$$

which can be rewritten as

$$s^{2m+1} + \left(2\sum_{i=1}^{m}\lambda_i - \mu_1\right)s^{2m} + \cdots + \left(\prod_{i=1}^{m}\lambda_i^2 + \mu_{2m}\right)s - 1 + a(s+\lambda_1)^2\cdots(s+\lambda_m)^2 = 0.$$

Thus, $u_0, u_1, \ldots, u_{2m-1}, u_{2m}$ are on the root loci of the characteristic equation of the closed loop transfer function

$$1 + KG(s) \triangleq 1 + \frac{K(s+\lambda_1)^2\cdots(s+\lambda_m)^2}{s^{2m+1} + \left(2\sum_{i=1}^{m}\lambda_i - \mu_1\right)s^{2m} + \cdots + \left(\prod_{i=1}^{m}\lambda_i^2 + \mu_{2m}\right)s - 1}. \tag{16}$$

If K is chosen sufficiently large, then the characteristic equation $1 + KG(s) = 0$ has all roots real.[4] Therefore, $u_0, u_1, \ldots, u_{2m-1}, u_{2m}$ can be computed by choosing a large enough K. Applying q groups of $u_0, u_1, \ldots, u_{2m-1}, u_{2m}$ yields

$$\begin{aligned} &[(A+u_{2m}I)(A+u_{2m-1}I)\cdots(A+u_1I)(A+u_0I)]^q\xi \\ &= [(A+u_{2m}I)\cdots(A+u_1I)(A+u_0I)]^{q-1}\Phi_{\xi\to\eta}^{\frac{1}{q}}\xi \\ &= \Phi_{\xi\to\eta}^{\frac{1}{q}}[(A+u_{2m}I)\cdots(A+u_1I)(A+u_0I)]^{q-1}\xi \\ &= \cdots = \left(\Phi_{\xi\to\eta}^{\frac{1}{q}}\right)^q\xi = \Phi_{\xi\to\eta}\xi = \eta. \end{aligned}$$

That is, controllability on each of the 2^m open orthants has been proved. The rest proof is the same as that given in Tie and Cai [2011]. ■

By using the new sufficiency proof of near-controllability, an algorithm is given to compute the required control inputs that steer the nearly controllable system (3) from one state to another, which both belong to (7).

Algorithm 1. *Steps on computing the required control inputs for initial state ξ and terminal state η:*

- *Transform A into the Jordan canonical form as that given in (6) by a nonsingular matrix P and choose a real number b distinct with every eigenvalue of A. ξ, η are thus transformed into $P\xi, P\eta$, respectively;*

[4] Note that $\left\|\Phi_{\xi\to\eta}^{\frac{1}{q}} - I\right\|$ is sufficiently small if q is sufficiently large. If $\xi = \eta$, then $\Phi_{\xi\to\eta}^{\frac{1}{q}} = I$ and $\mu_1 = 0, \ldots, \mu_m = 0$. We have

$$1 + KG(s) = 1 + \frac{K(s+\lambda_1)^2\cdots(s+\lambda_m)^2}{s(s+\lambda_1)^2\cdots(s+\lambda_m)^2 - 1} = 0,$$

which has been shown in Tie and Cai [2011] to have all the root loci finally lie on the real axis approaching the $2m$ zeros and $-\infty$.

- *Find the control inputs that transfer $P\xi$ to a state ζ that belongs to the same orthant as $P\eta$ belongs to (Lemma 2 in Appendix A is useful in this step);*
- *Get the transition matrix $\Phi_{\zeta\to P\eta}$ for $\zeta, P\eta$ from (9);*
- *Choose a positive integer q and compute $\Phi_{\zeta\to P\eta}^{\frac{1}{q}}$;*
- *Obtain the root loci of $1+KG(s)=0$ given in (16), where $\lambda_1,\ldots,\lambda_m$ are the eigenvalues of $(A-bI)$ and K increases from 0 to $+\infty$. If when K approaches $+\infty$, any of the root loci does not end at a zero of $1+KG(s)=0$ through the real axis, then return to the former step and choose another integer q greater than the previous one. Otherwise, choose a K large enough such that the roots of $1+KG(s)=0$ are all real. Then, the real roots are the control inputs that transfer ζ to $\Phi_{\zeta\to P\eta}^{\frac{1}{q}}\zeta$. q groups of such control inputs, together with the ones that transfer $P\xi$ to ζ, are the required control inputs steering the nearly controllable system (3) from ξ to η.*

An example will be provided in Section 3 to demonstrate the effectiveness of Algorithm 1.

3. EXAMPLE

Example 1. *Consider the system*

$$x(k+1)=(A+u(k)I)x(k)$$
$$=\left(\begin{bmatrix}-2&0&0&0&0\\0&-2&-3&0&-1\\1&0&1&0&1\\-1&0&-2&-1&-1\\3&0&0&0&1\end{bmatrix}+u(k)I\right)x(k),\quad(17)$$

where $x(k)\in\mathbb{R}^5$ and $u(k)\in\mathbb{R}$. Given $\xi=[\,1\ 0\ 0\ 1\ 0\,]^T$, $\eta=[\,-120\ -50\ 20\ -120\ 150\,]^T$. Find the control inputs such that ξ is transferred to η.

We apply Algorithm 1 to compute the required control inputs. **Step 1**: let $\bar{x}(k)=Px(k)$, where

$$P=\begin{bmatrix}0&0&1&0&0\\1&0&0&0&1\\0&1&1&0&0\\1&0&0&0&0\\0&0&1&1&0\end{bmatrix}.$$

It follows

$$\bar{x}(k+1)=\left(PAP^{-1}+u(k)I\right)\bar{x}(k)$$
$$=\left(\begin{bmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&-2&1&0\\0&0&0&-2&0\\0&0&0&0&-1\end{bmatrix}+u(k)I\right)\bar{x}(k)\quad(18)$$

and $P\xi=[\,0\ 1\ 0\ 1\ 1\,]^T$, $P\eta=[\,20\ 30\ -30\ -120\ -100\,]^T$. Since PAP^{-1} is cyclic and the dimension of its largest Jordan block is no greater than two, system (18) is nearly controllable according to Theorem 1, and system (17) is as well. Furthermore, A is nonsingular, so that it can be let $b=0$. **Step 2**: $P\xi, P\eta$ do not belong to the same orthants (the signs of $P\xi$'s latter two entries are respectively distinct with the signs of $P\eta$'s latter two entries). Let $u(0)=0$, then

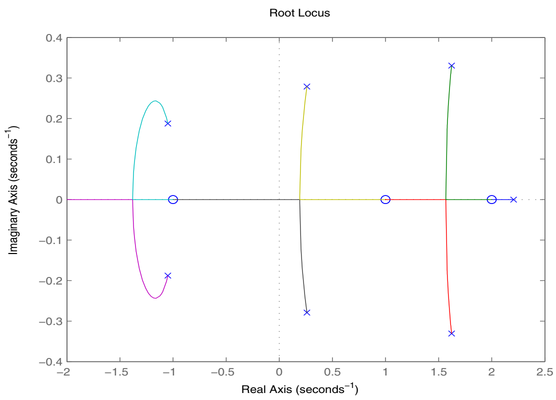

Fig. 1. Root loci of $1+KG(s)=0$.

$$\left(PAP^{-1}+u(0)I\right)P\xi=[\,1\ 1\ 1\ -2\ -1\,]^T\triangleq\zeta$$

that belongs to the same orthant as $P\eta$ belongs to. **Step 3**: obtain from (9) the transition matrix

$$\Phi_{\zeta\to P\eta}=\begin{bmatrix}30&-10&0&0&0\\0&30&0&0&0\\0&0&60&45&0\\0&0&0&60&0\\0&0&0&0&100\end{bmatrix}.$$

Step 4: choose $q=10$.

$$\Phi_{\zeta\to P\eta}^{\frac{1}{10}}=\begin{bmatrix}30^{\frac{1}{10}}&\dfrac{-10}{10\times30^{\frac{9}{10}}}&0&0&0\\0&30^{\frac{1}{10}}&0&0&0\\0&0&60^{\frac{1}{10}}&\dfrac{45}{10\times60^{\frac{9}{10}}}&0\\0&0&0&60^{\frac{1}{10}}&0\\0&0&0&0&100^{\frac{1}{10}}\end{bmatrix}.$$

Step 5: from (15),

$$\mu_1\approx-0.126597,\ \mu_2\approx-0.510202,\ \mu_3\approx-0.253520,$$
$$\mu_4\approx1.053639,\ \mu_5\approx0.875121,\ \mu_6\approx-0.633326.$$

Consider

$$1+KG(s)=1+\frac{K(s+1)^2(s-2)^2(s-1)^2}{s^{2m+1}+(-4-\mu_1)s^{2m}+\cdots+(4+\mu_{2m})s-1}.$$

By Matlab, the root loci of $1+KG(s)=0$ are shown in Fig. 1, in which "$\times$" and "$\circ$" respectively denote the poles and zeros of $G(s)$ and the colored curves are the root loci of $1+KG(s)=0$ starting at the poles and ending at the zeros. From Fig. 1, the root loci of $1+KG(s)=0$ finally lie on the real axis approaching the zeros. Choosing $K=500$ yields the roots of $1+KG(s)=0$:

$$u_0\approx-500.126605,\ u_1\approx-1.008777,$$
$$u_2\approx-0.991090,\ u_3\approx0.972255,\ u_4\approx1.028536,$$
$$u_5\approx1.981266,\ u_6\approx2.017817.$$

One can now verify that, by using 10 groups of the above control inputs with $u(0)=0$, system (17) is steered from ξ to η.

4. CONCLUSIONS

In this paper, a useful algorithm is proposed to compute the control inputs to achieve the state transition for a class of nearly controllable discrete-time bilinear systems. Accordingly, for such systems, near-controllability and the computability of control inputs for near-controllability are both shown. An example is also provided to demonstrate the effectiveness of the proposed algorithm. Future work should consider the near-controllability and controllability problems of more general discrete-time bilinear systems.

REFERENCES

F. Albertini and E.D. Sontag. Further results on controllability properties of discrete-time nonlinear systems. *Dynamics and Control*, 4:235–253, 1994.

C. Bruni, G.D. Pillo, and G. Koch. Bilinear systems: an appealing class of "nearly linear" systems in theory and applications. *IEEE Transactions on Automatic Control*, 19:334-348, 1974.

R.C. Dorf and R.H. Bishop. *Modern Control Systems.* Englewood Cliffs, Prentice-Hall, NJ, 2000.

D.L. Elliott. *Bilinear Control Systems: Matrices in Action.* Springer, Dordrecht, 2009.

M.E. Evans and D.N.P. Murthy. Controllability of a class of discrete time bilinear systems. *IEEE Transactions on Automatic Control*, 22:78-83, 1977.

M. Fliess. Fonctionnelles causales non linéaires et indéterminées non commutatives. *Bulletin de la Société Mathématique de France*, 109:3-40, 1981.

M. Fliess and D. Normand-Cyrot. A group-theoretic approach to discrete-time nonlinear controllability. In *the 20th IEEE Conference on Decision and Control*, pages 551-557. San Diego, 1981.

T. Goka, T.J. Tarn, and J. Zaborszky. On the controllability of a class of discrete bilinear systems, *Automatica*, 9:615-622, 1973.

R. Hermann and A.J. Krener. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22:728-740, 1977.

A. Isidori. *Nonlinear Control Systems*. Springer-Verlag, New York, 3rd edition, 1995.

B. Jakubczyk and E.D. Sontag. Controllability of nonlinear discrete-time systems: a Lie-algebraic approach. *SIAM Journal on Control and Optimization*, 28:1-33, 1990.

V. Jurdjevic. *Geometric Control Theory.* Cambridge University Press, New York, 1997.

R.E. Kalman, Y.C. Ho, and K.S. Narendra. Controllability of linear dynamical systems. *Contributions to Differential Equations*, 1:189-213, 1963.

J. Klamka. *Controllability of Dynamical Systems.* Kluwer Academic Publishers, Dordrecht, 1991.

R.R. Mohler and W.J. Kolodziej. An overview of bilinear system theory and applications. *IEEE Transactions on System, Man, and Cybernetics*, 10:683-688, 1980.

R.R. Mohler. *Nonlinear Systems: Vol. II, Applications to Bilinear Control.* Englewood Cliffs, Prentice-Hall, NJ, 1991.

H.J. Sussmann and V. Jurdjevic. Controllability of nonlinear systems. *Journal of Differential Equations*, 12: 95-116, 1972.

E.D. Sontag. *Mathematical Control Theory: Deterministic Finite-dimensional Systems.* Springer-Verlag, New York, 2nd edition, 1998.

T.J. Tarn, D.L. Elliott, and T. Goka. Controllability of discrete bilinear systems with bounded control. *IEEE Transactions on Automatic Control*, 18:298-301, 1973.

L. Tie. A root locus approach to near-controllability of a class of discrete-time bilinear systems with applications to Hermitian matrices. *Journal of the Franklin Institute*, 351:1810-1821, 2014a.

L. Tie. On near-controllability, nearly-controllable subspaces, and near-controllability index of a class of discrete-time bilinear systems: a root locus approach. *SIAM Journal on Control and Optimization*, in press, 2014b.

L. Tie. On nearly-controllable subspaces of a class of discrete-time bilinear systems. *Journal of Systems Science & Complexity*, 26:512-526, 2013a.

L. Tie. On necessary conditions and sufficient conditions for controllability of discrete-time bilinear systems. *IET Control Theory & Applications*, 7:143-150, 2013b.

L. Tie and K.-Y. Cai. On controllability of a class of discrete-time homogeneous bilinear systems with solvable controls. *International Journal of Robust and Nonlinear Control*, 22:591-603, 2012.

L. Tie and K.-Y. Cai. On near-controllability and stabilizability of a class of discrete-time bilinear systems. *Systems & Control Letters*, 60:650-657, 2011.

L. Tie, K.-Y. Cai, and Y. Lin. On uncontrollable discrete-time bilinear systems which are "nearly" controllable. *IEEE Transactions on Automatic Control*, 55:2853-2858, 2010.

L. Tie and Y. Lin. On controllability of two-dimensional discrete-time bilinear systems. *International Journal of Systems Science*, in press. Available on line: http://www.tandfonline.com/doi/abs/10.1080/00207721.2013.834999

L. Tie and Y. Lin. On near-controllability of a class of three-dimensional discrete-time bilinear systems. In *the 9th Asian Control Conference*, pages 1-6. Istanbul, 2013a.

L. Tie and Y. Lin. Remarks on controllability of discrete-time bilinear systems. In *the 25th Chinese Control and Decision Conference*, pages 1952-1955. Guiyang, 2013b.

F. Wirth. Dynamics and controllability of nonlinear discrete-time control systems. In *the Fourth IFAC Nonlinear Control Systems Design Symposium*, pages 269–275. Enschede, 1998.

W.M. Wonham. *Linear Multivariable Control: A Geometric Approach.* Springer-Verlag, New York, 3rd edition, 1985.

Appendix A

Lemma 2 (Tie et al. [2010]). *Let the set of m-dimensional sign vectors be denoted by $SV(m)$ (which contains 2^m elements). Then, all of the elements of $SV(m)$ can be produced from the following m sign pattern vectors*

$$\begin{bmatrix} - \\ + \\ \vdots \\ + \\ + \end{bmatrix} \begin{bmatrix} + \\ - \\ \vdots \\ + \\ + \end{bmatrix} \cdots \begin{bmatrix} + \\ + \\ \vdots \\ - \\ + \end{bmatrix} \begin{bmatrix} + \\ + \\ \vdots \\ + \\ - \end{bmatrix}$$

via the Hadamard product "$\circ$", where $+ \circ + = +$, $+ \circ - = - \circ + = -$, and $- \circ - = +$.