Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

$\mathscr{H}_\infty$ Filter Design through Multi-simplex Modeling for Discrete-time Markov Jump Linear Systems with Partly Unknown Transition Probability Matrix

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Abstract: This paper addresses the problem of $\mathscr{H}_\infty$ filter design for discrete-time Markov jump linear systems (MJLS) with transition probability matrix affected by uncertainties. The proposed methodology allows to take into account the different types of uncertainties usually adopted in MJLS in a systematic way. New conditions are given for $\mathscr{H}_\infty$ filter design with partial, complete or null Markov mode availability. Due to the presence of slack variables in the synthesis conditions and to the use of homogeneous polynomial solutions of arbitrary degrees, less conservative linear matrix inequality relaxations can be obtained. Numerical experiments illustrate the better performance and efficiency of the proposed approach when compared to other strategies available in the literature.

Keywords: Markov jump linear systems; Discrete-time systems; Uncertain transition probability; $\mathscr{H}_\infty$ filtering; Linear matrix inequalities.

1. INTRODUCTION

The $\mathscr{H}_\infty$ filtering problem is among the most studied topics in control and signal processing literature. The $\mathscr{H}_\infty$ performance is mainly used when there is no information about the statistics of the noise actuating in the system. In the last years, results have been reported for $\mathscr{H}_\infty$ filter design of uncertain linear systems [Duan et al., 2006], linear parameter varying systems [de Souza et al., 2006], time-delay systems [Lacerda et al., 2013], nonlinear systems [Li et al., 2012] and Markov jump linear systems – MJLS [de Souza and Fragoso, 2003]. Concerning the latter case, the filtering problem has been investigated in the discrete-time [de Souza and Fragoso, 2003] and continuous-time [de Souza et al., 2006] cases, considering mode-dependent [Zhang and Boukas, 2009b] and mode-independent [Li and Shi, 2012] filter design.

MJLS are a category of hybrid systems in which multiple operation modes can occur. Each individual mode is linear, described by difference equations, in the discrete-time case, or differential equations, in the continuous-time case, depending upon a random variable. The switching between different modes is governed by a stochastic process depicted by a Markov chain associated to a transition probability matrix. This dynamical system class can appropriately represent plants subject to abrupt changes in operation modes or structure (see Costa et al. [1999, 2005] and references therein). Regarding the transition probabilities, to obtain accurate information about them can be a very arduous or expensive work. Therefore, to overcome such challenge, some researches treat incomplete knowledge of transition probabilities, for instance considering that the uncertain probability matrix is: i) polytopic [Gonçalves et al., 2011]; ii) partly unknown with elements lying in known intervals [Luan et al., 2010]; iii) partly unknown without any particular structure in the unknown entries [Zhang and Boukas, 2009a,b]; while other results deal only with a completely known transition probability matrix [Gonçalves et al., 2009].

This paper handles the problem of designing full order filters for discrete-time MJLS with uncertain transition probability matrix, guaranteeing an upper bound to the $\mathscr{H}_\infty$ norm. The three types of uncertainties, usually studied in the literature, are modeled by the multi-simplex methodology [Oliveira et al., 2008]. In order to do that, each row of the uncertain transition probability matrix containing any kind of uncertainty is described in terms of parameters belonging to a unit simplex. Then, all the uncertain parameters are combined into one single domain generated by the Cartesian product of simplexes. Exploiting this representation, sufficient conditions for $\mathscr{H}_\infty$ filtering are proposed in terms of linear matrix inequalities (LMIs) with scalar parameters. Such conditions can cope with $\mathscr{H}_\infty$ filter design of MJLS under the assumptions of complete, partial or no mode observation. As illustrated by numerical examples, the proposed conditions can be less conservative than the others available in the literature. At the price of increasing the computational effort using higher degrees in the Lyapunov matrices and in the slack variables, or even performing line searches of the scalar parameters, less conservative outcomes can be obtained in terms of $\mathscr{H}_\infty$ guaranteed performances.

The remainder of the paper is structured as follows. Section 2 outlines the notations and preliminary results related to the LMI conditions, the multi-simplex modeling of uncertain transition probabilities and some machinery to handle matrix polynomials. Section 3 introduces the main results for $\mathscr{H}_\infty$ filtering design. Section 4 provides numerical examples and comparisons with other conditions from the literature. Section 5 summarizes the paper.

2. PRELIMINARIES

Consider the fundamental probability space $(\Omega, \mathcal{F}, \Gamma)$ and the discrete-time MJLS $\mathscr{G}$ defined by the following stochastic equations

$$\mathscr{G} = \begin{cases} x(k+1) = A(\theta_k)x(k) + E(\theta_k)w(k) \\ z(k) = C_z(\theta_k)x(k) + E_z(\theta_k)w(k) \\ y(k) = C_y(\theta_k)x(k) + E_y(\theta_k)w(k) \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^{n_x}$ is the system state, $z(k) \in \mathbb{R}^{n_z}$ is the signal to be estimated, $y(k) \in \mathbb{R}^{n_y}$ is the measured output, and $w(k) \in \mathbb{R}^{n_w}$ is the external perturbation, which is supposedly energy bounded, that is $w(k) \in \ell_2$. Furthermore, $\{\theta(k);\ k \geq 0\}$ is a discrete-time homogeneous Markov chain with finite state-space $\mathbb{K} \triangleq \{1, \ldots, \sigma\}$, which comprises the operation modes of system $\mathscr{G}$, and a associated stationary transition probability matrix $\Gamma = [p_{ij}]$, $\forall i, j \in \mathbb{K}$, where

$$p_{ij} = \Pr(\theta(k+1) = j \mid \theta(k) = i), \ \forall k \geq 0.$$

Whenever possible, $\theta(k)$ is replaced by i, $\forall i \in \mathbb{K}$, such that the system matrices are given and, for ease of notation, written as $A_i \in \mathbb{R}^{n_x \times n_x}$, $E_i \in \mathbb{R}^{n_x \times n_w}$, $C_{zi} \in \mathbb{R}^{n_z \times n_x}$, $E_{zi} \in \mathbb{R}^{n_z \times n_w}$, $C_{y_i} \in \mathbb{R}^{n_y \times n_x}$, $E_{y_i} \in \mathbb{R}^{n_y \times n_w}$, $\forall i \in \mathbb{K}$.

One definition that generalizes the concept of stability applied to MJLS is the mean square stability (MSS), that is

$$\mathscr{E}[\|x(k)\|] \to 0 \text{ as } k \to \infty$$

for any initial condition $x(0) \in \mathbb{R}^{n_x}$, $\theta_0 \in \mathbb{K}$. Necessary and sufficient conditions for MSS are proved in Costa and Fragoso [1993], Costa et al. [2005].

To handle the $\mathscr{H}_\infty$ filtering problem for system $\mathscr{G}$, some definitions are required. The $\mathscr{H}_\infty$ norm to the model (1), denoted as $\|\mathscr{G}\|_\infty$, is formally characterized by the following definition [Costa and do Val, 1996].

Definition 1. Assume that $\mathscr{G}$ is stable. The $\mathscr{H}_\infty$ norm of system (1) from the input $w(k)$ to the output $z(k)$ is given by

$$\|\mathscr{G}\|_\infty^2 = \sup_{w(k)\in\ell_2, \theta_0 \in \mathbb{K}} \frac{\|z(k)\|_2^2}{\|w(k)\|_2^2}. \quad (2)$$

The problem to be studied in this paper is: find a robust causal full order mode-dependent linear filter $\mathscr{F}$ given by

$$\mathscr{F} = \begin{cases} x_f(k+1) = A_f(\theta_k)x_f(k) + B_f(\theta_k)y(k) \\ z_f(k) = C_f(\theta_k)x_f(k) + D_f(\theta_k)y(k) \end{cases} \quad (3)$$

where $x_f(k) \in \mathbb{R}^{n_f}$, $n_f = n_x$, is the estimated state, $z_f(k) \in \mathbb{R}^{n_z}$ is the estimated output and the matrices $A_{f_i} \in \mathbb{R}^{n_x \times n_x}$, $B_{f_i} \in \mathbb{R}^{n_x \times n_y}$, $C_{f_i} \in \mathbb{R}^{n_z \times n_x}$, and $D_{f_i} \in \mathbb{R}^{n_z \times n_y}$ are sought. Additionally, the dynamics of the error, $e(k) = z(k) - z_f(k)$, is MSS and the energy gain from the external perturbation input $w(k)$ to the error $e(k)$, that is a bound to the $\mathscr{H}_\infty$ norm, is minimized.

Connecting the filter (3) to the MJLS (1) the dynamics of the estimation error satisfies the following augmented state-space system model:

$$\mathscr{G}_{aug} = \begin{cases} \tilde{x}(k+1) = \tilde{A}(\theta_k)x(k) + \tilde{B}(\theta_k)w(k) \\ e(k) = \tilde{C}(\theta_k)x(k) + \tilde{D}(\theta_k)w(k) \end{cases} \quad (4)$$

with $\tilde{x}(k) = \begin{bmatrix} x(k)^T & x_f(k)^T \end{bmatrix}^T$ and

$$\begin{aligned} \tilde{A}_i &= \begin{bmatrix} A_i & 0 \\ B_{f_i}C_{y_i} & A_{f_i} \end{bmatrix}, & \tilde{B}_i &= \begin{bmatrix} E_i \\ B_{f_i}E_{y_i} \end{bmatrix}, \\ \tilde{C}_i &= \begin{bmatrix} C_{zi} - D_{f_i}C_{y_i} & -C_{f_i} \end{bmatrix}, & \tilde{D}_i &= E_{zi} - D_{f_i}E_{y_i}. \end{aligned} \quad (5)$$

This paper considers a scenario where the transition probability matrix $\Gamma = [p_{ij}]$ can be affected by different types of uncertainty. Similarly to Gonçalves et al. [2012], each element p_{ij} can vary between two known bounds, i.e., $0 \leq \underline{p}_{ij} \leq p_{ij} \leq \overline{p}_{ij} \leq 1$, or, as in Zhang and Boukas [2009a,b], where the entries are completely unknown, i.e., $p_{ij} =?$. Note that the latter can be seen as a particular case of the known bounded assumption, since the minimum and maximum bounds of each element can be inferred.

The procedure to construct a generic representation that can cope with all types of uncertainties is performed in two steps. At first, similarly to Gonçalves et al. [2012], each uncertain row of Γ is modeled by uncertain parameters belonging to a unit simplex (Λ_{N_r}), given by

$$\Lambda_{N_r} \triangleq \left\{ \zeta \in \mathbb{R}^{N_r} : \sum_{i=1}^{N_r} \zeta_i = 1,\ \zeta_i \geq 0,\ i = 1, \ldots, N_r \right\}.$$

Next, using the methodology presented in Morais et al. [2013] to treat uncertain transition probability matrices, the parameters of the m uncertain rows are combined into one single domain, created by the Cartesian product of m unit simplexes $\Lambda = \Lambda_{N_1} \times \cdots \times \Lambda_{N_m}$, called a multi-simplex [Oliveira et al., 2008]. The dimension of Λ is defined as the index $N = (N_1, \ldots, N_m)$. It is noteworthy that the proposed approach can also deal with polytopic uncertain probability matrix [de Souza, 2003, Gonçalves et al., 2011].

The computation of an $\mathscr{H}_\infty$ guaranteed cost for an MJLS with uncertain transition probability matrix $\Gamma(\alpha)$ is presented in the next lemma. The result can be viewed as an extension of the bounded real lemma for discrete-time MJLS [Seiler and Sengupta, 2003, Costa and do Val, 1996, Gonçalves et al., 2012] to cope with uncertain parameters belonging to the multi-simplex domain.

Lemma 1. System (4) is MSS and $\|\mathscr{G}\|_\infty < \gamma$ if there exist symmetric positive definite parameter-dependent matrices $P_i(\alpha) \in \mathbb{R}^{2n_x \times 2n_x}$, $i = 1, \ldots, \sigma$, such that the parameter-dependent inequalities

$$\begin{bmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i \end{bmatrix}^T \begin{bmatrix} P_{pi}(\alpha) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i \end{bmatrix} - \begin{bmatrix} P_i(\alpha) & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0 \quad (6)$$

hold for each $i = 1, \ldots, \sigma$ and for all $\alpha \in \Lambda$, being $P_{pi}(\alpha) = \sum_{j=1}^{\sigma} p_{ij}(\alpha)P_j(\alpha)$. By minimizing γ under the robust LMIs constraints (6), one gets the worst case $\mathscr{H}_\infty$ norm of system (4).

Considering only the block (1,1) in (6), $i = 1, \ldots, \sigma$, the inequalities in Lemma 1 can also be used to deal with robust MSS of MJLS.

Given filtering matrices A_{f_i}, B_{f_i}, C_{f_i} and D_{f_i}, Lemma 1 presents an infinite dimensional problem to compute a bound to the $\mathscr{H}_\infty$ norm of the augmented system (4) with uncertainties in the transition probability matrix (the parameter-dependent inequalities must be verified for all $\alpha \in \Lambda$). Whenever one solution exists, the computation of the minimum value of γ in terms of a finite set of LMIs can be obtained by applying relaxation techniques based on homogeneous polynomials of arbitrary degrees g [Bliman, 2004]. When the parameters belong to the multi-simplex Λ, the solution can be approximated by a homogeneous polynomial solution of sufficiently large partial degrees g_r, $r=1,\ldots,m$, without loss of generality [Oliveira et al., 2008].

The notation and definitions related to homogeneous polynomials and a systematic procedure to compute the vertices of the transition probability matrix [1] in the multi-simplex domain used in this paper follow the same lines presented in Morais et al. [2013]. Additionally, the LMI conditions proposed in Section 3 require a homogeneous polynomial representation of $\Gamma(\alpha)$ matrix with degree one in Λ_{N_r}, $r=1,\ldots,m$. For this purpose, the s-th row of each vertex v of the $\Gamma(\alpha)$ is grouped as

$$\Upsilon_s^{(v)} = \begin{bmatrix} p_{s1}^{(v)}\mathrm{I} & p_{s2}^{(v)}\mathrm{I} & \cdots & p_{s\sigma}^{(v)}\mathrm{I} \end{bmatrix}$$

where $v \in \mathscr{K}_N(\mathbb{1}) = \mathscr{K}_{N_1}(1)\times\ldots\times\mathscr{K}_{N_m}(1)$ and $\mathbb{1}$ is defined as $\mathbb{1}=(1,\ldots,1)$, with m elements, and the coefficient $\pi(k)$ is defined as

$$\pi(k) = (k_{11}!)\cdots(k_{1N_1}!)\cdots(k_{m1}!)\cdots(k_{mN_m}!).$$

When $P_i(\alpha)$ of degree g_r, for each $i=1,\ldots,\sigma$ and $r=1,\ldots,m$, are considered, the following coefficient matrices are used

$$\mathscr{X}_k = \begin{bmatrix} P_{1_k} & P_{2_k} & \cdots & P_{\sigma_k} \end{bmatrix}^T.$$

Next section presents a systematic procedure, based on a sequence of LMI problems, which searches for homogeneous polynomial solutions of arbitrary degree in the multi-simplex [Oliveira and Peres, 2007, Oliveira et al., 2008].

3. MAIN RESULTS

The following theorem presents sufficient LMI conditions with scalar parameters for the existence of a mode-dependent filter assuring robust MSS and an upper bound to the $\mathscr{H}_\infty$ norm for system (4) with uncertain transition probability matrix. The LMI conditions are given in terms of the partial degrees of the homogeneous solutions and the level d of Pólya's relaxations [Scherer, 2003, 2005, Oliveira and Peres, 2007, Oliveira et al., 2008]. Moreover, structural constraints are imposed to the slack variables to derive tractable conditions [Duan et al., 2006, Lacerda et al., 2011].

Theorem 1. If there exist symmetric matrices $P_{i_k} \in \mathbb{R}^{2n_x\times 2n_x}$, $k\in\mathscr{K}_N(g)$, $i=1,\ldots,\sigma$, matrices $K_{11_{i_k}} \in \mathbb{R}^{n_x\times n_x}$, $K_{21_{i_k}} \in \mathbb{R}^{n_x\times n_x}$, $Q_{1_{i_k}} \in \mathbb{R}^{n_w\times n_x}$, $F_{1_{i_k}} \in \mathbb{R}^{n_z\times n_x}$, $G_{1_{i_k}} \in \mathbb{R}^{n_x\times n_x}$ and $G_{2_{i_k}} \in \mathbb{R}^{n_x\times n_x}$, $k\in\mathscr{K}_N(h)$, $i=1,\ldots,\sigma$, $H_i\in\mathbb{R}^{n_x\times n_x}$, $Z_i\in\mathbb{R}^{n_x\times n_y}$, $\hat{K}_i\in\mathbb{R}^{n_x\times n_x}$, $C_{f_i}\in\mathbb{R}^{n_z\times n_x}$, $D_{f_i}\in\mathbb{R}^{n_z\times n_y}$, $i=1,\ldots,\sigma$, partial degrees $g=(g_1,\ldots,g_m)$, $h=(h_1,\ldots,h_m)$ with g_r and $h_r\in\mathbb{N}$, a degree $d\in\mathbb{N}$ and given scalars λ_1 and λ_2 such that for $i=1,\ldots,\sigma$, the following LMIs hold [2]

$$\Xi_k = \sum_{\substack{k'\in\mathscr{K}_N(\mathbb{1}d)\\ k\geq k'}} \frac{(d!)^m}{\pi(k')}(P_{i_{k-k'}}) > 0, \qquad \forall k\in\mathscr{K}_N(g+\mathbb{1}d) \quad (7)$$

$$\Theta_k + \Psi_k + \Phi_k > 0, \qquad \forall k\in\mathscr{K}_N(w) \quad (8)$$

with

$$\Theta_k = \sum_{\substack{k'\in\mathscr{K}_N(w-g-\mathbb{1})\\ k\geq k'}} \sum_{\substack{\hat{k}\in\mathscr{K}_N(\mathbb{1})\\ k\geq k'+\hat{k}}} \frac{\prod_{j=1}^{m}(w_j-g_j-1)!}{\pi(k')} \begin{bmatrix} P_{i_{k-k'-\tilde{k}}} & \star & \star & \star \\ 0 & -\Upsilon_i^{(\hat{k})}\mathscr{X}_{k-k'-\hat{k}} & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (9)$$

Ψ_k and Φ_k given by (10) and (11), respectively, where $w=\max\{g+\mathbb{1}(d+1), h+\mathbb{1}d\}$, then

$$A_{f_i} = \hat{K}_i^{-1}H_i, \quad B_{f_i} = \hat{K}_i^{-1}Z_i, \quad C_{f_i} \text{ and } D_{f_i} \quad (12)$$

are the filter matrices assuring robust MSS and an $\mathscr{H}_\infty$ guaranteed cost, given by γ, for the augmented system (4).

Proof. First, note that the term $\Upsilon_i^{(\tilde{k})}\mathscr{X}_{k-k'-\tilde{k}}$ of (9) can be rewritten as

$$\begin{aligned}\Upsilon_i^{(\tilde{k})}\mathscr{X}_{k-k'-\tilde{k}} &= \begin{bmatrix} p_{i1}^{(\tilde{k})}\mathrm{I} & \cdots & p_{i\sigma}^{(\tilde{k})}\mathrm{I}\end{bmatrix} \times \begin{bmatrix} P^T_{1_{k-k'-\tilde{k}}} & \cdots & P^T_{\sigma_{k-k'-\tilde{k}}}\end{bmatrix}^T \\ &= p_{i1}^{(\tilde{k})}P_{1_{k-k'-\tilde{k}}} + p_{i2}^{(\tilde{k})}P_{2_{k-k'-\tilde{k}}} + \ldots + p_{i\sigma}^{(\tilde{k})}P_{\sigma_{k-k'-\tilde{k}}} \\ &= \sum_{i=1}^{\sigma} p_{ij}^{(\tilde{k})}P_{i_{k-k'-\tilde{k}}}\end{aligned}$$

and $\prod_{r=1}^m\left(\sum_{t=1}^{N_r}\alpha_{rt}\right)^d = 1$ for any $d\in\mathbb{N}$, then matrix (7) can be equivalently rewritten as

$$\prod_{r=1}^{m}\left(\sum_{t=1}^{N_r}\alpha_{rt}\right)^d P_i(\alpha) = \sum_{k\in\mathscr{K}_N(g+\mathbb{1}d)} \alpha^k\Xi_k, \quad i=1,\ldots,\sigma. \quad (13)$$

Now with $P_i(\alpha)=P_i^T(\alpha)$, $P_{pi}(\alpha)=\sum_{j=1}^{\sigma}p_{ij}(\alpha)P_j(\alpha)$, H_i, Z_i, $\hat{K}_i$, C_{f_i}, D_{f_i},

$$\begin{aligned} K_i(\alpha) &= \begin{bmatrix} K_{11i}(\alpha) & \lambda_1\hat{K}_i \\ K_{21i}(\alpha) & \lambda_2\hat{K}_i\end{bmatrix}, \quad & G_i(\alpha) &= \begin{bmatrix} G_{1i}(\alpha) & \hat{K}_i \\ G_{2i}(\alpha) & \hat{K}_i\end{bmatrix}, \\ Q_i(\alpha) &= \begin{bmatrix} Q_{1i}(\alpha) & 0\end{bmatrix}, & F_i(\alpha) &= \begin{bmatrix} F_{1i}(\alpha) & 0\end{bmatrix},\end{aligned} \quad (14)$$

one has

$$\begin{bmatrix} P_i(\alpha)+K_i(\alpha)\tilde{A}_i+\tilde{A}_i^TK_i^T(\alpha) & \star & \star & \star \\ G_i(\alpha)\tilde{A}_i - K_i^T(\alpha) & -P_{pi}(\alpha)-G_i(\alpha)-G_i^T(\alpha) & \star & \star \\ \tilde{B}_i'K_i^T(\alpha)+Q_i(\alpha)\tilde{A}_i & \tilde{B}_i^TG_i^T(\alpha)-Q_i(\alpha) & \tilde{B}_i^TQ_i^T(\alpha)+Q_i(\alpha)\tilde{B}_i+\gamma^2\mathrm{I} & \star \\ F_i(\alpha)\tilde{A}_i+\tilde{C}_i & -F_i(\alpha) & F_i(\alpha)\tilde{B}_i+\tilde{D}_i & \mathrm{I}\end{bmatrix} > 0, \quad (15)$$

which is (8), multiplied by α^k, summed up for $k\in\mathscr{K}_N(w)$. Then, multiplying it by T on the left and by T^T on the right, with

$$T = \begin{bmatrix} \mathrm{I} & \tilde{A}_i^T & 0 & 0 \\ 0 & \tilde{B}_i^T & \mathrm{I} & 0 \\ 0 & 0 & 0 & \mathrm{I}\end{bmatrix}, \quad (16)$$

yields

$$\begin{bmatrix} P_i(\alpha)-\tilde{A}_i^TP_{pi}(\alpha)\tilde{A}_i & \star & \star \\ -\tilde{B}_i^TP_{pi}(\alpha)\tilde{A}_i & -\tilde{B}_i^TP_{pi}(\alpha)\tilde{B}_i+\gamma^2\mathrm{I} & \star \\ \tilde{C}_i & \tilde{D}_i & \mathrm{I}\end{bmatrix} > 0 \quad (17)$$

or, by Schur complement,

$$\begin{bmatrix}\tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i\end{bmatrix}^T \begin{bmatrix} P_{pi}(\alpha) & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix}\tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i\end{bmatrix} - \begin{bmatrix} P_i(\alpha) & 0 \\ 0 & \gamma^2 I\end{bmatrix} < 0 \quad (18)$$

[1] The routine that automatically generates the vertices of $\Gamma(\alpha)$ is available for download at http://www.dt.fee.unicamp.br/~ricfow/programs/Gamma_Multi_Simplex.m.

[2] The symbol $\star$ represents a symmetric block in the LMI.

$$\Psi_k = \sum_{\substack{\breve{k}\in\mathscr{K}_N(w-h)\\ k\geq\breve{k}}} \frac{\prod_{j=1}^{m}(w_j-h_j)!}{\pi(\breve{k})} \begin{bmatrix} K_{11_{i_{k-\breve{k}}}}A_i + A_i^T K_{11_{i_{k-\breve{k}}}}^T & \star & \star & \star & \star & \star \\ K_{21_{i_{k-\breve{k}}}}A_i & 0 & \star & \star & \star & \star \\ G_{11_{i_{k-\breve{k}}}}A_i - K_{11_{i_{k-\breve{k}}}}^T & -K_{21_{i_{k-\breve{k}}}}^T & -G_{11_{i_{k-\breve{k}}}} - G_{11_{i_{k-\breve{k}}}}^T & \star & \star & \star \\ G_{21_{i_{k-\breve{k}}}}A_i & 0 & -G_{21_{i_{k-\breve{k}}}} & 0 & \star & \star \\ E_i^T K_{11_{i_{k-\breve{k}}}}^T + Q_{1_{i_{k-\breve{k}}}}A_i & E_i^T K_{21_{i_{k-\breve{k}}}}^T & E_i^T G_{11_{i_{k-\breve{k}}}}^T - Q_{1_{i_{k-\breve{k}}}} & E_i^T G_{21_{i_{k-\breve{k}}}}^T & E_i^T Q_{1_{i_{k-\breve{k}}}}^T + Q_{1_{i_{k-\breve{k}}}}E_i & \star \\ F_{1_{i_{k-\breve{k}}}}A_i & 0 & -F_{1_{i_{k-\breve{k}}}} & 0 & F_{1_{i_{k-\breve{k}}}}E_i & 0 \end{bmatrix} \quad (10)$$

$$\Phi_k = \frac{\prod_{j=1}^{m} w_j!}{\pi(k)} \begin{bmatrix} \lambda_1\left(C_{y_i}^T Z_i^T + Z_i C_{y_i}\right) & \star & \star & \star & \star & \star \\ \lambda_2 Z_i C_{y_i} + \lambda_1 H_i & \lambda_2\left(H_i + H_i^T\right) & \star & \star & \star & \star \\ Z_i C_{y_i} & H_i & 0 & \star & \star & \star \\ -\lambda_1 \hat{K}_i + Z_i C_{y_i} & -\lambda_2 \hat{K}_i^T + H_i & -\hat{K}_i^T & -\hat{K}_i - \hat{K}_i^T & \star & \star \\ \lambda_1 E_{y_i}^T Z_i^T & \lambda_2 E_{y_i}^T Z_i^T & E_{y_i}^T Z_i^T & E_{y_i}^T Z_i^T & \gamma^2 \mathrm{I} & \star \\ C_{z_i} - D_{f_i} C_{y_i} & -C_{f_i} & 0 & 0 & E_{z_i} - D_{f_i} E_{y_i} & \mathrm{I} \end{bmatrix} \quad (11)$$

which is the bounded real lemma for discrete-time MJLS (6). □

As a by-product, the condition of Theorem 1 can be straightforwardly adapted to deal with the case of partial observation of the Markov state (observations by clusters [do Val et al., 2002]), providing partly mode-dependent filters. In order to do that, consider the set $\mathbb{Q} = \{1,2,\dots,\sigma_c\}$, $\sigma_c \leq \sigma$, that contains the indexes q of system clusters, and set $\mathbb{U}_q$, that gathers the modes belonging to the cluster q, such that the clusters are mutually exclusive groups whose union generates the set of states $\mathbb{K}$. In other words, $\mathbb{K} \equiv \cup_{q\in\mathbb{Q}}\mathbb{U}_q$ such that $\cap_{q\in\mathbb{Q}}\mathbb{U}_q \equiv \varnothing$. The result is presented in the corollary below.

Corollary 1. If the conditions of Theorem 1 are satisfied with H_i, Z_i, $\hat{K}_i$, C_{f_i} and D_{f_i} replaced by H_q, Z_q, $\hat{K}_q$, C_{f_q} and D_{f_q} for all $q \in \mathbb{Q}$ and $i \in \mathbb{U}_q \subset \mathbb{K}$, respectively, then $A_{f_i} = \hat{K}_q^{-1}H_q$, $B_{f_i} = \hat{K}_q^{-1}Z_q$, C_{f_i} and D_{f_i} are the partly mode-dependent filter matrices. Moreover, γ is an upper bound to the $\mathscr{H}_\infty$ norm, assuring the MSS, of system (4).

It is worthy to mention that if $\sigma_c = 1$ (i.e., no observations of the Markov state chain) and $\mathbb{Q} = \{1\}$ with $\mathbb{U}_{q=1} = \{1,2,\dots,\sigma\}$, the attained filter matrices will be the same for all operation modes, which is called a mode-independent filter. On the other hand, if $\sigma_c = \sigma$, Corollary 1 reproduces exactly the conditions in Theorem 1.

4. NUMERICAL EXAMPLES

This section presents numerical comparisons between the approach proposed in this paper and other methods from the literature. All routines were implemented in Matlab, version 7.10 (R2010a) using Yalmip [Löfberg, 2004] and SeDuMi [Sturm, 1999]. The computer used was an AMD Phenon II X6 1090T (3.2GHz), 4GB RAM, Windows 7.

Example 1

Consider the system borrowed from Zhang and Boukas [2009b], where the specific data can be found, which deals with four types of transition probability matrix: Completely Known (CK), Partly Known Case 1 (C1), Partly Known Case 2 (C2) and Completely Unknown (CUK).

The aim here is to design, mode-dependent (MD) and mode-independent (MI), proper (P) and strictly proper (SP), $\mathscr{H}_\infty$ filters to the four types of uncertain transition probability matrix. Table 1 compares the $\mathscr{H}_\infty$ guaranteed costs obtained with Theorem 1 (T1) and Corollary 1 (C1) proposed in this paper with Theorem 1 from Zhang and Boukas [2009b], called in this example as T1 [ZB:09].

Table 1. $\mathscr{H}_\infty$ guaranteed costs (γ) of mode-dependent (MD) and mode-independent (MI), proper (P) and strictly proper (SP) filters for Example 1 using Theorem 1 (T1), Corollary 1 (C1), both with $d = g = h = \lambda_1 = \lambda_2 = 0$, and Theorem 1 (T1 [ZB:09]) from Zhang and Boukas [2009b].

		Method	CK	C1	C2	CUK
MD	P	T1 [ZB:09]	1.8556	3.8215	4.7293	4.4624
MD	P	T1	1.1893	1.6422	1.6436	3.8104
MD	SP	T1	1.2465	1.7554	1.7559	3.8118
MI	P	C1	1.2360	1.7361	1.7625	4.0250
MI	SP	C1	1.2923	1.8690	1.9011	4.3416

As can be seen, Theorem 1 and Corollary 1 can provide smaller $\mathscr{H}_\infty$ attenuation levels. Therefore, it is important to note that, even mode-independent strictly proper filters designed with Corollary 1 outperform the results from Zhang and Boukas [2009b] for mode-dependent proper filters, emphasizing the superiority of the proposed approach.

Example 2

Consider the MJLS with two operation modes, taken from de Souza [2003], whose system matrices are

$$A_1 = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -0.33 \\ 1 & 1.40 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_{z1} = C_{z2} = [0\ 1], \qquad E_{z1} = E_{z2} = [0\ 0],$$
$$C_{y_1} = C_{y_2} = [1\ 0], \qquad E_{y_1} = E_{y_2} = [0\ 1],$$

with a transition probability matrix belonging to a convex polytope given by two vertices

$$\Gamma_1 = \begin{bmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{bmatrix} \qquad \text{and} \qquad \Gamma_2 = \begin{bmatrix} 0.85 & 0.15 \\ 0.30 & 0.70 \end{bmatrix}.$$

The minimum upper bound for $\mathscr{H}_\infty$ norm obtained with the mode-independent strictly proper filter designed by Theorem 3.3 from de Souza [2003] is $\gamma = 9.8247$. The result obtained by Theorem 3 from Gonçalves et al. [2011] is $\gamma = 7.8252$. Using Corollary 1, with $d = g = \lambda_1 = \lambda_2 = 0$, the value of $\mathscr{H}_\infty$ guaranteed cost is given by $\gamma = 7.8171$. This outcome can be improved increasing the degrees of the Lyapunov matrices. For instance, employing $g = 1$, Corollary 1 yields $\gamma = 7.2940$ with the following filter matrices

$$A_f = \begin{bmatrix} -1.3636 & -0.5626 \\ 2.4208 & 0.9118 \end{bmatrix}, \qquad B_f = \begin{bmatrix} -1.0678 \\ 1.9274 \end{bmatrix},$$
$$C_f = [0.0006 \;\; -0.9995].$$

Although the conditions presented in de Souza [2003] deal only with strictly proper filters, Theorem 3 from Gonçalves et al. [2011] and the proposed approach are capable to handle proper filters, which can reduce the $\mathscr{H}_\infty$ guaranteed cost. In this case, Theorem 3 from Gonçalves et al. [2011] provides a proper filter with $\gamma = 5.9681$, while Corollary 1 with $\lambda_1 = \lambda_2 = h = d = 0$ and $g = 1$ yields $\mathscr{H}_\infty$ bound equal to 5.6103, with filter matrices given by

$$A_f = \begin{bmatrix} -1.1042 & -0.4580 \\ 2.2215 & 0.9144 \end{bmatrix}, \qquad B_f = \begin{bmatrix} -0.9872 \\ 1.6916 \end{bmatrix},$$
$$C_f = [-2.3217 \;\; -0.9994], \qquad D_f = -2.3227.$$

Example 3

Consider the system borrowed from de Souza and Fragoso [2003], also investigated in Gonçalves et al. [2009], with system matrices given by

$$A_1 = \begin{bmatrix} 1 & 5.2529 \times 10^{-2} \\ 1.5146 \times 10^{-3} & 1.1022 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.9955 & 4.9660 \times 10^{-2} \\ -0.2669 & 0.8075 \end{bmatrix},$$
$$E_1 = E_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix},$$
$$C_{z1} = C_{z2} = [0 \;\; 1], \qquad E_{z1} = E_{z2} = [0 \;\; 0],$$
$$C_{y_1} = C_{y_2} = [-1 \;\; 1], \qquad E_{y_1} = E_{y_2} = [0 \;\; 1]$$

and completely known transition probability matrix

$$\Gamma = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}.$$

Regarding the design of mode-independent strictly proper filters, for this particular problem, the LMI conditions presented in de Souza and Fragoso [2003] are infeasible, while the approach in Gonçalves et al. [2009] provides an $\mathscr{H}_\infty$ guaranteed cost of 25.3786. On the other hand, Corollary 1 gets 22.5464 with $\lambda_1 = \lambda_2 = g = h = d = 0$. Better results can be achieved by performing a search of the scalar parameters, for instance, choosing $\lambda_1 = -0.37$ and $\lambda_2 = -0.50$, the $\mathscr{H}_\infty$ guaranteed cost provided by Corollary 1 is 21.3977 ($g = h = d = 0$), with filter matrices given by

$$A_f = \begin{bmatrix} 0.2121 & 0.8058 \\ 0.5960 & 0.2641 \end{bmatrix},$$
$$B_f = \begin{bmatrix} 0.9757 \\ -0.8892 \end{bmatrix}, \qquad C_f = \begin{bmatrix} 0.0414 \\ -0.7809 \end{bmatrix}^T. \tag{19}$$

To evaluate the dynamical behavior of the augmented system (4), a Monte Carlo time simulation for a total of 500 possible realizations of the Markov chain has been performed under the disturbances

$$w_1(k) = w_2(k) = e^{-0.2k} \sin(1.8k).$$

Figure 1 presents a comparison between the average error signal $e(k)$ of (4), with the filter matrices (19) and the results obtained with the approach introduced in Gonçalves et al. [2009]. It is noteworthy that Corollary 1 provides a filter that produces smaller errors than the one obtained by the condition in Gonçalves et al. [2009].

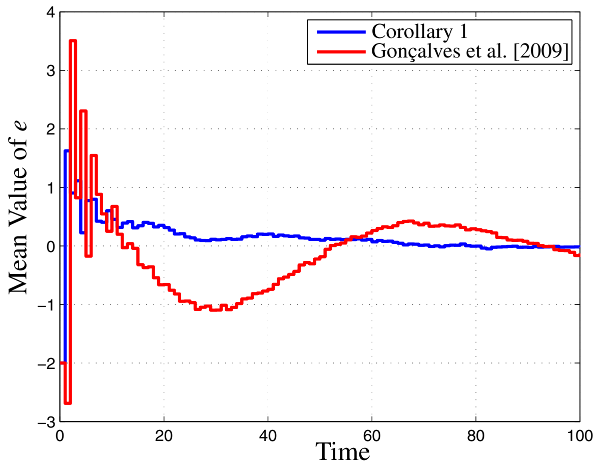

Fig. 1. Comparison between the dynamical behavior of the average error signal $e(k)$ yielded by filters obtained with Corollary 1 and the condition from Gonçalves et al. [2009] for a total of 500 possible realizations of the Markov chain .

5. CONCLUSION

The problem of $\mathscr{H}_\infty$ filtering for discrete-time MJLS with uncertain transition probability matrix was studied in this paper. By conveniently using the multi-simplex representation, different types of uncertainties in the transition probability matrix can be considered in a systematic way. Differently from other existing approaches, LMI relaxations based on homogeneous polynomials of arbitrary degrees have been used in this work. Numerical experiments illustrated the advantages of the proposed conditions, which can design filters that provide smaller $\mathscr{H}_\infty$ attenuation levels than the ones obtained through other approaches available in the literature and improve even more the results when the degrees in the decision variables are increased and searches of the scalar parameters are performed.

ACKNOWLEDGEMENTS

The work was supported by the Brazilian agencies CAPES, CNPq, and grants 2011/08312-6 and 2010/10118-0 of the São Paulo Research Foundation (FAPESP).

REFERENCES

P.-A. Bliman. An existence result for polynomial solutions of parameter-dependent LMIs. *Systems & Control Letters*, 51 (3-4):165–169, March 2004.

O. L. V. Costa and J. B. R. do Val. Full information $\mathscr{H}_\infty$-control for discrete-time infinite Markov jump parameter systems. *Journal of Mathematical Analysis and Applications*, 202(2): 578–603, September 1996.

O. L. V. Costa and M. D. Fragoso. Stability results for discrete-time linear systems with Markovian jumping parameters. *Journal of Mathematical Analysis and Applications*, 179: 154–178, 1993.

O. L. V. Costa, E. O. Assumpção Filho, E. K. Boukas, and R. P. Marques. Constrained quadratic state feedback control of discrete-time Markovian jump linear systems. *Automatica*, 35(4):617–626, April 1999.

O. L. V. Costa, M. D. Fragoso, and R. P. Marques. *Discrete-Time Markovian Jump Linear Systems*. Springer-Verlag, New York, NY, USA, 2005.

C. E. de Souza. A mode-independent $\mathscr{H}_\infty$ filter design for discrete-time Markovian jump linear systems. In *Proceedings of the 42nd IEEE Conference on Decision and Control*, volume 3, pages 2811–2816, Maui, HI, USA, December 2003.

C. E. de Souza and M. D. Fragoso. $\mathscr{H}_\infty$ filtering for discrete-time linear systems with Markovian jumping parameters. *International Journal of Robust and Nonlinear Control*, 13 (14):1299–1316, December 2003.

C. E. de Souza, K. A. Barbosa, and A. Trofino. Robust $\mathscr{H}_\infty$ filtering for discrete-time linear systems with uncertain time-varying parameters. *IEEE Transactions on Signal Processing*, 54(6):2110–2118, June 2006.

C. E. de Souza, A. Trofino, and K. A. Barbosa. Mode-independent $\mathscr{H}_\infty$ filters for Markovian jump linear systems. *IEEE Transactions on Automatic Control*, 51(11): 1837–1841, November 2006.

J. B. R. do Val, J. C. Geromel, and A. P. C. Gonçalves. The $\mathscr{H}_2$-control for jump linear systems: cluster observations of the Markov state. *Automatica*, 38(2):343–349, February 2002.

Z. S. Duan, J. X. Zhang, C. S. Zhang, and E. Mosca. Robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filtering for uncertain linear systems. *Automatica*, 42 (11):1919–1926, November 2006.

A. P. C. Gonçalves, A. R. Fioravanti, and J. C. Geromel. $\mathscr{H}_\infty$ filtering of discrete-time Markov jump linear systems through linear matrix inequalities. *IEEE Transactions on Automatic Control*, 54(6):1347–1351, June 2009.

A. P. C. Gonçalves, A. R. Fioravanti, and J. C. Geromel. Filtering of discrete-time Markov jump linear systems with uncertain transition probabilities. *International Journal of Robust and Nonlinear Control*, 21(6):613–624, April 2011.

A. P. C. Gonçalves, A. R. Fioravanti, and J. C. Geromel. $\mathscr{H}_\infty$ robust and networked control of discrete-time MJLS through LMIs. *Journal of The Franklin Institute*, 349(6):2171–2181, August 2012.

M. J. Lacerda, R. C. L. F. Oliveira, and P. L. D. Peres. Robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filter design for uncertain linear systems via LMIs and polynomial matrices. *Signal Processing*, 91(5): 1115–1122, May 2011.

M. J. Lacerda, V. J. S. Leite, R. C. L. F. Oliveira, and P. L. D. Peres. Delay-dependent robust $\mathscr{H}_\infty$ filter design for state-delayed discrete-time linear systems via homogeneous polynomial matrices. *IET Control Theory & Applications*, 7(1): 125–135, January 2013.

H. Li and Y. Shi. Robust $\mathscr{H}_\infty$ filtering for nonlinear stochastic systems with uncertainties and Markov delays. *Automatica*, 48(1):159–166, January 2012.

P. Li, J. Lam, and G. Chesi. On the synthesis of linear $\mathscr{H}_\infty$ filters for polynomial systems. *Systems & Control Letters*, 61(1):31–36, January 2012.

J. Löfberg. YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the 2004 IEEE International Symposium on Computer Aided Control Systems Design*, pages 284–289, Taipei, Taiwan, September 2004.

X. Luan, F. Liu, and P. Shi. Finite-time filtering for non-linear stochastic systems with partially known transition jump rates. *IET Control Theory & Applications*, 4(5):735–745, May 2010.

C. F. Morais, M. F. Braga, R. C. L. F. Oliveira, and P. L. D. Peres. $\mathscr{H}_2$ control of discrete-time Markov jump linear systems with uncertain transition probability matrix: Improved linear matrix inequality relaxations and multi-simplex modeling. *IET Control Theory & Applications*, 7:1665–1674, August 2013.

R. C. L. F. Oliveira and P. L. D. Peres. Parameter-dependent LMIs in robust analysis: Characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations. *IEEE Transactions on Automatic Control*, 52(7): 1334–1340, July 2007.

R. C. L. F. Oliveira, P.-A. Bliman, and P. L. D. Peres. Robust LMIs with parameters in multi-simplex: Existence of solutions and applications. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 2226–2231, Cancun, Mexico, December 2008.

C. W. Scherer. Higher-order relaxations for robust LMI problems with verifications for exactness. In *Proceedings of the 42nd IEEE Conference on Decision and Control*, pages 4652–4657, Maui, HI, USA, December 2003.

C. W. Scherer. Relaxations for robust linear matrix inequality problems with verifications for exactness. *SIAM Journal on Matrix Analysis and Applications*, 27(2):365–395, June 2005.

P. Seiler and R. Sengupta. A bounded real lemma for jump systems. *IEEE Transactions on Automatic Control*, 48(9): 1651–1654, September 2003.

J. F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1–4):625–653, 1999. `http://sedumi.ie.lehigh.edu/`.

L. Zhang and E. K. Boukas. Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities. *Automatica*, 45(2):463–468, February 2009a.

L. Zhang and E. K. Boukas. Mode-dependent filtering for discrete-time Markovian jump linear systems with partly unknown transition probabilities. *Automatica*, 45(6):1462–1467, June 2009b.