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Robust tracking control for a class of perturbed and uncertain reaction-diffusion equations

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Abstract: This paper is focused on the design of robust tracking control for a class of reaction-diffusion partial differential equations. Both uncertainties in the model parameters and bounded external perturbations are considered. Global practical stabilization is addressed through a regularized sliding-mode approach; in addition, a scheduled controller is proposed in order to achieve asymptotic stability if some conditions on the reference to be tracked are met. Numerical simulations support theoretical results.

Keywords Control of partial differential equations, Infinite-dimensional sliding-mode control, Robust tracking.

1. INTRODUCTION

Reaction-diffusion equations are quasilinear second-order partial differential equations which are typically used to describe chemical reactions, pattern formation and population dynamics [Grindrod, 1996], [Jones et al., 2010]. In recent years, partial differential equations have attracted the attention of the control community (see [Fridman and Orlov, 2008], [Krstic and Smyshlyaev, 2008] among several others) since many plant models are described by infinite-dimensional systems and hence involve PDEs or systems of PDEs: examples can be found in robotics (haptic controllers and flexible manipulators), in industrial processes (manufacturing, reactors and heat transfer plants) as well as in biomedical applications (tissue engineering).
Due to the high complexity of such models, it could be necessary to handle several sources of uncertainty, this enforcing the interest in the analysis and synthesis of robust control strategies. Sliding-mode [Utkin, 1992] is a well established robust control technique having the advantage of constraining the state of the controlled system in a region which results to be invariant with respect to external disturbances. Sliding-mode controllers have also been proposed as possible solution to the problem of robust control for PDEs [Sira-Ramirez, 1989], [Orlov, 2009], [Pisano et al., 2011]. In particular in [Sira-Ramirez, 1989] the problem of distributed control for quasilinear first-order parabolic equations is addressed and a variable-structure control policy is proposed, while in [Pisano et al., 2011] the authors focus on the design of sliding-mode controllers for robust tracking in the case of unidimensional heat equation and wave equation.
In the framework of reaction-diffusion equations, both boundary control [Barthel et al., 2010] and distributed control [Kishida and Braatz, 2010] have been investigated. This paper proposes the extension of some results from [Pisano et al., 2011] to the case of uncertain and perturbed reaction-diffusion equations. In particular the problem of robustly tracking a reference profile is considered for equations incorporating both parameter uncertainties and external perturbations. The considered class of equations is characterized by a nonhomogeneous term with a time-varying and possibly uncertain coefficient, this corresponding to diffusion rates with a non-constant behavior. Robust global practical stability of the tracking error system is proved via a regularized sliding-mode controller. In addition, in the case of a reference profile with a decay behavior, a scheduled controller is proved to ensure global asymptotic stability of the system. The main advantage of the proposed approach is that, thanks to the regularization of the sliding-mode, no discontinuity appears in the control variable, this allowing to avoid the introduction of approximated solutions. On the other hand this procedure leads in general to practical stabilization only; however, by suitably tuning the controller parameters, the reference profile can be robustly tracked with the desired accuracy.
The paper is structured as follows. In Section 2 the model setup is presented, while Sections 3 and 4 contain the main results: robust tracking control in the presence of model uncertainties and bounded external perturbations, respectively. Numerical tests supporting theoretical developments are reported in Section 5.

2. MODEL AND SETUP

Adopting the setting introduced in [Pisano et al., 2011], let us consider the following reaction-diffusion partial differential equation:

$$h_t(x,t) = \lambda h_{xx}(x,t) + f(t, h(x,t)) + u(x,t) + b(x,t)$$
$$(x,t) \in [0,1] \times [0,\infty)$$

with Dirichlet boundary conditions

$$h(0,t) = \omega_0(t), \quad h(1,t) = \omega_1(t)$$

and initial condition

$$h(x,0) = h_0(x).$$

The parameter $\lambda > 0$ is assumed to be a known positive real constant and the function $f(t,\xi)$, representing the diffusion term, is supposed to be in the following class:

$$f(t,\xi) = (f_0(t) + \delta(t))\xi,$$

where $f(t) \in \mathcal{L}^\infty(0,\infty)$ and $\delta(\cdot)$ is an uncertain parameter. We only assume to know a bound $\delta_0 > 0$ on its size:

$$\sup_{t\in(0,\infty)} |\delta(t)| \le \delta_0.$$

The distributed external source $b(x,t)$ is an unmeasured perturbation term. The description of the class of admissible perturbations is given in Section 4.

For the reader convenience, we recall the following classical definitions of functional spaces:

$$\mathcal{L}^2(a,b) := \left\{ g(\cdot) : ||g||_2 := \int_a^b |g(x)|^2 dx < \infty \right\},$$

$$\mathcal{L}^\infty(a,b) : \left\{ g(\cdot) : ||g||_\infty := \text{ess}\sup_{(a,b)} |g(x)| < \infty \right\},$$

$$W^{k,2}(a,b) := \left\{ g(\cdot) : ||g^{(j)}||_2 < \infty \; \forall j = 0,1,...,k \right\},$$

where $g^{(j)}(x)$ denotes the (weak) j^{th}-derivative of the function $g(x)$.

Let us consider a reference function

$$h^\star(x,t) : [0,1] \times [0,\infty) \to \mathbb{R}$$

with

$$h^\star(\cdot,t) \in W^{2,2}(0,1) \; \forall t \in [0,\infty),$$

$$h^\star(x,\cdot) \in W^{1,2}(0,\infty) \; \forall x \in [0,1]$$

and such that the boundary conditions are consisent:

$$h^\star(0,t) = \omega_0(t), \quad h^\star(1,t) = \omega_1(t).$$

The basic task is to design the control input $u(x,t)$ in order to track the reference $h_\star(x,t)$, i.e. such that the following asymptotic condition is verified

$$\lim_{t\to\infty} \int_0^1 \|h(x,t) - h^\star(x,t)\|^2 dx = 0$$

Let us introduce the tracking error $q(x,t)$:

$$q(x,t) = h(x,t) - h^\star(x,t).$$

The error dynamics is assigned by

$$\begin{aligned} q_t(x,t) &= \lambda q_{xx}(x,t) + (f_0(t) + \delta(t))(q(x,t) + h^\star(x,t)) \\ &\quad -h_t^\star(x,t) + \lambda h_{xx}^\star(x,t) + u(x,t) + b(x,t). \end{aligned}$$

We point out that, by construction, the function $q(x,t)$ verifies Dirichlet boundary conditions

$$q(0,t) = q(1,t) = 0 \; \forall t \in [0,\infty).$$

Prior to introduce the main sections of the paper, we recall the following classical result for parabolic partial differential equations (see [Evans, 1998], Thm 2.3.2 and Thm 2.3.5):

Theorem 1. Let us fix $T > 0$ and set $\Omega_T := (0,1) \times (0,T)$. Assume that the function $g(x,t) : \Omega_T \to \mathbb{R}$ verifies

$$g(x,t) \in C(\overline{\Omega}_T),$$

$$g(\cdot,t) \in C^2(0,1) \; \forall t \in [0,\infty),$$

$$g(x,\cdot) \in C^1(0,T) \; \forall x \in \mathbb{R}.$$

Then, for any $z_0(x) \in C^2(0,1)$, there exists at most one solution $z(x,t) \in C^2(\Omega_T)$ of the parabolic problem

$$\begin{cases} z_t(x,t) = z_{xx}(x,t) + g(x,t) & (x,t) \in \Omega_T \\ z(0,t) = z(1,t) = 0 & \forall t \in [0,T] \\ z(x,0) = z_0(x) & \forall x \in [0,1]. \end{cases}$$

3. ROBUST CONTROL: UNPERTURBED EQUATION

Let us consider the case $b(x,t) \equiv 0$ first. The error dynamics reduces to

$$\begin{aligned} q_t(x,t) &= \lambda q_{xx}(x,t) + f(t,(q(x,t) + h^\star(x,t))) \\ &\quad -h_t^\star(x,t) + \lambda h_{xx}^\star(x,t) + u(x,t). \end{aligned} \tag{1}$$

For any fixed $\epsilon > 0$, we define the continuously differentiable function $\varrho_\epsilon : [0,\infty) \to [0,1]$ as

$$\varrho_\epsilon(s) = \begin{cases} 1 & s \ge \epsilon \\ \dfrac{-16(s-\epsilon)^2}{\epsilon^2} + 1 & s \in (3\epsilon/4, \epsilon) \\ \dfrac{16(s-\epsilon/2)^2}{\epsilon^2} & s \in (\epsilon/2, 3\epsilon/4] \\ 0 & s \le \epsilon/2 \end{cases} \tag{2}$$

The proposed control strategy is then given by the following family of state-feedback laws, obtained as a parameter-dependent regularization of an infinite-dimensional sliding mode incorporating a feedforward term:

$$u_\epsilon(x,t) = u_{eq}(x,t) + \delta_0 \tilde{u}_\epsilon(x,t), \tag{3}$$

where

$$\begin{aligned} u_{eq}(x,t) &= h_t^\star(x,t) - h_{xx}^\star(x,t) - \theta \frac{\text{sign}(q(x,t))|q(x,t)|^\beta}{||q(\cdot,t)||_2} \\ &\quad -f_0(t)(q(x,t) + h^\star(x,t)) \end{aligned}$$

with $\theta > 0$, $\beta \in (1,2)$ and

$$\tilde{u}_\epsilon(x,t) = -q(x,t) - \varrho_\epsilon(||q(x,t)||_2) \frac{q(x,t)||h^\star(\cdot,t)||_2}{||q(\cdot,t)||_2}.$$

Theorem 2. The error system driven by the family of control inputs given by (3) is globally practically stable, i.e. for any $\eta > 0$ and for any initial condition $q(x,0) = q_0(x) \in \mathcal{L}^2(0,1)$ with $||q_0(\cdot)||_2 \ne 0$, there exist $\tau_\eta \ge 0$ and $\epsilon = \epsilon(\eta) > 0$ such that the corresponding control function $u_\epsilon(x,t)$ ensures

$$||q(\cdot,t)||_2 \le \eta \; \forall \; t \ge \tau_\eta.$$

Proof. Following [Pisano et al., 2011], let us consider the following simple Lyapunov functional

$$V(t) = \frac{1}{2}\int_0^1 q^2(x,t)dx.$$

The time derivative along the system (strong) solution satisfies

$$\dot{V}(t) = \int_0^1 q(x,t)q_t(x,t)dx$$

$$= \int_0^1 q(x,t)q_{xx}(x,t)dx + \int_0^1 q(x,t)f(t,q(x,t) + h^\star(x,t))dx$$

$$+ \int_0^1 q(x,t)u(x,t)dx + \int_0^1 q(x,t)(-h_t^\star(x,t) + h_{xx}^\star(x,t))dx.$$

Choosing arbitrarily $0 < \epsilon < \eta$ and using the explicit expression of the control $u(x,t)$ one gets

$$\dot{V}(t) = \int_0^1 q(x,t)q_{xx}(x,t)dx - \theta\int_0^1 \frac{|q(x,t)|^{\beta+1}}{||q(\cdot,t)||_2}dx$$

$$+(\delta(t) - \delta_0)\int_0^1 q^2(x,t)dx + \delta(t)\int_0^1 q(x,t)h^\star(x,t)dx$$

$$-\delta_0\rho_\epsilon(||q(\cdot,t)||_2)||q(\cdot,t)||_2||h^\star(\cdot,t)||_2.$$

Integrating by parts one has

$$\lambda\int_0^1 q(x,t)q_{xx}(x,t)dx = -\lambda\int_0^1 q_x^2(x,t)dx;$$

the application of Cauchy-Schwartz inequality yields

$$\delta(t)\int_0^1 q(x,t)h^\star(x,t)dx \le |\delta(t)|||q(\cdot,t)||_2||h^\star(\cdot,t)||_2$$

and hence, recalling that $|\delta(t)| - \delta_0 \le 0 \ \forall t \ge 0$, the following condition holds

$$\dot V(t) \le -\theta \int_0^1 \frac{|q(x,t)|^{\beta+1}}{||q(\cdot,t)||_2}dx \le 0 \tag{4}$$

whenever $||q(\cdot,t)||_2 \ge \epsilon$. In order to conclude, one can observe that, by a standard Lyapunov argument, for any $\zeta > \epsilon$, the solution $q(x,t)$ enters the positively invariant sets $\{z \in \mathcal{L}^2(0,1) : ||z(\cdot)||_2 < \zeta\}$ in finite time and in particular this holds true for $\zeta = \eta$. ♢

Remark 1. We point out that, for $\epsilon \to 0$, the proposed control law converges to a classical sliding-mode. In fact, the feedback control $u_0(x,t)$ incorporates the term

$$\tilde u_0(x,t) = -\frac{q(x,t)||h^\star(\cdot,t)||_2}{||q(\cdot,t)||_2}$$

that is ill-defined for $||q(x,t)|| = 0$. In particular, although one has

$$||\tilde u_0(\cdot,t)||_2 = ||h^\star(\cdot,t)||_2 \ \forall t \ge 0,$$

in general the limit

$$\lim_{t\to\infty} ||u_0(\cdot,t) - h_\star(\cdot,t)||$$

does not exist. The use of such control technique requires to deal with approximate solutions (see for instance), while the proposed strategy (3) enables to consider classical strong solutions only and moreover, by suitably tuning the parameter ϵ, it guarantees to reach the target reference $h^\star(x,t)$ with any desired accuracy.

An explicit quantitative estimation of the time-step τ_η in Theorem 2 can be easily obtained by the comparison principle, as it is illustrated in the next statement.

Proposition 2. Given $\eta > 0$, an upper bound $\tau_{\eta,\max} \ge \tau_\eta$ can be found; $\tau_{\eta,\max}$ only depends on the initial conditions and on the controller parameters β, θ and it is given by

$$\tau_{\eta,\max} = \frac{||q(\cdot,0)||_2^{2-\beta} - \eta^{2-\beta}}{2\theta(1-\beta/2)}.$$

Proof. As shown in the proof of Theorem 2, choosing arbitrarily $\epsilon < \eta$, one gets $\dot V(q(x,t)) \le 0$ for any $q(x,t)$ with $||q(\cdot,t)||_2 \ge \epsilon$. In particular from (4) one has

$$\dot V(t) \le -\theta \int_0^1 \frac{|q(x,t)|^{\beta+1}}{||q(\cdot,t)||_2}dx$$

and applying Hölder inequality one gets

$$\dot V(t) \le -\theta\left(\int_0^1 q^2(x,t)dx\right)^{\beta/2} = -\theta 2^{\beta/2}V(t)^{\beta/2},$$

where $\beta/2 < 1$. Now, setting

$$W(0) = V(0), \quad \dot W(t) = -2^{\beta/2}\theta W(t)^{\beta/2},$$

one has $V(t) \le W(t)$ and by integration

$$W(t) = (V(0)^{1-\beta/2} - 2^{\beta/2}\theta(1-\beta/2)t)^{2/(2-\beta)}.$$

Recalling that, by definition, $2V(t) = ||q(\cdot,t)||_2^2$, the bound $\tau_{\eta,\max}$ can be computed imposing $\sqrt{2W(t)} = \eta$, which gives

$$\eta = 2\left(\frac{||q(\cdot,0)||_2^{2-\beta}}{2^{1-\beta/2}} - 2^{\beta/2}\theta(1-\beta/2)t\right)^{1/(2-\beta)} \tag{5}$$

and therefore

$$t = \frac{||q(\cdot,0)||_2^{2-\beta} - \eta^{2-\beta}}{2\theta(1-\beta/2)} =: \tau_{\eta,\max}.$$

By construction, for any $t \le \tau_{\eta,\max}$, one has

$$\begin{aligned}||q(\cdot,t)||_2 &= \sqrt{2V(t)} \le \sqrt{2V(\tau_{\eta,\max})}\\ &\le \sqrt{2W(\tau_{\eta,\max})} = \eta. \quad ♢\end{aligned}$$

Corollary 1. If no uncertainty is considered, i.e. if $\delta_0 = 0$, the proposed control technique guarantees finite-time global asymptotic stability [Bhat and Bernstein, 2000]. In particular the stabilization time is given by

$$\tau_0 = \frac{||q(\cdot,0)||_2^{2-\beta}}{2\theta(1-\beta/2)}.$$

Proof. In this case the derivative of the Lyapunov functional reduces to

$$\dot V(t) = -\lambda\int_0^1 q_x^2(x,t)dx - \theta\int_0^1 \frac{|q(x,t)|^{\beta+1}}{||q(\cdot,t)||_2}dx \quad \forall t \ge 0$$

and hence condition (5) can be imposed with $\eta = 0$, this allowing to obtain the stabilization time τ_0.

3.1 A scheduled controller

Let us assume the reference profile $h^\star(x,t)$ to be characterized by a decay behavior, i.e. let us suppose that

$$\lim_{t\to\infty} ||h^\star(\cdot,t)||_2 = 0. \tag{6}$$

In this case, for any fixed $\epsilon > 0$, the control $u_\epsilon(x,t)$ guarantees global asymptotic stability. In order to prove that, we recall the following classical result.

Lemma 3. (Poincaré Inequality) Let $v(\cdot) \in W^{1,2}(0,1)$ such that $v(0) = v(1) = 0$. The following condition holds

$$||v||_2^2 = \int_0^1 v^2(x)dx \le \int_0^1 v_x^2(x)dx = ||v_x||_2^2.$$

Set the Lyapunov functional $V(t)$ as in the proof of Theorem 2. Now, differentiating $V(t)$ along the solution $q(x,t)$ yields the following estimate:

$$\dot V(t) \le -\lambda\int_0^1 q_x^2(x,t)dx + \delta_0\int_0^1 |q(x,t)h^\star(x,t)|dx. \tag{7}$$

Setting $\mu_0 = 2\delta_0/\lambda$, whenever $||q(\cdot,t)||_2 > \mu_0||h^\star(\cdot,t)||_2$ one has

$$\begin{aligned}\frac{\lambda}{2}\int_0^1 q^2(x,t)dx &\ge \frac{\lambda\mu_0}{2}||q(\cdot,t)||_2||h^\star(x,t)||_2\\ &\ge \delta_0\int_0^1 |q(x,t)h^\star(x,t)|dx,\end{aligned}$$

and hence

$$\dot V(t) \le -\lambda V(t) \tag{8}$$

if $||q(\cdot,t)||_2 \ge \mu_0||h^\star(\cdot,t)||_2$. On the other hand, due to (6), for any $k \ge 1$, there exist $t_k > 0$ such that

$$||h^\star(\cdot,t)||_2 \le \eta/2^k =: \eta_k \ \forall t \ge t_k$$

and hence, thanks to (8), there exists $t'_k > 0$ such that $||q(\cdot,t)||_2 \le \mu_0\eta_k \ \forall t > t'_k$, this proving asymptotic stability. We notice that, since no assumption on the monotonicity of $||h^\star(\cdot,t)||_2$ has been considered, this is allowed to have arbitrary variations and, as a consequence, the time step t'_k may be very large. Moreover, estimating the time t'_k requires in general a complete knowledge on the evolution of the profile $h^\star(x,t)$. In order to avoid such problems and prevent the related negative effects, a scheduled controller $u_\epsilon(x,t)$ with $\epsilon = \epsilon(||h^\star(\cdot,t)||_2)$ can be designed such that global asymptotic stability is ensured together with the invariance of prescribed sets. Prior to define the controller we state the following simple result.

Lemma 4. For any real constant $\mu > 0$, the operator $F_\mu : \mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1) \to \mathcal{L}^2(0,1)$ given by

$$F_\mu(v_1,v_2) = \varrho_{\mu||v_2||}(||v_1||)\frac{v_1||v_2||}{||v_1||}, \quad F_\mu(0,0) = 0$$

is well-defined and it verifies

$$\lim_{(||v_1||,||v_2||)\to(0,0)} ||F_\mu(v_1,v_2)|| = 0.$$

Proof. Let us fix $\mu > 0$ and let us suppose that two sequences $\{v_1^{(k)}\}$, $\{v_2^{(k)}\} \subset \mathcal{L}^2(0,1)$ can be found such that

$$\lim_{k\to\infty} ||v_1^{(k)}|| = \lim_{k\to\infty} ||v_2^{(k)}|| = 0,$$

$$\limsup_{k\to\infty} \left\| F_\mu\left(v_1^{(k)}, v_2^{(k)}\right)\right\| = \alpha > 0.$$

We notice that for any $w_1, w_2 \in \mathcal{L}^2(0,1)$ with $||w_1|| < \mu||w_2||/2$ one has $F_\mu(w_1,w_2) = 0$; without loss of generality, up to a subsequence, we can therefore assume that $2||v_1^{(k)}|| \geq \mu||v_2^{(k)}||$ for any $k \geq 0$. Now one has

$$\left\| F_\mu\left(v_1^{(k)}, v_2^{(k)}\right)\right\| \leq ||v_2^{(k)}|| \leq 2||v_1^{(k)}||/\mu \quad \forall k \geq 0,$$

this proving that $\alpha = 0$ necessarily. $\diamondsuit$

Remark 5. It is worth to note that the operator $F_\mu(\cdot,\cdot)$ satisfies

$$F_\mu(0,v) = 0 \ \forall v \in \mathcal{L}^2(0,1).$$

In addition let us set

$$H(t) = \inf_{s\in[0,t]}\{||h^\star(\cdot,s)||_2\}$$

and

$$u_\sharp(x,t) = u_{eq}(x,t) - \delta_0(q(x,t) - \tilde{F}_\mu(q(x,t),h^\star(x,t))) \quad (9)$$

where $\tilde{F}_\mu(\cdot,\cdot)$ is obtained by the following modification of the operator $F_\mu(\cdot,\cdot)$:

$$\tilde{F}_\mu(q(x,t),h^\star(x,t)) = \varrho_{\mu H(t)}(||q(\cdot,t)||_2)\frac{q(x,t)||h^\star(\cdot,t)||_2}{||q(\cdot,t)||_2}.$$

We need the following assumption.

Assumption 1. The reference profile $h^\star(x,t)$ satisfies one of the following conditions:

(1) $||h^\star(\cdot,t)||_2 > 0 \ \forall t \in [0,\infty)$;

(2) if $||h^\star(\cdot,t_0)||_2 = 0$ then $||h^\star(\cdot,t)||_2 = 0 \ \forall t > t_0$.

Theorem 3. If the reference profile $h^\star(x,t)$ satisfies (6) and Assumption 1, the error system driven by the scheduled controller $u_\sharp(x,t)$ defined by (9) is globally asymptotically stable, i.e. for any $q(x,0) = q_0(x) \in \mathcal{L}^2(0,1)$ one has

$$\lim_{t\to\infty} ||q(\cdot,t)||_2 = 0.$$

Proof. First we note that the origin is a stationary point for the system driven by the controller $u_\sharp(x,t)$ (due to Remark 5 and Assumption 1). Set the Lyapunov functional $V(t)$ as in the proof of Theorem 2. Now, differentiating $V(t)$ along the solution $q(x,t)$ yields the following estimates:

$$\dot{V}(t) \leq -\theta 2^{\beta/2}V(t)^{\beta/2} \quad (10)$$

if $||q(\cdot,t)||_2 \geq \mu_1 H(t)$, where $\mu = \mu_1$ is a free design parameter. Let us set $\eta = ||h_\star(\cdot,0)||_2$; without loss of generality one can assume $Q_0 = ||q(\cdot,0)||_2 > \mu_1\eta$. Due to (6), for any $k \geq 1$, there exists $t_k > 0$ such that

$$H(t) \leq \eta/2^k = \eta_k \ \forall t \geq t_k.$$

Moreover, thanks to (10), for any $\bar{t} > 0$ the sets

$$E_{\bar{t}} := \{q(\cdot) \in \mathcal{L}^2(0,1) : ||q||_2 \leq \mu_1 H(\bar{t})\}$$

are invariant for the solution $q(x,t)$ for $t > \bar{t}$. On the other hand, for any $k > 0$ fixed, there exists $\tau_k > 0$ such that the solution $q(x,t)$ driven by the scheduled controller $u_\epsilon(x,t)$ with $\epsilon = \mu_1 H(t_k)$ satisfies

$$||q(\cdot,\tau_k)||_2 = \mu_0\eta_k;$$

in particular τ_k verifies

$$\tau_k \leq \tau_{k,\max} := \frac{Q_0^{2-\beta} - (\mu_1\eta_k)^{2-\beta}}{2\theta(1-\beta/2)}.$$

In conclusion, setting $\sigma_k := \tau_{k,\max} + t_k$, it has been proved that

$$q(x,\sigma_k) \in E_{t_k},$$

that implies

$$||q(\cdot,t)||_2 \leq \mu_1\eta_k \ \forall t \geq \sigma_k.$$

Since by definition η_k converges to zero as k tends to ∞, the asymptotic stability follows.

4. ROBUST CONTROL: BOUNDED PERTURBATIONS

This section is focused on the design of robust tracking control in the presence of a bounded external perturbation $b(x,t) \neq 0$. For sake of simplicity we do not take into account parameter uncertainty, i.e. $\delta_0 = 0$ throughout the section. In order to avoid confusion with already used notations, in this section the control input will be denoted by $v(x,t)$. We assume that the perturbation $b(x,t)$ verifies the following conditions.

Assumption 2. A positive number M_b can be (a priori) determined such that

$$||b(\cdot,t)||_2 \leq M_b \quad \forall t \geq 0.$$

Assumption 3. There exists a closed linear subspace $\mathcal{W} \subset \mathcal{L}^2(0,1)$ such that

$$b(\cdot,t) \in \mathcal{W} \quad \forall t \geq 0.$$

Let us denote by $P_\mathcal{W}$ and $P_\mathcal{W}^\perp$ the linear projection operators associated to the subspace $\mathcal{W}$; in particular given an arbitrary function $z(\cdot) \in \mathcal{L}^2(0,1)$, it admits a unique decomposition [Brezis, 2010]

$$z(\cdot) = P_\mathcal{W}z(\cdot) + P_\mathcal{W}^\perp z(\cdot),$$

with $P_\mathcal{W}z(\cdot) \in \mathcal{W}$ and $\int_0^1 P_\mathcal{W}^\perp z(x)p(x)dx = 0 \ \forall p(\cdot) \in \mathcal{W}$. With respect to the error dynamics

$$\begin{aligned} q_t(x,t) &= \lambda q_{xx}(x,t) + f_0(t)(q(x,t) + h^\star(x,t)) \\ &\quad - h_t^\star(x,t) + \lambda h_{xx}^\star(x,t) + v(x,t) + b(x,t), \end{aligned} \quad (11)$$

let us define the sliding-surface

$$s(q(\cdot,t)) = P_\mathcal{W}q(\cdot)$$

and, accordingly, the equivalent control

$$\begin{aligned} v_{eq}(x,t) &:= -f_0(t)(q(x,t)+h^\star(x,t)) + h^\star_t(x,t) \\ &\quad -\lambda h^\star_{xx}(x,t) + v_0(x,t), \end{aligned} \tag{12}$$

where $v_0(x,t)$ is an arbitrary function belonging to the subspace $\mathcal{W}^\perp$ $\forall t \geq 0$. A simple and helpful choice is to set

$$v_0(x,t) = -P^\perp_{\mathcal{W}} q(x,t). \tag{13}$$

We propose a control law based on the decomposition

$$v(x,t) = v_{eq}(x,t) + v_b(x,t),$$

where the term $v_b(x,t)$ is responsible to enforce the sliding-mode and it is given by the discontinuous function:

$$v_b(x,t) := -M_b \frac{P_{\mathcal{W}} q(x,t)}{||P_{\mathcal{W}} q(\cdot,t)||_2} - P_{\mathcal{W}} q(x,t). \tag{14}$$

Let us consider the function $\varrho_\epsilon(s)$ introduced in (2) and, for $\epsilon > 0$, define the family of differentiable controllers

$$v_\epsilon(x,t) = v_{eq}(x,t) + \varrho_\epsilon(||P_{\mathcal{W}} q(\cdot,t)||_2) v_b(x,t). \tag{15}$$

It will proved now that the above family of controllers ensures global robust practical stabilization of system (11).

Theorem 4. For any $\eta > 0$, there exist $\epsilon > 0, T_\eta > 0$ such that system (11) driven by the controller $v_\epsilon(x,t)$ given by (12)-(13)-(14)-(15) verifies the asymptotic boundedness condition

$$||q(\cdot,t)||_2 \leq \eta \quad \forall t \geq T_\eta. \tag{16}$$

Proof. Let us consider the usual Lyapunov functional

$$V(t) = \frac{1}{2} \int_0^1 q^2(x,t) dx.$$

Computing the derivative one gets

$$\begin{aligned} \dot V(t) &= \int_0^1 q(x,t) q_t(x,t) dx = \lambda \int q(x,t) q_{xx}(x,t) dx \\ &+ \int_0^1 q(x,t)(b(x,t) + v_0(x,t) + v_{b,\epsilon}(x,t)) dx \\ &= -\lambda \int q_x^2(x,t) dx + I_1(t) + I_2(t) \end{aligned}$$

where we have set $v_{b,\epsilon}(x,t) = \varrho_\epsilon(||P_{\mathcal{W}} q(\cdot,t)||_2) v_b(x,t)$ and

$$I_1(t) = \int_0^1 q(x,t) v_0(x,t) dx,$$

$$I_2(t) = \int_0^1 q(x,t)(b(x,t) + v_{b,\epsilon}(x,t)) dx.$$

Let us treat the two terms separately. Using the expression of $v_0(x,t)$ (13) one has

$$\begin{aligned} I_1(t) &= -\int_0^1 q(x,t) P^\perp_{\mathcal{W}} q(x,t) dx \\ &= -\int_0^1 (P^\perp_{\mathcal{W}} q(x,t))^2 dx = -||P^\perp_{\mathcal{W}} q(\cdot,t)||_2^2; \end{aligned}$$

regarding the second integral, for $||P_{\mathcal{W}} q(\cdot,t)||_2 \geq \epsilon$ one has

$$\begin{aligned} I_2(t) &= \int_0^1 q(x,t) b(x,t) dx \\ &- \left(\frac{M_b}{||P_{\mathcal{W}} q(\cdot,t)||_2} + 1 \right) \int_0^1 (P_{\mathcal{W}} q(x,t))^2 dx. \end{aligned}$$

Observing that, by Assumption 3, the identity

$$\int_0^1 q(x,t) b(x,t) dx = \int_0^1 P_{\mathcal{W}} q(x,t) b(x,t) dx$$

holds, the application of Cauchy-Schwartz inequality and the bound given in Assumption 2 yield

$$I_2(t) \leq -||P_{\mathcal{W}} q(\cdot,t)||_2^2 \quad \text{for } ||P_{\mathcal{W}} q(\cdot,t)||_2 \geq \epsilon.$$

We have shown that

$$\dot V(t) \leq -||q_x(\cdot,t)||_2^2 - ||P_{\mathcal{W}} q(\cdot,t)||_2^2 - ||P^\perp_{\mathcal{W}} q(\cdot,t)||_2^2 \leq -2V(t)$$

as long as $||P_{\mathcal{W}} q(\cdot,t)||_2 \geq \epsilon$. Now it is straightforward to verify that, due to orthogonal decomposition, the following expressions can be obtained from the previous inequality:

$$||P_{\mathcal{W}} q(\cdot,t)||_2^2 \leq e^{-2t} ||P_{\mathcal{W}} q_0(\cdot)||_2^2,$$

$$||P^\perp_{\mathcal{W}} q(\cdot,t)||_2^2 \leq e^{-2t} ||P^\perp_{\mathcal{W}} q_0(\cdot)||_2^2.$$

Set $\epsilon = \eta/\sqrt{2}$ and compute $T_{\eta,1}$, $T_{\eta,2}$ such that

$$||P_{\mathcal{W}} q(\cdot, T_{1,\eta})||_2^2 = ||P^\perp_{\mathcal{W}} q(\cdot, T_{\eta,2})||_2^2 = \frac{\eta^2}{2};$$

in particular one has

$$T_{\eta,1} \leq \log \frac{\sqrt{2}||P_{\mathcal{W}} q_0(\cdot)||_2}{\eta}, \quad T_{\eta,2} \leq \log \frac{\sqrt{2}||P^\perp_{\mathcal{W}} q_0(\cdot)||_2}{\eta}.$$

The conclusion then follows observing that for $t \geq T_\eta$

$$||q(\cdot,t)||_2^2 = ||P_{\mathcal{W}} q(\cdot,t)||_2^2 + ||P^\perp_{\mathcal{W}} q(\cdot,t)||_2^2 \leq \frac{\eta^2}{2} + \frac{\eta^2}{2} = \eta^2,$$

where $T_\eta = \max\{T_{\eta,1}, T_{\eta,2}\}$. ♢

Remark 6. It is worth to note that the family of controllers (15) guarantees indeed a stability condition which is stronger than asymptotic boundedness (16): in particular, for any $\epsilon > 0$, the system (11) driven by $v_\epsilon(x,t)$ satisfies

$$\lim_{t\to\infty} ||P^\perp_{\mathcal{W}} q(\cdot,t)||_2 = 0.$$

5. NUMERICAL TESTS

In this section we present several numerical simulations in order to illustrate different scenarios [Estep et al., 2000].

Example 1. In this first example we present the case of uncontrolled dynamics, that is $u(x,t) \equiv 0$. We assume initial condition $q(x,0) = 0$ and Dirichlet boundary conditions $q(0,t) = 0$, $q(1,t) = \sin t$. We set $\lambda = 1$ and $f_0(t) = 3 + 0.005 \cos t$. The considered reference profile is $h^\star(t) = x^2 \sin t$ and no perturbation is considered, i.e. $b(x,t) \equiv 0$. The evolution of the solution $h(x,t)$ and the evolution of the reference $h^\star(x,t)$ to be tracked are depicted in Fig. 1.

Example 2. In this second example we consider again the system above, which is now supposed to be driven by the control input $u_\epsilon(x,t)$ for $\theta = 1$, $\beta = 1.5$ and $\epsilon = 0.1$. We set $\delta_0 = 0.01$ as upper bound for the uncertain term $\delta(t)$. The behavior of the system is shown in Fig. 2: as the time t increases the solution $h(x,t)$ approaches $h^\star(x,t)$ and the estimate $||h(\cdot,t) - h^\star(\cdot,t)||_2 \leq \epsilon = 0.1$ holds definitely.

Example 3. Let us consider the same system and take the following reference profile

$$h^\star(x,t) = x^2 e^{-0.03t} \sin t;$$

the boundary condition $h(1,t)$ is updated accordingly. Since $||h^\star(\cdot,t)||_2$ tends to 0 as t tends to ∞, applying the scheduled controller $u_\sharp(x,t)$ we can ensure the robust asymptotic tracking of the reference. Fig. 3 shows the behavior of the norm $||h(\cdot,t) - h^\star(\cdot,t)||_2$ as the time t increases.

Example 4. In the last example we consider the case of an external perturbation $b(x,t) = 0.3\cos 2t \cos 2\pi x$, with bound $||b(\cdot,t)|| \leq M_b = 0.15$. The perturbation $b(x,t)$ belongs for $t \geq 0$ to the closed subspace $\mathcal{W}$ of zero-mean functions, i.e.

$$\mathcal{W} = \left\{ g(\cdot) \in \mathcal{L}^2(0,1) : \int_0^1 g(x)dx = 0 \right\}.$$

Setting $\overline{g} = \int_0^1 g(x)dx$, the linear operators $P_{\mathcal{W}}$ and $P_{\mathcal{W}}^{\perp}$ are defined as follows:

$$P_{\mathcal{W}} g(x) = g(x) - \overline{g}, \quad P_{\mathcal{W}}^{\perp} = \overline{g}.$$

Assuming $f_0(t) = 1$ and $h^\star(x,t) = 3x^2 \sin 2t$, the control law $v_\epsilon(x,t)$ has been implemented with $\epsilon = 0.1$. Results are shown in Fig. 4: the norm $||h(\cdot,t) - h^\star(\cdot,t)||_2$ is definitely bounded by $0.142 \simeq \sqrt{2}\epsilon$.

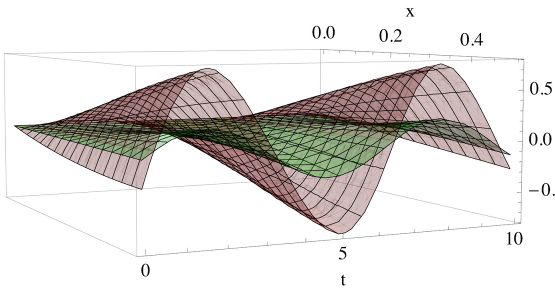

Fig. 1. Uncontrolled case: evolution of $h(x,t)$ (red) and $h^\star(x,t)$ (green)

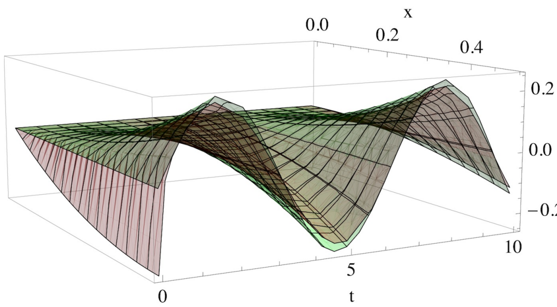

Fig. 2. Robust control $u_\epsilon(x,t)$: evolution of $h(x,t)$ (red) and $h^\star(x,t)$ (green)

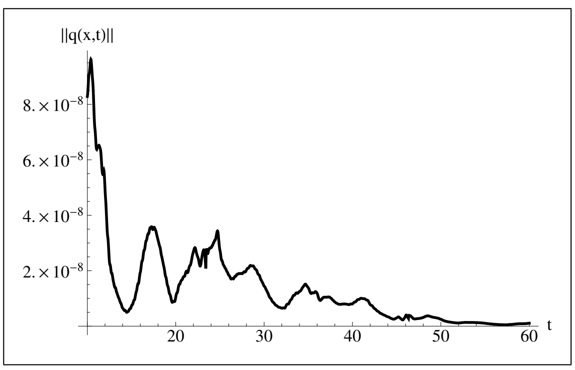

Fig. 3. Scheduled control $u_\sharp(x,t)$: evolution of $||q(\cdot,t)||_2$

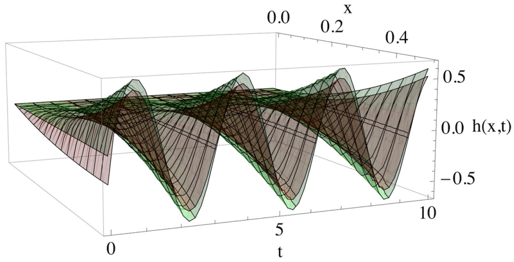

Fig. 4. Robust control $v_\epsilon(x,t)$: evolution of $h(x,t)$ (red) and $h^\star(x,t)$ (green)

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