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Quadratic Equality Constrained Tracking Algorithm Using TDOA measurements

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Abstract: Moving target tracking using time difference of arrival measurements has received significant attention in recent years. When the target trajectory satisfies quadratic equality constraints, second-order nonlinear equations can be incorporated in the tracking algorithm to improve the accuracy. Existing methods dealing with constraints may suffer from a lack of convergence or large computation, and are often affected by the initial value of the iteration. The proposed algorithm first utilizes standard Kalman filter to update state estimation and then refines it with a maximum likelihood estimator. When solving the constrained maximum likelihood problem, a kind of generalized trust region sub-problem is incorporated to obtain the global optimal solution. Computer simulation results show that the proposed algorithm outperforms the existing methods in tracking accuracy and do not diverge when the initial state is unknown.

1. INTRODUCTION

Moving target tracking has received significant attention in the signal processing literature including teleconferencing, wireless communications, surveillance, and wireless sensor networks. The target trajectory is estimated by measuring the signal parameters and the integration. One common technique is based on measuring the time difference of arrival (TDOA) measurements between sensor receivers. This paper studies the target tracking algorithm using the Kalman filtering (KF), which is recursive and suitable for computer implementation. It contains two parts of prediction and updating. The prediction part firstly utilizes the process equation to obtain the predicted values of the current time. The updating part utilizes the measurement equation to fix the predicted estimation. When dealing with linear problems, KF can give the minimum mean square error (MMSE) estimation.

In practical tracking problems, we can usually get some priori information about the target motion. For example, when the vehicle is traveling in a known way, the travel trajectory will be constrained by the road shape. In addition, when tracking a moving target in the Earth surface by satellites, we can approximate that the target is at the sphere with the geocentric as the center and a known radius. How to use priori constraints in the KF to improve tracking accuracy is concerned in this paper.

For linear equality constraints, the common methods are model reduction, perfect measurements, probability density function truncation, and projection method. The projection methods include estimator projection, gain projection and system projection. Estimator projection projects the unconstrained KF estimates to the constrained surface. Gain projection projects the Kalman gain in order to satisfy the constraints. System projection constrains the processing noise and then modifies the KF equations. It has been proved that for linear dynamic systems with linear equality constraints, estimator projection can give the MMSE estimation.

For nonlinear equality constraints, first-order Taylor series can be used to approximate the nonlinear constraints as linear constraints, which is similar to the idea of extended Kalman filter (EKF). This method has the disadvantage that the estimation accuracy reduces greatly when the constraints include strong nonlinearity. To reduce the impact of linearization, smoothly constraint Kalman filter (SCKF) treats the nonlinear equality constraints as measurement equations, and applies iterated Kalman filter (IKF) to improve the performance. Moving horizon estimation (MHE) realizes state estimation by solving nonlinear constrained optimization problems with large amount of calculation and the global convergence is not guaranteed. Teixeira combined nonlinear equality constraints and unscented Kalman filter (UKF) and proposed three algorithms, as projected UKF (PUKF), equality-constrained UKF (ECUKF), measurement-augmentation UKF (MAUKF), which can be applied in the nonlinear state estimation with constraints.

In different forms of nonlinear constraints, the quadratic constraint is the most common one. Yang and Blasch proposed a quadratic equality constrained KF algorithm, which uses the basic idea that projects unconstrained KF results onto the quadratic constrained surface by solving a maximum likelihood estimator (MLE) with a quadratic equality constraint. This algorithm calculates the Lagrange multiplier to obtain a MLE based on iteration. However, if the initial state is selected away from the true, the algorithm is not easy to converge to the global optimal and the accuracy will be reduced.

This paper presents a quadratic constrained KF algorithm that can be applied in target tracking. It is also based on the idea of projection, and updates the state estimation by solving a MLE problem with quadratic equation constraints. To ensure the global convergence, this paper proposes to formulate it as a generalized trust region sub-problem (GTRS) and applies bisection search to solve it. The proposed algorithm has an advantage of global convergence. Simulations show that the proposed algorithm performs better than Yang's algorithm and PUKF, ECUKF, MAUKF in target tracking accuracy.

The paper is organized as follows. Section 2 presents the tracking scenario using TDOA and a brief summary on nonlinear equality constrained KF algorithm, including Yang's method and PUKF, ECUKF, MAUKF. Section 3 presents a new quadratic constrained KF algorithm. In section 4, computer simulation results are presented to illustrate the algorithm. Finally, Section 5 provides conclusions and extensions for future work.

2. PRELIMINARIES

2.1 Tracking Scenario

Consider the system model

$$\boldsymbol{\theta}_k = \mathbf{F}(\boldsymbol{\theta}_{k-1}) + \mathbf{v}_k \tag{1}$$

$$\mathbf{z}_k = \mathbf{H}(\boldsymbol{\theta}_k) + \mathbf{w}_k \tag{2}$$

where the subscript k is the time index, $\boldsymbol{\theta}_k$ is the state, $\mathbf{z}_k$ is the measurement, $\mathbf{v}_k$ and $\mathbf{w}_k$ are the zero-mean process noise and measurement noise with covariance $\mathbf{Q}_v$ and $\mathbf{Q}_w$ respectively, $\mathbf{F}(\cdot)$ and $\mathbf{H}(\cdot)$ are the transition and measurement matrices.

A scenario of M sensors is considered in a two-dimensional (2-D) space to track the target using TDOA measurements and priori information. The sensor positions $\mathbf{s}_i=[\mathrm{x}_i \quad y_i]^T$ are assumed known. Let r_{ki} be the true distance between the target and sensor i at time k:

$$r_{ki} = \|\boldsymbol{\theta}_k - \mathbf{s}_i\|_2, \quad i = 1,2,\cdots,M \tag{3}$$

where $\|\cdot\|_2$ represents the 2-norm, and

$$r_{ki1} = r_{ki} - r_{k1}, \quad i = 2,3,\cdots,M \tag{4}$$

Let c be the signal propagation speed. Then cd_{ki1} is the true range difference so that the true TDOAs and measurements are

$$d_{ki1} = \frac{r_{ki1}}{c} = \frac{r_{ki} - r_{k1}}{c}, \quad i = 2,3,\cdots,M$$

$$\mathbf{z}_k = \mathbf{d}_k^0 + \Delta\mathbf{d}, \quad E[\Delta\mathbf{d}] = \mathbf{0}, \quad E[\Delta\mathbf{d}\Delta\mathbf{d}^T] = \mathbf{Q}_w \tag{5}$$

where $\mathbf{d}_k^0 = [d_{k21}, d_{k31}, \cdots, d_{kM1}]^T$, $\{\}^0$ is noise-free quantity of $\{\}$ and $\Delta\{\}$ is its random component.

As the measurement equations are nonlinear, UKF can be used to estimate the state. Below we will give common state estimation algorithm when the state vectors satisfy nonlinear equality constraints.

2.2 Existing Algorithm

The basic idea of PUKF firstly obtains the current state $\hat{\boldsymbol{\theta}}_k$ by unconstrained UKF algorithm, and then uses the equality constraints $\mathbf{g}(\boldsymbol{\theta}_k) = \mathbf{b}$ as measurement equation of the state vector $\boldsymbol{\theta}_k$. UKF filtering algorithm is used again to get the filter output.

The difference between ECUKF and PUKF is: ECUKF calculate the covariance matrix of $\hat{\boldsymbol{\theta}}_k^b$ by the following formula after PUKF outputs the estimation.

$$\mathbf{P}_k^b = \mathbf{P}_k - \mathbf{K}_k^b \mathbf{P}_k^{bb} (\mathbf{K}_k^b)^T \tag{6}$$

Then it chooses $\hat{\boldsymbol{\theta}}_k^b$ and $\mathbf{P}_k^b$ as the input at the next time, hoping to improve filtering convergence speed.

MAUKF uses the equality constraints $\mathbf{g}(\boldsymbol{\theta}_k) = \mathbf{b}$ to enhance the measurement equation (1-b):

$$\tilde{\mathbf{z}}_k = \tilde{\mathbf{H}}(\boldsymbol{\theta}_k) + \tilde{\mathbf{w}}_k \tag{7}$$

where $\tilde{\mathbf{z}}_k = \begin{bmatrix} \mathbf{z}_k \\ \mathbf{b} \end{bmatrix}$, $\tilde{\mathbf{H}}(\boldsymbol{\theta}_\mathbf{k}) = \begin{bmatrix} \mathbf{H}(\theta_k) \\ \mathbf{g}(\theta_k) \end{bmatrix}$, $\tilde{\mathbf{w}}_k = \begin{bmatrix} \mathbf{w}_k \\ \mathbf{0} \end{bmatrix}$, and the covariance matrix of $\tilde{\mathbf{w}}_k$ is $\tilde{\mathbf{Q}}_w = \begin{bmatrix} \mathbf{Q}_w & \mathbf{0}_{m\times s} \\ \mathbf{0}_{s\times m} & \mathbf{0}_{s\times s} \end{bmatrix}$. Replace the standard UKF with (7) will get measurement-augmentation unscented Kalman filter algorithm.

Yang's method assumes that quadratic equality constraint $\mathbf{g}(\boldsymbol{\theta}_k) = \mathbf{b}$ is a function $\boldsymbol{\theta}_k^T \mathbf{M} \boldsymbol{\theta}_k + \mathbf{m}^T \boldsymbol{\theta}_k + \boldsymbol{\theta}_k^T \mathbf{m} - b = 0$ of $\boldsymbol{\theta}_k$.

It solves the following constrained MLE problem by projecting the unconstrained KF state estimate onto the constrained surface and outputs

$$\begin{aligned} \boldsymbol{\theta}_k^b &= \arg\min_{\boldsymbol{\theta}_k} (\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_k)^T \boldsymbol{\Omega}_k (\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_k) \\ s.t.\ & \boldsymbol{\theta}_k^T \mathbf{M} \boldsymbol{\theta}_k + \mathbf{m}^T \boldsymbol{\theta}_k + \boldsymbol{\theta}_k^T \mathbf{m} - b = 0 \end{aligned} \tag{8}$$

where $\boldsymbol{\Omega}_k = \mathbf{P}_k^{-1}$, $\mathbf{M}$ and $\mathbf{m}$ are known symmetric matrix and column vector.

Using the Cholesky decomposition and $\boldsymbol{\Omega}$ will be represented as $\boldsymbol{\Omega} = \mathbf{B}^T \mathbf{B}$, where $\mathbf{B}$ is a reversible upper triangular matrix. Then the constrained solution is the output of the following Lagrange function

$$\mathbf{J}(\boldsymbol{\theta}, \lambda) = (\boldsymbol{\zeta} - \mathbf{B}\boldsymbol{\theta})^T (\boldsymbol{\zeta} - \mathbf{B}\boldsymbol{\theta}) + \lambda \left[\boldsymbol{\theta}^T \mathbf{M} \boldsymbol{\theta} + \mathbf{m}^T \boldsymbol{\theta} + \boldsymbol{\theta}^T \mathbf{m} - b \right] \tag{9}$$

An iterative method may be used to solve the non-convex problem. So iterations may not converge or converge to a local solution, which means the iterative results are not the global optimal solution of (9). To avoid the above problems, initial iterative value requires being chosen close to the optimal values. However, a good initial choice is often difficult to be obtained in practical applications.

3. NEW ALGORITHM

3.1 Tracking Part

A new algorithm with quadratic equality constraints for MLE problems is proposed in this section. The main idea of the new algorithm is that (8) can be treated as a kind of generalized trust region sub-problem (GTRS) in order to guarantee obtaining the global optimal solution.

Using the Cholesky decomposition and setting $\boldsymbol{\zeta} = \mathbf{B}\hat{\boldsymbol{\theta}}$, we can get a MLE problem with quadratic constraints completely equivalent with (9). Then we can get

$$\min_{\boldsymbol{\theta}\in R^n}\{\|\boldsymbol{\zeta} - \mathbf{B}\boldsymbol{\theta}\|^2 : \boldsymbol{\theta}^T\mathbf{M}\boldsymbol{\theta} + \mathbf{m}^T\boldsymbol{\theta} + \boldsymbol{\theta}^T\mathbf{m} - b = 0\} \tag{10}$$

The optimization problem above can be treated as a GTRS problem, the optimal solution needs to meet the following three necessary and sufficient conditions

$$(\mathbf{B}^T\mathbf{B} + \lambda\mathbf{M})\boldsymbol{\theta} = \mathbf{B}^T\boldsymbol{\zeta} - \lambda\mathbf{m} \tag{11-1}$$

$$\boldsymbol{\theta}^T\mathbf{M}\boldsymbol{\theta} + \mathbf{m}^T\boldsymbol{\theta} + \boldsymbol{\theta}^T\mathbf{m} - b = 0 \tag{11-2}$$

$$\mathbf{B}^T\mathbf{B} + \lambda\mathbf{M} \succeq \mathbf{0} \tag{11-3}$$

Where $\mathbf{A} \succeq \mathbf{0}$ means that matrix $\mathbf{A}$ must be positive semi-definite. Notice that (11-1), (11-2) are necessary conditions of the Karush-Kuhn-Tacker (KKT) Conditions, and (11-3) is a necessary and sufficient condition of GTRS problems. In order to get the output from (11), we firstly solve the Lagrange multiplier λ, which is the solution of the following equation

$$\varphi(\lambda) = \boldsymbol{\theta}^T(\lambda)\mathbf{M}\boldsymbol{\theta}(\lambda) + \mathbf{m}^T\boldsymbol{\theta}(\lambda) + \boldsymbol{\theta}^T(\lambda)\mathbf{m} - b = 0,\ \lambda \in I \tag{12}$$

The interval I consists of all λ for which $\mathbf{B}^T\mathbf{B} + \lambda\mathbf{M}$ is positive semi-definite, which immediately implies that

$$I = \left(-1/\lambda_1(\mathbf{M}, \mathbf{B}^T\mathbf{B}), +\infty\right) \tag{13}$$

where $\lambda_1(\mathbf{X},\mathbf{Y}) = \lambda_1(\mathbf{Y}^{-1/2}\mathbf{X}\mathbf{Y}^{-1/2})$ represents the biggest eigenvalue of $\mathbf{Y}^{-1/2}\mathbf{X}\mathbf{Y}^{-1/2}$.

It has been proved that $\varphi(\lambda)$ is strictly decreasing over I and therefore a simple kind of bisection algorithm can be used to find the optimal λ over the interval I. Putting the root of (12) into (11-1) will lead to the global optimal solution of MLE with quadratic equality constraints.

We summarize the steps of solving (10) based on bisection algorithm as follows:

Step 1. Calculate $\lambda_1(\mathbf{M}, \mathbf{B}^T\mathbf{B})$ and set $a = -1/\lambda_1(\mathbf{M}, \mathbf{B}^T\mathbf{B})$, $b = 1/\lambda_1(\mathbf{M}, \mathbf{B}^T\mathbf{B})$.

Step 2. Calculate upper and lower bound of the interval I.

Step 3. Use bisection algorithm to solve (12) between the interval I.

Step 4. Bring λ into $\boldsymbol{\theta}(\lambda) = (\mathbf{B}^T\mathbf{B} + \lambda\mathbf{M})^{-1}(\mathbf{B}^T\boldsymbol{\zeta} - \lambda\mathbf{m})$ and output a new optimal solution of estimator.

It must be noted that the solving process based on bisection method is iterative. But the fundamental difference between the proposed algorithm and Yang's method is that the new algorithm for solving the Lagrange multiplier is globally optimal and unrelated to initial choice. This is because the introduction of necessary and sufficient condition (13-3) makes the solution unique.

3.2 Initialization Part

As we know, the initial values of the state and covariance matrix are important for Kalman filtering. In Yang's paper, the initial state is selected to be the same as the true state, which cannot be guaranteed in practical applications. So a new algorithm with quadratic constraints for initialization of tracking is proposed in this section.

This is an over-determined situation in which the number of equations is larger than the number of unknowns. In this case, introducing Lagrange multipliers λ_1 and λ_2, the augmented cost function is

$$\begin{aligned}\xi \equiv (\mathbf{h} - \mathbf{G}\boldsymbol{\theta} - \mathbf{g}r_1)^T\mathbf{W}(\mathbf{h} - \mathbf{G}\boldsymbol{\theta} - \mathbf{g}r_1) + \lambda_1(2\mathbf{s}_1^T\boldsymbol{\theta} - \mathbf{s}_1^T\mathbf{s}_1 - r^2 + r_1^2) \\ + \lambda_2(\boldsymbol{\theta}^T\mathbf{M}\boldsymbol{\theta} + \mathbf{m}^T\boldsymbol{\theta} + \boldsymbol{\theta}^T\mathbf{m} - b)\end{aligned} \tag{14}$$

where $\mathbf{h} = \begin{bmatrix} r_{2,1}^2 - \mathbf{s}_2^T\mathbf{s}_2 + \mathbf{s}_1^T\mathbf{s}_1 \\ r_{3,1}^2 - \mathbf{s}_3^T\mathbf{s}_3 + \mathbf{s}_1^T\mathbf{s}_1 \end{bmatrix}$, $\mathbf{G} = -2\begin{bmatrix} \mathbf{s}_2^T - \mathbf{s}_1^T \\ \mathbf{s}_3^T - \mathbf{s}_1^T \end{bmatrix}$, $\mathbf{g} = -2\begin{bmatrix} r_{2,1} \\ r_{3,1} \end{bmatrix}$

Newton's method is effective for finding the root of λ_2. But this method is too complicated to calculate. If we make $\mathbf{C}^T\mathbf{C}$=chol($\mathbf{W}$) and then we can get

$$\begin{aligned}\min_{\mathbf{u}\in R^n}\{\|-\mathbf{C}\mathbf{G}\boldsymbol{\theta} - \mathbf{C}(\mathbf{g}r_1 - \mathbf{h})\|^2 : \\ \boldsymbol{\theta}^T\mathbf{M}\boldsymbol{\theta} + \mathbf{m}^T\boldsymbol{\theta} + \boldsymbol{\theta}^T\mathbf{m} - b = 0\}\end{aligned} \tag{15}$$

The optimization can also be seen as a GTRS problem, which means it can also be solved by the algorithm proposed in section 3.1 as we set $\xi_2 = -\mathbf{C}(\mathbf{g}_1 r_1 - \mathbf{h})$ and $\mathbf{B}_2 = \mathbf{C}\mathbf{G}_1$.

It should be noted that this method requires an initial value of r_1, the accuracy of the initialization will be greatly affected by the accuracy of r_1.

4. SIMULATION

This section describes the effectiveness of the new algorithm under different simulations. The simulation scene is set similar with Yang's paper. The target is moving with uniform circular motion in two-dimensional x-y plane. Trajectory center is at the origin and radius is 100 meters. The target maintains a constant turn rate of 5.7 deg/s with an equivalent linear speed of 10 m/s. Set state vector $\boldsymbol{\theta}_k = \begin{bmatrix} x_k & v_{x,k} & y_k & v_{y,k} \end{bmatrix}^T$, where x_k and $v_{x,k}$ respectively denote the coordinate and velocity in x-axis of the current time, y_k and $v_{y,k}$ respectively denote the coordinate and velocity in y-axis. The number of sensors is 3 and the sensors location are $\mathbf{s}_1 = [50,0]^T$, $\mathbf{s}_2 = [-50,-50]^T$, $\mathbf{s}_3 = [0,50]^T$. The vehicle is tracked by sensors with a sampling interval of T=1s. The discrete-time second-order kinematic model is

$$\boldsymbol{\theta}_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{\theta}_{k-1} + \begin{bmatrix} \frac{1}{2}T^2 & 0 \\ T & 0 \\ 0 & \frac{1}{2}T^2 \\ 0 & T \end{bmatrix} \mathbf{v}_{k-1} \tag{16}$$

where $\mathbf{Q}_v = \mathrm{diag}([\sigma_{\ddot{x}}^2 \quad \sigma_{\ddot{y}}^2])$ uses the particular values of $\sigma_{\ddot{x}}^2 = \sigma_{\ddot{y}}^2 = 0.32 m/s^2$. And $\mathbf{Q}_w = c^2\delta_d^2\mathbf{R}$, whose elements are $c^2\delta_d^2$ in the diagonal and $0.5c^2\delta_d^2$ otherwise, where δ_d^2 is the TDOA variance.

Under scenario 1, we set the real value of the target initial state vector $\boldsymbol{\theta}_0 = \begin{bmatrix}100\, m, 0\, m/s, 0\, m, 10\, m/s\end{bmatrix}^T$. Set the initial estimation error covariance to be $\mathbf{P}_0 = \mathrm{diag}(\begin{bmatrix}5^2 & 1^2 & 5^2 & 1^2\end{bmatrix})$.

We compare the proposed projection method between MAUKF, PUKF, ECUKF in the target tracking accuracy. The Monte Carlo simulation times are 1000. In figure 1, we plot the mean square error (MSE) of target position estimation in different methods.

As the initial estimation is selected as the true value, the MSE at the beginning tracking are nearly zero. From the results, we can see ECUKF and MAUKF outperform PUKF. That is because the use of equality constraints accelerates the convergence speed. Compared with PUKF, MAUKF and ECUKF, the proposed new algorithm can obtain the highest positioning accuracy and get a better tracking stability.

Under scenario 2, we use the initial algorithm proposed in 3.2 to get the initial state vector and set the velocity to be zero. Set the initial estimation error covariance to be $\mathbf{P}_0 = \mathrm{diag}(\begin{bmatrix}15^2 & 7^2 & 15^2 & 7^2\end{bmatrix})$.

We compare the proposed projection method with MAUKF, PUKF, ECUKF in target tracking accuracy. The Monte Carlo simulation times are 1000. In Figure 2, we plot the mean square error (MSE) of target position estimation in the four methods.

From figure 2 we can see that the proposed new algorithm can achieve the highest positioning accuracy with nonlinear measurement equations and do not diverge when the initial state is unknown.

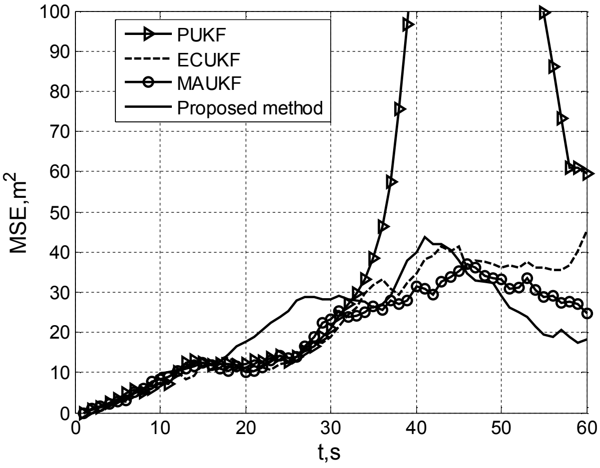

Fig. 1. Performance comparison of different constrained UKFs under scenario 1

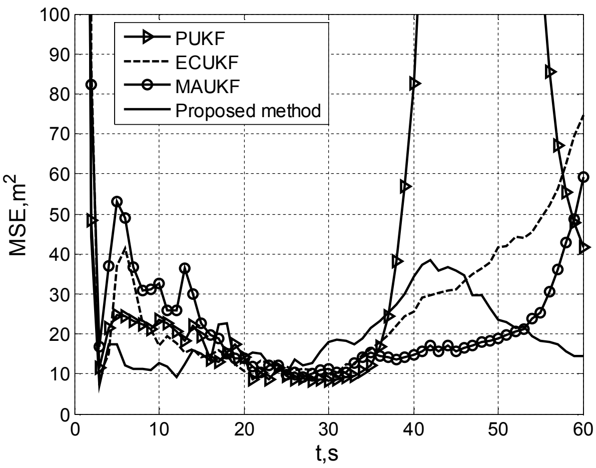

Fig. 2. Performance comparison of different constrained UKFs under scenario 2

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