A Decomposition Result in Linear-Quadratic Coordinated Control*

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Abstract: Scalability is a fundamental requirement in control design for large-scale systems. Typically, it needs to be considered explicitly at the expense of performance degradation and more complicated design procedures. In this paper, we present a class of large scale systems where scalability is an inherent property of the optimal centralized solution. More specifically, we study a coordination problem where a group of identical subsystems are required to satisfy an equality constraint on the sum of their inputs. We show that the problem can be completely decomposed in terms of the unconstrained problems associated with each subsystem. In particular, the computational effort required to obtain the optimal solution is independent of the number of subsystems and the only global information processing required to execute the optimal control law is a simple summation, which scales well when the number of subsystems grows large.

Keywords: Large scale systems, distributed control, coordinated control, constrained control, linear systems, optimal control.

1. INTRODUCTION

In control design for large-scale systems, scalability is of prime importance. Within the control community, considerable effort has been devoted to reducing the complexity of the control design procedure and execution of the resulting control law.

Perhaps the most common approach to control design for large scale systems is to impose a sparsity structure on the controller. However, while a sparse structure can ensure scalability of the control law by limiting the amount of information processing in the controller, it typically complicates the design procedure. This was pointed out by Witsenhausen (1968) when he showed that some useful properties in classical linear-quadratic control do not extend to problems with structural constraints. Rotkowitz and Lall (2006) identified a tractable subclass of problems by showing that convexity in the Youla-parameter is preserved under a certain condition, but even then, well-understood solutions are currently only available for a handful of instances (Lessard and Lall, 2012; Shah and Parrilo, 2013). When faced with non-convexity, standard approaches are to either search for locally optimal solutions, e.g. as in Mårtensson and Rantzer (2012), or introduce additional restrictions to make the problem convex (see Šiljak and Stipanović, 2000; Tanaka and Langbort, 2011).

Here, we present a class of large-scale systems where scalability does not need to be addressed explicitly. Instead, we show that it is an inherent property of the *centralized* solution. More specifically, we study group of physically

independent subsystems with identical dynamics that are required to coordinate their inputs. We show that when the objective is to minimize a weighted sum of the individual system costs, the coordination problem decomposes in terms of unconstrained problems where each system minimizes its own cost. This decomposition has several useful implications. First, the computational effort needed to obtain the optimal solution is independent of the number of subsystems: the solution is obtained by simply solving the associated unconstrained problem for a *single* subsystem. Second, the only global information processing required to execute the control law is a simple summation, which scales well when the number of subsystems grows large. Third, it enables us to derive closed form expressions for the optimal control law and performance in terms of the control laws and performances of the associated unconstrained problems. This provides insight into the optimal control policy and makes it possible to extend the results to problems with additional constraints on subsystem performance.

2. PROBLEM FORMULATION

In this paper we study how well a group of m *stable* LTI subsystems with identical dynamics is able to perform under a joint input constraint. The subsystems are given by

$$x_i(t+1) = Ax_i(t) + Bu_i(t) + w_i(t), \ i = 1,\ldots,m \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^p$, and the w_i:s are independent stationary zero-mean Gaussian white noise processes with covariance matrix W_i. Associated with each subsystem i is the cost function

$$J_i(u_i) = \mathbf{E}\Big(|x_i(t)|^2 + |u_i(t)|^2\Big). \quad (2)$$

Note that the subscript i in the definition of J_i is needed because in general $W_i \neq W_j$.

Let $x = [x_1^T \cdots x_m^T]^T$, and $u = [u_1^T \cdots u_m^T]^T$ be the vector of stacked states and control signals, respectively. The objective of the group is to solve the following constrained state feedback problem:

Problem 1. (Constrained problem). Given $\gamma_i > 0$, $i = 1, \ldots, m$, such that $\sum_{i=1}^m \gamma_i = 1$, find a state feedback law $u = -\bar{L}x$ that minimizes

$$\sum_{i=1}^m \gamma_i J_i(u_i)$$

and satisfies the exchange constraint

$$\sum_{i=1}^m u_i = 0. \qquad (3)$$

A larger group implies more freedom in choosing a control signal allocation $(u_1, \ldots, u_m)$ that satisfies (3). Thus, it is intuitively clear that increasing the group size leads to better performance. In Section 3 we provide closed form expressions for the performance and the optimal control signal allocation in Problem 1 in terms of the performance and solution of the following unconstrained problems:

Problem 2. (Unconstrained problem). Find a state feedback law $u_i = -L^* x_i$ that minimizes (2).

Remark 1. Since the systems in (1) have identical A and B matrices, the optimal feedback gain in Problem 2 is the same for $i = 1, \ldots, m$.

Remark 2. Note that by setting $u_m = -\sum_{i=1}^{m-1} u_i$, the constraint (3) is eliminated and Problem 1 becomes a standard state feedback problem of order mn. Our aim in this paper is to show that the problem has several useful and scalable properties.

Remark 3. If we would allow zero weights in the statement of Problem 1, say $\gamma_1 = 0$, the solution would be to solve Problem 2 for $i = 2, \ldots, m$ and set $u_1 = -\sum_{j=2}^m u_j$. The requirement $\gamma_i > 0$, $1, \ldots, m$ excludes such trivialities.

Remark 4. The requirement that the systems are stable is necessary in order to stabilize the system with a control law that satisfies (3). This can be seen by introducing $z = \sum_1^m x_i$. Then

$$z(t+1) = \sum_i^m x_i(t+1) = A\sum_1^m x_i(t) + B\sum_1^m u_i(t) = Az(t),$$

which cannot be stabilized unless A is stable.

3. SOLUTION

For notational purposes we define

$$P_i(u_i) = J_i(0) - J_i(u_i)$$

as the performance improvement obtained in system i by applying the control signal trajectory u_i. Henceforth, we use u_i^* to denote the optimal control trajectory obtained by running (1) with the optimal state feedback law in Problem 2. The resulting performance improvements are denoted $p_i^* = P_i(u_i^*)$.

We now state the main result of this paper. It states that performance and optimal control signal trajectories in the Problem 1 are completely characterized by the performance and control signal trajectories in Problem 2.

Theorem 1. Let $\bar{u}_i$ and u_i^* be the optimal control signal trajectories obtained by running (1) with the optimal state feedback laws in Problem 1 and Problem 2, respectively. Set $p_i^* = P_i(u_i^*)$. Then,

$$\bar{u}_i = u_i^* - \beta_i \sum_{j=1}^m u_j^*, \qquad (4)$$

and

$$P_i(\bar{u}_i) = p_i^* - \beta_i^2 \sum_{j=1}^m p_j^* \qquad (5)$$

where

$$\beta_i = \frac{\gamma_i^{-1}}{\sum_{j=1}^m \gamma_j^{-1}}. \qquad (6)$$

Proof. The proof is given in the Appendix.

Corollary 1. If $\gamma_1 = \ldots = \gamma_m = 1/m$, then

$$\bar{u}_i = u_i^* - \text{avg}\{u_1^*, \ldots, u_m^*\}$$

$$P_i(\bar{u}_i) = p_i^* - \frac{1}{m}\text{avg}_j\{p_1^*, \ldots, p_m^*\}$$

$$\sum_{i=1}^m P_i(\bar{u}_i) = (1 - \frac{1}{m})\sum_{i=1}^m p_i^*,$$

where $\text{avg}\{x_1, \ldots, x_m\} = \frac{1}{m}\sum_1^m x_i$.

Corollary 1 states that if the objective is to minimize the aggregated cost of the group, satisfying the constraint (3) will cost the group $1/m$ in relative performance improvement. This cost is evenly distributed among the systems. That is, each system pays a fraction, $1/m$, of the avergae uncoordinated performance improvement for satisfying the constraint. Note that in general $p_i^* \neq p_j^*$ because the noise covariance matrices are allowed to differ (i.e. $W_i \neq W_j$ if $i \neq j$). This implies that satisfying the constraint (3) might lead to a negative performance improvement compared to zero control effort at some system i (i.e. $P_i(\bar{u}_i) < 0$).

Corollary 1 also implies that all systems should spend an equal amount of control effort in order to satisfy the constraint. Aside from this effort they should behave exactly as they would without the constraint.

Note that (4) is a relation between the optimal *signal trajectories* in the constrained and unconstrained problems. It does not offer a solution to Problem 1 because it requires a priori knowledge of the control signal trajectories in Problem 2. The next result remedies this issue, by establishing a relation between the *control laws* of the two problems. In fact, the solution to Problem 1 is obtained simply by solving Problem 2. Since the system have identical dynamics, this amounts to solving the unconstrained problem for a single system in the group.

Theorem 2. Let L^* be the optimal state feedback gain in Problem 2. Then the solution to Problem 1 is given by

$$u_i(t) = -L^*\left(x_i - \beta_i \sum_{j=1}^m x_j\right), \qquad (7)$$

where $\beta_i = \frac{\gamma_i^{-1}}{\sum_{j=1}^m \gamma_j^{-1}}$.

Proof. The optimal control signals in Problem 2 can be expressed as $u_i^* = H w_i$, where H is defined by the following system:

$$H : \begin{cases} x(t+1) = (A - BL^*)x(t) + w(t) \\ u(t) = -L^*x(t) \end{cases}$$

Introduce $z = \beta_i \sum_{j=1}^{m} x_j$ and $z_i = x_i - z$. In the new coordinates, control law (7) reads $u_i = -Lz_i$ and it is easily verified that under this control law we have $z(t+1) = Az(t) + \beta_i \sum_{j=1}^{m} w_j(t)$. Using this it is straightforward to verify that

$$z_i(t+1) = (A - BL^*)z_i + w_i(t) - \beta_i \sum_{j=1}^{m} w_j(t).$$

Hence,

$$u_i = -Lz_i = H\Big(w_i - \beta_i \sum_{j=1}^{m} w_j\Big) = u_i^* - \beta_i \sum_{j} u_j^*,$$

which by Theorem 1 is the optimal control trajectory. □

The control law in (7) simple to implement: At time t each subsystem computes $\nu_i(t) = -L^*x_i(t)$ and submits it to a central entity. The central entity computes $\bar{\nu}(t) = \sum \nu_i(t)$ and sends it back to the subsystems who then apply the control signal $u_i(t) = \nu_i(t) - \beta_i \bar{\nu}(t)$. Note that the subsystems do not need to know the state of other systems, only $\bar{\nu}(t)$.

4. EXTENSIONS

The results presented so far can be used on more general problem formulations than the one in Problem 1. Two such cases are given in Problem 3 and Problem 4 below:

Problem 3. Let u_i^* be the control signal trajectory obtained by running (1) with the optimal feedback gain in Problem 2 and set $p_i^* = P_i(u_i^*)$. Given $a_i \in [0,1)$, $i = 1 \ldots, m$, find a state feedback law $u = -\bar{L}x$ that maximizes

$$\sum_{i=1}^{m} P_i(u_i) \qquad (8)$$

subject to $\sum_{i=1}^{m} u_i = 0$ and
$$P_i(u_i) \geq \alpha_i p_i^*, \; i = 1, \ldots, m. \qquad (9)$$

The quantity quantity α_i in (9) can be interpreted as a pay-off required by system i in order to contribute to the coordination effort.

Theorem 3. Suppose that there is a control law $u = -Lx$ such that $P_i(u_i) > \alpha_i p_i^*$, $i = 1, \ldots, m$. Then, the optimal control law and performance improvements in Problem 3 are given by (7) and (5), respectively, where the vector $\beta = [\beta_1 \cdots \beta_m]$ is the solution to the quadratic program:

$$\min \beta^T \beta$$

subject to $\mathbf{1}^T \beta = 1$ and

$$0 < \beta_i \leq \sqrt{\frac{p_i^*}{\sum_{j=1}^{m} p_j^*}(1 - \alpha_i)}.$$

Proof. See the Appendix.

Problem 4. Let u_i^* be the control signal trajectory obtained by running (1) with the optimal feedback gain in Problem 2 and set $p_i^* = P_i(u_i^*)$. Find a state feedback law $u = -\bar{L}x$ that maximizes

$$\min\Big\{\frac{P_i(u_1)}{p_1^*}, \ldots, \frac{P_m(u_m)}{p_m^*}\Big\} \qquad (10)$$

subject to $\sum_{i=1}^{m} u_i = 0$.

Table 1. Resulting β for the different cases in Section 5.

β_i	1	2	3	4	5	6
Case 1	1/6	1/6	1/6	1/6	1/6	1/6
Case 2	0.04	0.18	0.20	0.20	0.20	0.20
Case 3	0.06	0.08	0.12	0.17	0.24	0.33

Theorem 4. Suppose that $p_i^* > 0$, $i = 1, \ldots, m$. Then, the optimal control law and performance improvements in Problem 4 are given by (7) and (5), respectively, where the vector $\beta \in \mathbb{R}^m$ is given by

$$\beta_i = \frac{\sqrt{p_i^*}}{\sum \sqrt{p_j^*}}.$$

Moreover,

$$\frac{P_i(u_i)}{p_i^*} = \frac{P_j(u_j)}{p_j^*}, \; \forall i, j = 1, \ldots, m.$$

Proof. See the Appendix.

5. NUMERICAL EXAMPLE

Consider $m = 6$ systems given by (1) with

$$A = 0.8, \; B = 1, \; W_i = (\sqrt{2})^{i-1}, \; i = 1, \ldots, 6.$$

Each system i wishes to minimize $J_i(u_i)$ in (2). However, together they must satisfy the constraint (3). Consider the following cases:

Case 1 The systems agree on minimizing the aggregate cost:

$$\sum_{i=1}^{6} J_i(u_i).$$

Case 2: The systems agree on minimizing the aggregate cost. However, system $i = 2, \ldots, 6$ only agree to participate in the coordination effort if they are guaranteed not to be worse off than if they were not participating. System 1 requires a performance improvement of $0.9p_1^*$.

Case 3: The systems agree to distribute the relative benefits of coordination, $\frac{P_i(\bar{u}_i)}{p_i^*}$, evenly.

The solution to Case 1 is given by Corollary 1, the solution to Case 2 is given by Theorem 3 with $\alpha_1 = 0.9$, and $\alpha_2, \ldots, \alpha_6 = 0$, and the solution to Case 3 is given by Theorem 4. For each case, the corresponding β is presented in Table 1 and the relative performance improvements, $\frac{P_i(\bar{u}_i)}{p_i^*}$, are shown in Figure 1. The results show that in Case 1 systems 2–6 benefit at the expense of system 1. This is not the situation in Case 2 due to the constraint (9). In Case 3 all systems obtain a relative improvement of 0.78. Note that the identical distribution in relative improvement is a feature of Problem 4.

6. SUMMARY

We considered a stochastic linear quadratic problem formulation, where a group of identical systems coordinate their actions in order to reduce their costs. The need for coordination is due to an equality constraint on the sum of their inputs.
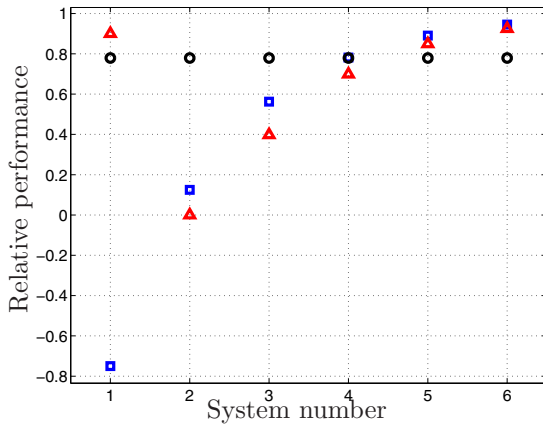
Fig. 1. Relative performance, $P_i(u_i)/p_i^*$, for the different systems. Case 1 (blue squares), Case 2 (red triangles) and Case 3 (black circles).

In Theorem 1, we showed that when the objective of the group is to minimize a weighted sum of the individual system costs, the coordination problem (Problem 1) is completely characterized by the unconstrained problems for the individual systems (Problem 2). Both performance and control trajectories under the optimal coordination policy are decomposed into two terms. The first term is the same as in the unconstrained problem and the second term describes the effects of coordination. In Theorem 2 we showed that this is true also for the optimal control law. The result is a coordination policy that is transparent and scalable. Moreover, since the systems in the group have identical dynamics, the optimal control law is obtained by solving the unconstrained problem for a single system in the group.

An insightful special case of Problem 1 is when the costs of the individual systems are weighted equally (Corollary 1). Then, satisfying the constraint will cost the group $1/m$ in relative performance improvement, where m is the number of systems in the group. This cost is distributed equally among the systems.

Theorem 1 and Theorem 2 can also the used as a tool to solve more general coordination problems with additional constraints on performance. In Problem 3 we introduced constraints on minimum performance bounds for the individual systems. By using Theorem 1 we were able to restate the problem as quadratic program in standard form1 (Theorem 3). In Problem 4, we look for a solution that maximizes the minimum performance improvement among the systems. In this case, we were able express the solution in closed form (Theorem 4).

APPENDIX–PROOFS

We first introduce needed notation and state some lemmas that will be used in proving Theorem 1. Let $v = \{v(t)\}_{t=-\infty}^{\infty}$ be a stationary zero-mean Gaussian white noise process. Introduce the space of processes obtained by running v through a stable and strictly causal LTI system:

$$\mathcal{X}(v) = \left\{ x : x(t) = \sum_{\tau=1}^{\infty} g_\tau v(t-\tau), \ \sum_{\tau=1}^{\infty} |g_\tau|^2 < \infty \right\}.$$

Lemma 1. The space $\mathcal{X}(v)$ is a Hilbert space under the scalar product $\langle x, y \rangle = \mathbf{E} x^T(t) y(t)$.

Proof. This follows from the fact that $\mathcal{X}(v)$ is isometric to $\ell_2(0, \infty)$: the space of sequences $\{g_k\}_0^\infty$ that satisfy $\sum_0^\infty |g_k|^2 < \infty$. To see this, let V denote the covariance matrix of v and define $T : \mathcal{X}(v) \to \ell_2(0, \infty)$ as $(Tx)_k = g_{k+1} V^{\frac{1}{2}}$. Then T is a bijective isometry. $\square$

Let w_i, $i = 1, \ldots, m$ be independent stationary zero-mean Gaussian white noise processes, and let $w = \begin{bmatrix} w_1^T \ldots w_m^T \end{bmatrix}^T$ be the vector of stacked noise processes. Then

$$\mathcal{X}(w) = \mathcal{X}(w_1) \oplus \mathcal{X}(w_2) \oplus \ldots \oplus \mathcal{X}(w_m).$$

where the $\mathcal{X}(w_i)$:s are mutually orthogonal subspaces of $\mathcal{X}(w)$. This can also be stated as:

$$\mathcal{X}(w) = \left\{ x : x = \sum_{i=1}^m x_i, \ x_i \in \mathcal{X}(w_i) \right\} \quad \text{and}$$

$$\langle x, y \rangle_{\mathcal{X}(w)} = \sum_{i=1}^m \langle x_i, y_i \rangle_{\mathcal{X}(w_i)}.$$

Lemma 2. Let $\bar{u}_i$ and u_i^* be the optimal control signal trajectories obtained by running (1) with the optimal state feedback laws in Problem 1 and Problem 2, respectively. Then

$$J_i(u_i^*) = \min_{u_i \in \mathcal{X}(w_i)} J_i(u_i), \ i = 1, \ldots, m, \qquad (.1)$$

and

$$\sum_1^m J_i(\bar{u}_i) = \min_{u \in \mathcal{T}} \sum_1^m \gamma_i J_i(u_i), \qquad (.2)$$

where

$$\mathcal{T} = \{ u \in \mathcal{X}(w) : \sum_{i=1}^m u_i = 0 \}, \qquad (.3)$$

Proof. It is a well known fact that the solution to the minimization in (.1) results in a state feedback law (see Åström and Wittenmark, 1997, Chapter 11). After eliminating the constraint $u \in \mathcal{T}$ (e.g. by setting $u_1 = -\sum_{j=2}^m u_j$) we note that the same is true for the minimization in (.2). $\square$

Define $y_i \in \mathcal{X}(w_i)$, and $N : \mathcal{X}(w) \to \mathcal{X}(w)$ as

$$y_i(t) = \sum_{\tau=1}^\infty A^{\tau-1} w_i(t-\tau)$$

$$(Nu)(t) = \sum_{\tau=1}^\infty A^{\tau-1} B u(t-\tau).$$

With these definitions (1) can be restated as

$$x_i = y_i + N u_i, \ i = 1 \ldots, m. \qquad (.4)$$

The adjoint operator to N is denoted $N^* : \mathcal{X}(w) \to \mathcal{X}(w)$.

Lemma 3. Let u_i^* be the optimal control signal trajectory obtained by running (1) with the solution to Problem 2. Then u_i^* and $p_i^* = P_i(u_i^*)$ can be expressed as

$$u_i^* = -\Phi^{-1} N^* y_i \quad \text{and} \quad p_i^* = \langle y_i, N\Phi^{-1}N^* y_i \rangle,$$

where $\Phi = I + N^* N$.

Proof. By (.1) in Lemma 2:

$$u_i^* = \operatorname*{argmin}_{u_i \in \mathcal{X}(w_i)} J_i(u_i)$$

Using Lemma 1 and the notation in (.4) we have

$$J_i(u_i) = \|y_i + Nu_i\|^2 + \|u_i\|^2 = \|y_i\|^2 + 2\langle u_i, N^*y_i\rangle$$
$$+ \langle u_i, (I + N^*N)u_i\rangle = \|y_i\|^2$$
$$+ \|\Phi^{\frac{1}{2}}(u_i - \Phi^{-1}N^*y_i)\|^2 - \langle y_i, N\Phi^{-1}N^*y_i\rangle.$$

Proof of Theorem 1:

Since $\sum_{i=1}^m \beta_i = 1$, we have

$$\sum_{i=1}^m \bar{u}_i^* = \sum_{i=1}^m u_i^* - \sum_{i=1}^m (\beta_i \sum_{j=1}^m u_j^*) = (\sum_{i=1}^m u_i^*)(1 - \sum_{j=1}^m \beta_j) = 0,$$

and hence $\bar{u}^* \in \mathcal{T}$, where $\mathcal{T}$ is defined in (.3) and denotes the space of feasible control signal trajectories in Problem 1.

Using the notation in (.4), we introduce the operator $\Theta_i : \mathcal{X}(w) \to (\mathcal{X}(w), \mathcal{X}(w))$ and $z_i \in \mathcal{X}(w_i)$ as:

$$z_i = \sqrt{\gamma_i} \begin{bmatrix} y_i \\ 0 \end{bmatrix}, \; \Theta_i u_i = \sqrt{\gamma_i} \begin{bmatrix} Nu_i \\ u_i \end{bmatrix}.$$

Then by Lemma 2, Problem 1 can be stated as

$$\min_{u \in \mathcal{T}} \|z + \Theta u\|^2$$

where

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}, \; \Theta u = \begin{bmatrix} \Theta_1 u_1 \\ \vdots \\ \Theta_m u_m \end{bmatrix}.$$

Let $u \in \mathcal{T}$. Then,

$$c = \langle z + \Theta \bar{u}^*, \Theta u \rangle = \sum_{i=1}^m \langle z_i + \Theta_i \bar{u}_i^*, \Theta_i u_i \rangle$$

$$= \sum_{i=1}^m \gamma_i \Big(\langle y_i + N\bar{u}_i^*, Nu_i \rangle + \langle \bar{u}_i^*, u_i \rangle \Big)$$

$$= \sum_{i=1}^m \gamma_i \langle N^*y_i + \Phi \bar{u}_i^*, u_i \rangle$$

$$= \sum_{i=1}^m \gamma_i \langle N^*y_i + \Phi u_i^*, u_i \rangle - \sum_{i=1}^m \gamma_i \beta_i \langle \sum_{j=1}^m \Phi u_j^*, u_i \rangle.$$

Since $N^*y_i + \Phi u_i^* = 0$ by Lemma 3 and $\gamma_i \beta_i = \gamma_j \beta_j = $ **const**, we have

$$c = \textbf{const} \langle \sum_{j=1}^m \Phi u_j^*, \sum_{i=1}^m u_i \rangle = \textbf{const} \langle \sum_{j=1}^m \Phi u_j^*, 0 \rangle = 0.$$

As u is arbitrary the first assertion follows from the projection theorem. For the second assertion, first note that for $u_i \in \mathcal{X}(w)$, we have

$$J_i(u_i) = \|y_i\|^2 + 2\langle u_i, N^*y_i \rangle + \langle u_i, \Phi u_i \rangle,$$

which implies $P_i(u_i) = -2\langle u_i, N^*y_i \rangle - \langle u_i, \Phi u_i \rangle$. Now,

$$P(\bar{u}_i^*) = -2\langle \bar{u}_i^*, N^*y_i \rangle - \langle \bar{u}_i^*, \Phi \bar{u}_i^* \rangle$$

$$= -2(1 - \beta_i)\langle u_i^*, N^*y_i \rangle + 2\beta_i \langle \sum_{j \neq 1} u_j^*, N^*y_i \rangle$$

$$- (1 - \beta_i)^2 \langle u_i^*, \Phi u_i^* \rangle - 2(1 - \beta_i)\beta_i \langle u_i^*, \Phi \sum_{j \neq i} u_j^* \rangle$$

$$- \beta_i^2 \langle \sum_{j \neq i} u_j^*, \sum_{j \neq i} \Phi u_j^* \rangle.$$

By (.2) in Lemma 3 and since $\langle y_i, Ty_j \rangle = 0$ for any T on $\mathcal{X}(w_j)$ and $i \neq j$, we have

$$\langle u_i^*, \Phi u_j^* \rangle = -\langle u_i^*, N^*y_j \rangle = \langle y_i, N\Phi^{-1}N^*y_j \rangle$$
$$= \begin{cases} 0 & \text{if } i \neq j \\ p_i^* & \text{if } i = j \end{cases}$$

Thus,

$$P(\bar{u}_i^*) = -2(1 - \beta_i)\langle u_i^*, N^*y_i \rangle - (1 - \beta_i)^2 \langle u_i^*, \Phi u_i^* \rangle$$

$$- \beta_i^2 \sum_{j \neq i} \langle u_j^*, \Phi u_j^* \rangle = (1 - \beta_i)(1 + \beta_i)p_i^*$$

$$- \beta_i^2 \sum_{j \neq i} p_j^* = p_i^* - \beta_i^2 \sum_{j=1}^m p_j^*$$

Proof of Theorem 3

We claim that Problem 3 is equivalent to Problem 1 for some $\gamma \in \Omega$, where

$$\Omega = \Big\{ x \in \mathbb{R}^m : x_i > 0, \; \sum_{i=1}^m x_i = 1 \Big\}. \tag{.5}$$

To see this, we restate Problem 3 as

$$\delta = \max_{\substack{u \in \mathcal{T} \\ P_i(u_i) \geq \alpha_i p_i^*}} \sum_{i=1}^m P(u_i) \tag{.6}$$

where $\mathcal{T}$ is defined in (.3). This problem is convex in u and strictly feasible by assumption. Hence, according to the S-procedure losslessness criterion for quadratic functionals in Yakubovich (1992, Theorem 2) there exists $\tau_i \geq 0$, $i = 1, \ldots, m$ such that

$$\delta = \max_{u \in \mathcal{T}} \left(\sum_{i=1}^m P_i(u_i) + \sum_{i=1}^m \tau_i \Big(P_i(u_i) - \alpha_i p_i^* \Big) \right)$$

$$= \left(\max_{u \in \mathcal{T}} \sum_{i=1}^m (1 + \tau_i)P(u_i) \right) - \sum_{i=1}^m \tau_i \alpha_i p_i^*$$

Therefore, the maximization in (.6) is equivalent to

$$\max_{u \in \mathcal{T}} \sum_{i=1}^m \gamma_i P(u_i) \Longleftrightarrow \min_{u \in \mathcal{T}} \sum_{i=1}^m \gamma_i J(u_i) \tag{.7}$$

for some $\gamma \in \Omega$. The claim then follows from Lemma 2 which states that a solution to (.7) is of the form $u = -Lx$. A solution to Problem 3 can thus be found by iteratively solving Problem 1 for different $\gamma \in \Omega$. However, by Theorem 1 a solution to Problem 1 results in

$$\sum_{i=1}^m P_i(u_i) = \sum_{i=1}^m \left(p_i^* - \beta_i^2 \sum_{j=1}^m p_j^* \right)$$

$$= \left(\sum_{i=1}^m p_i^* \right) \left(1 - \sum_{i=1}^m \beta_i^2 \right) \tag{.8}$$

for some $\beta \in \Omega$. Since the map defined by (6) is a bijection on Ω (the map is its own inverse), the iteration over γ can be replaced by minimizing (.8) subject to $\beta \in \Omega$, and

$$p_i^* - \beta_i^2 \sum_{j=1}^m p_j^* \geq \alpha_i p_i^*.$$

This is the quadratic program in Theorem 3.

Proof of Theorem 4

We claim that Problem 4 is equivalent to Problem 1 for some $\gamma \in \Omega$, where Ω is defined in (.5). To see this, we restate Problem 4 as:

$$\delta = \max_{\substack{u \in \mathcal{T}, \, \alpha \in \mathbb{R} \\ P_i(u_i) \geq \alpha p_i^*}} \alpha \qquad (.9)$$

where $\mathcal{T}$ is defined in (.3). It follows from Theorem 1 that there is a $\nu \in \mathcal{T}$ such that $P_i(\nu_i) > 0$, $\forall i$. Indeed, such a ν can be obtained by solving Problem 1 with $\gamma_i = (p_i^*)^{-\frac{1}{2}} / \sum_j (p_j^*)^{-\frac{1}{2}}$. This gives

$$\beta_i = \frac{\sqrt{p_i}}{\sum_{j=1}^m \sqrt{p_j^*}} \Rightarrow P_i(\nu_i) = \left(1 - \frac{\sum_{i=1}^m p_i^*}{(\sum_{i=1}^m \sqrt{p_j^*})^2}\right) p_i^* > 0.$$

This implies that $(u, \alpha) = (\nu, \, 0)$ is a strictly feasible point for (.9). Hence by Yakubovich (1992, Theorem 2) we have

$$\delta = \min_{\tau \succeq 0} \left(\max_{u \in \mathcal{T}, \, \alpha \in \mathbb{R}} \alpha + \sum_{i=1}^m \tau_i \left(P(u_i) - \alpha p_i^* \right) \right)$$

$$= \min_{\tau \succeq 0} \left(\max_{u \in \mathcal{T}, \alpha \in \mathbb{R}} \left(\left(1 - \sum_{i=1}^m p_i^* \tau_i\right) \alpha + \sum_{i=1}^m \tau_i P_i(u_i) \right) \right)$$

$$= \min_{\substack{\tau \succeq 0 \\ \sum p_i^* \tau_i = 1}} \left(\max_{u \in \mathcal{T}} \sum_{i=1}^m \tau_i P_i(u_i) \right) \qquad (.10)$$

Now we claim that the minimization in (.10) can be restricted to a search over $\tau \succ 0$. If not, then without loss of generality we can assume that the optimal τ has $\tau_1 = 0$. Then the solution to the inner maximization is to make $P(u_i) = p_i^*, i = 2 \ldots, m$ by setting $u_1 = -\sum_{i=2}^m u_i^*$, which results in $\delta = 1$. However, by Theorem 1 and Lemma 2, setting $\tau_i = \frac{1}{m p_i^*}$, $i = 1, \ldots, m$ gives

$$P(u_i) = p_i^* \left(1 - \frac{1}{\sum_{j=1}^m p_j^*}\right)$$

and thus

$$\max_{u \in \mathcal{T}} \sum_{i=1}^m \tau_i P_i(u_i) = 1 - \frac{1}{m \sum_{1}^m p_i^*} < 1,$$

which is a contradiction. Therefore, the maximization in (.9) is equivalent to

$$\max_{u \in \mathcal{T}} \sum_{i=1}^m \tau_i P(u_i) \iff \min_{u \in \mathcal{T}} \sum_{i=1}^m \tau_i J(u_i) \qquad (.11)$$

for some $\tau \in \Sigma$, where

$$\Sigma = \left\{ x \in \mathbb{R}^m : x_i > 0, \ \sum_{i=1}^m p_i^* x_i = 1 \right\}.$$

The claim stated in the beginning of the proof follows from Lemma 2 which states that a solution to (.11) is of the form $u = -Lx$. A solution to Problem 3 can thus be found by iteratively solving Problem 1 for different $\gamma \in \Sigma$. However, by Theorem 1 a solution to Problem 1 results in

$$\min_i \left\{ \frac{P_i(u_i)}{p_i^*} \right\} = \min_i \left\{ 1 - \beta_i^2 \frac{\sum p_j^*}{p_i^*} \right\} \qquad (.12)$$

for some $\beta \in \Omega$. Since (6) defines a bijection $T : \Sigma \to \Omega$ with

$$(T^{-1}x)_i = \frac{x_i^{-1}}{\sum_{j=1}^m p_j^* x_j^{-1}},$$

the iteration over γ can be replaced by the following linear program:

$$\max_{\alpha, \beta} \alpha$$

subject to $\beta \in \Omega$ and

$$1 - \frac{\beta_i^2}{c_i} \geq \alpha, \ i = 1, \ldots, m, \qquad (.13)$$

where $c_i = \frac{p_i^*}{\sum_j p_j^*}$. We claim that the inequality in (.13) can be replaced by an equality. To see this, suppose that $1 - \beta_i^2/c_i > \alpha$ and $1 - \beta_j^2/c_j = \alpha$ for some i, j. Since it is always possible to slightly decrease β_j by a corresponding increase in β_i we could increase α. This proves the claim and the second assertion in Theorem 4. It also implies that the optimal β must satisfy $\beta \in \Omega$ and

$$1 - \frac{\beta_i^2}{c_i} = \alpha = 1 - \frac{\beta_j^2}{c_j}, \ \Rightarrow \beta_i = \sqrt{\frac{p_i^*}{p_j^*}} \beta_j, \ \forall i, j. \qquad (.14)$$

Hence,

$$\sum_{i=1}^m \beta_i = \sum_{i=1}^m \sqrt{\frac{p_i^*}{p_i^*}} \beta_1 = 1 \Rightarrow \beta_1 = \frac{\sqrt{p_1^*}}{\sum_{j=1}^m \sqrt{p_j^*}},$$

which proves the first assertion for $i = 1$. For $i = 2, \ldots, m$ set $\beta_j = \beta_1$ in (.14). $\qquad \square$

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