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Robust Controller Design for Feedback Systems with Uncertain Backlash and Plant Uncertainties Subject to Inputs Satisfying Bounding Conditions ⋆

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Abstract: This paper develops a method for designing unity feedback systems where an uncertain linear time-invariant (possibly distributed-parameter) plant is in cascade connection with an uncertain backlash and a controller. The design problem considered is to determine a robust controller so as to ensure that, despite plant uncertainties, the error and the controller output stay within prescribed bounds for all time and for all inputs whose magnitude and whose slope satisfy given bounding conditions. In essence, the backlash is replaced with a constant gain and a bounded disturbance, thereby resulting in an auxiliary uncertain linear system. Then, by applying the multi-valued version of the fixed-point theorem and the extended version of the theory of majorants, we derive a practical condition in the form of inequalities that can be solved in practice. Further we show that if such inequalities are satisfied for a chosen nominal system, then the original design problem is solved. The usefulness of the method is illustrated by a design example where the plant has a time-delay.

Keywords: Robust control, nonlinear systems, backlash, computer-aided control system design, process control, theory of majorants.

1. INTRODUCTION

Backlash exists in many practical applications, and it has long been known that it can severely limit system performance. Moreover, the model of a backlash is often not known accurately, and the plant often has uncertainties in its parameters.

There are several ways to alleviate, or ideally eliminate, undesirable effects of backlash on the performance of the system. One among them is to design an adaptive controller for the system (see, e.g., Tao and Kokotovic 1995). However, this approach often results in a complicated controller. An alternative way is to design a robust controller where the uncertainty of the backlash is taken into account (see, e.g., Barreiro and Baños 2006). The advantage of this method is that although there may be a certain amount of conservatism, the controller obtained (if exists) is much simpler and easier to implement.

The purpose of this paper is to develop a computational method for designing a robust controller for feedback systems in which an uncertain backlash model and the plant uncertainties are explicitly taken into account in the design formulation. Moreover, in the formulation, all inputs that can happen or are likely to happen in practice are explicitly taken into account as a set of functions whose magnitude and whose slope satisfy bounding conditions.

Specifically, the paper considers the feedback control system displayed in Fig. 1, which is described by

$$\left.\begin{array}{l} u = g_c * e \\ e = f - u_s * g_p = f - \psi(u) * g_p,\ G_p(s) \in \mathcal{G}_p \end{array}\right\}, \qquad (1)$$

where ψ is a backlash, $g_p : [0,\infty) \to \mathbb{R}$ represents the plant (which can be either lumped or distributed) and has Laplace transform equal to $G_p(s)$, $g_c : [0,\infty) \to \mathbb{R}$ represents the controller and has Laplace transform equal to $G_c(s, \mathbf{p})$, and $\mathbf{p} \in \mathbb{R}^n$ denotes a design parameter vector. Assume that the plant has uncertainties such that $G_p(s)$ is known to belong to a set $\mathcal{G}_p$. As usual, the asterisk denotes the convolution; that is,

$$(g_c * e)(t) = \int_0^t g_c(t-\tau)e(\tau)d\tau, \qquad t \geq 0. \qquad (2)$$

The backlash $\psi(\cdot)$ in the system (1) is described by the uncertain band model (Barreiro and Baños 2006)

$$\left.\begin{array}{l} \psi(x) = Kx + n(x) \\ n(x) = [-h, h] \quad \forall x \end{array}\right\}, \qquad (3)$$

where K is a constant gain and $n(\cdot)$ denotes the interval valued function mapping $\mathbb{R}$ to $2^{\mathbb{R}}$ (see Fig. 2). From the above, it follows that for the backlash with bandwidth γ,

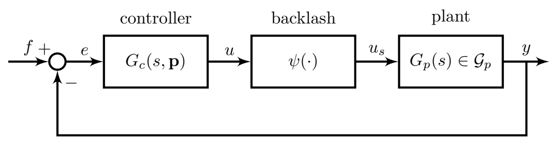

Fig. 1. Uncertain feedback system with backlash

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$$||n(\cdot)||_\infty \le h \quad \text{where} \quad h = K\gamma. \tag{4}$$

Now assume that the input f is known only to the extent that it belongs to a *possible set* $\mathcal{P}$ (defined as the set of inputs that can or are likely to happen in practice). For clarity, assume that the possible set $\mathcal{P}$ considered throughout the paper is defined by

$$\mathcal{P} \triangleq \{f : \mathbb{R}_+ \to \mathbb{R} |\ ||f||_\infty \le M, ||\dot{f}||_\infty \le D\}, \tag{5}$$

where the bounds M and D are given. However, it is important to note that the method to be developed is applicable to any possible set of bounded signals. For different ways of characterizing the possible set and the detailed discussion on this, see Zakian (2005); Silpsrikul and Arunsawatwong (2010) and the references therein.

The design problem considered in the paper is to determine a design parameter $\mathbf{p}$ (or equivalently the controller transfer function $G_c(s, \mathbf{p})$) such that the following design criteria are satisfied:

$$\sup_{G_p \in \mathcal{G}_p} \hat{e} \le E_{max} \quad \text{and} \quad \sup_{G_p \in \mathcal{G}_p} \hat{u} \le U_{max}, \tag{6}$$

where the bounds E_{max} and U_{max} are given. The numbers $\hat{e}$ and $\hat{u}$ are sometimes called the peak error and the peak controller output, respectively, for the possible set $\mathcal{P}$ and defined by

$$\hat{e} \triangleq \sup_{f \in \mathcal{P}} \|e\|_\infty \quad \text{and} \quad \hat{u} \triangleq \sup_{f \in \mathcal{P}} \|u\|_\infty. \tag{7}$$

Clearly, $\hat{e}$ and $\hat{u}$ depend upon $\mathbf{p}$ and the plant $G_p(s)$.

Note further that the criteria (6) are equivalent to

$$\left.\begin{array}{l} |e(t,f)| \le E_{max},\ \forall f \in \mathcal{P}\ \forall t \in \mathbb{R}_+\ \forall G_p(s) \in \mathcal{G}_p \\ |u(t,f)| \le U_{max},\ \forall f \in \mathcal{P}\ \forall t \in \mathbb{R}_+\ \forall G_p(s) \in \mathcal{G}_p \end{array}\right\}. \tag{8}$$

Once (6) are satisfied, one can ensure that despite the plant uncertainties, both $|e(t, f)|$ and $|u(t, f)|$ lie within the respective bounds E_{max} and U_{max} for all time t and for all inputs $f \in \mathcal{P}$.

In connection with the criteria (6) or (8), note in passing that the design criterion of the form

$$\hat{v} \le \varepsilon \quad (\hat{v} \triangleq \sup_{f \in \mathcal{P}} \|v\|_\infty), \tag{9}$$

where v is a system's response and ε is a given bound, captures accurately the real meaning of control and has been used for monitoring the performance of control systems in practice. Furthermore, it has long been investigated by many researchers (see, e.g., Birch and Jackson 1959; Zakian 1979, 1989, 1991, 1996, 2005; Silpsrikul and Arunsawatwong 2010 and the references therein). By applying the approach used in Mai et al. (2011) (see also Mai 2010), this work can be considered as the extension of Zakian's principle of matching (Zakian 1996, 2005) to the case of nonlinear systems (1) in which the nonlinearity $\psi(\cdot)$ is the uncertain backlash (3).

It should be noted that, the uncertain band model (3) is useful in the sense that the backlash width does not need to be known exactly (see Fig. 2). Hence, the work presented here can be used to find a robust controller to compensate the backlash effect which may not be known accurately. In addition, the uncertain model can be used to represent some other backlash models such as a friction-driven hysteresis model (see Fig. 3).

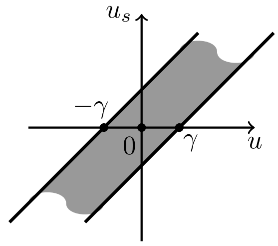

Fig. 2. The uncertainty model of backlash.

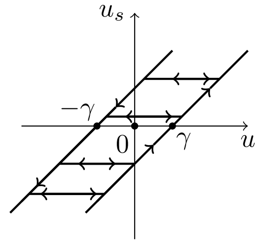

Fig. 3. The friction-driven hysteresis model of backlash.

The key tools used in the paper are the multivalued version of the fixed point theorem which is known as Kakutani's theorem (see, e.g., Barreiro and Baños 2006) and the extended version of the theory of majorants (Zakian 1984, 2005; Mai 2010; Mai et al 2011). By decomposing the backlash as a constant gain and a bounded disturbance (Oldak et al. 1994), we obtain an auxiliary linear system with uncertainties. Then, by applying the fixed-point theorem and the theory of majorants, we derive a practical condition in the form of inequalities that can be solved in practice. Further we show that if such inequalities are satisfied for a chosen nominal system, then the original design problem is solved.

The organization of the paper is as follows. Section 2 recapitulates the version of the theory of majorants that was extended by Mai et al. (2011). Section 3 derives the surrogate design criteria for (6), thereby providing practical inequalities for designing the system (1) so that the original criteria (6) are satisfied; this is indeed the main contribution of the paper. Section 4 provides the stability conditions for ensuring that the associated performance measures are finite, so as to enable a numerical algorithm to search for a design solution in the space of design parameters. To illustrate the usefulness and the potential of the developed method, a design example for a time-delay plant is carried out in Section 5. Finally, the discussion and conclusions are given.

2. THEORY OF MAJORANTS FOR UNCERTAIN LINEAR SYSTEMS

The theory of majorants (Zakian 1983, 1984, 1996, 2005) has been used for the design of robust control systems in Taiwo (2005, 1986); Bada (1987). The theory have been extended by Mai et al. (2011) (see also Mai (2010)) in a straightforward manner to the case of uncertain linear feedback systems with inputs f and d (see below). The following summarizes the theory to be used in Section 3.

Consider the uncertain linear system that is described by

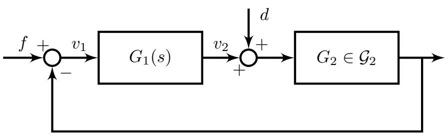

Fig. 4. An uncertain linear system with two inputs.

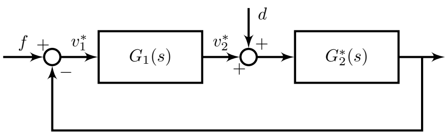

Fig. 5. The nominal system for the system (10).

$$\left.\begin{array}{l} v_2 = v_1 * g_1 \\ v_1 = f - g_2 * (d + v_2), \quad G_2(s) \in \mathcal{G}_2 \end{array}\right\}, \tag{10}$$

where $G_2(s)$ belongs to a set of plant transfer functions $\mathcal{G}_2$ (see Fig. 4). As before, the Laplace transforms of g_1 and g_2 are $G_1(s)$ and $G_2(s)$, respectively. The inputs f and d are assumed to belong to the sets $\mathcal{P}$ and $\mathcal{D}$, respectively, where

$$\mathcal{D} \triangleq \{d \in \mathbb{L}_\infty | ||d||_\infty \leq h\}. \tag{11}$$

Let the nominal system be obtained by replacing $G_2(s)$ in the system (10) by a fixed transfer function $G_2^*(s) = \mathcal{L}[g_2^*]$ (see Fig. 5). Then, the nominal system is given by

$$\left.\begin{array}{l} v_2^* = v_1^* * g_1 \\ v_1^* = f - g_2^* * (d + v_2^*) \end{array}\right\}. \tag{12}$$

Define the peak outputs for the systems (10) and (12) as follows.

$$\left.\begin{array}{l} \hat{v}_i \triangleq \sup\limits_{f \in \mathcal{P},\, d \in \mathcal{D}} ||v_i||_\infty \\ \hat{v}_i^* \triangleq \sup\limits_{f \in \mathcal{P},\, d \in \mathcal{D}} ||v_i^*||_\infty \end{array}\right\}, \quad i = 1, 2. \tag{13}$$

In relation to the nominal system (12), the following theorem provides useful sufficient conditions for ensuring

$$\sup_{G_2 \in \mathcal{G}_2} \hat{v}_i \leq V_i \quad (i = 1, 2), \tag{14}$$

where the bounds V_1 and V_2 are given.

Theorem 1. (Mai et al. 2011). *Consider the nominal system (12). Let $v_i^*(\mathbf{1})$ be the waveform of v_i^* in response to f being the unit step function $\mathbf{1}$ and $d = 0$, and let $v_i^*(t, \mathbf{1})$ be the value of $v_i^*(\mathbf{1})$ at time t. For a given $G_2^*(s)$, let $z \triangleq g_2 - g_2^*$ and define*

$$\tilde{\mu}_i \triangleq A|\sigma_i| + B||v_i^*(\mathbf{1}) - \sigma_i||_1, \quad \sigma_i = \lim_{t \to \infty} v_i^*(t, \mathbf{1}), \tag{15}$$

where

$$\left.\begin{array}{l} A = \sup\{||z||_1 : \ G_2 \in \mathcal{G}_2\} \\ B = \sup\{|z(0)| + ||\dot{z}||_1 : \ G_2 \in \mathcal{G}_2\} \end{array}\right\}. \tag{16}$$

Suppose that the system (12) is BIBO stable, and let $\tilde{\mu}_1 < \infty$ and $\tilde{\mu}_2 < 1$. The design criteria (14) are satisfied if, for $i = 1, 2$,

$$\hat{\phi}_i \leq V_i, \quad \text{where } \hat{\phi}_i \triangleq \frac{\hat{v}_i^* + \tilde{\mu}_i h}{1 - \tilde{\mu}_2}. \tag{17}$$

Now let $G_1(s) = G_1(s, \mathbf{p})$ be characterized by a design parameter vector $\mathbf{p}$. Zakian (1984, 1996, 2005) advocates that in solving the inequalities (17) by numerical methods, the numbers v_i^*, σ_i and $\tilde{\mu}_i$ have to be computed repeatedly for different values of $\mathbf{p}$. However, it is clear from (16) that the numbers A and B do not depend on $G_1(s)$ and thus need to be computed only once.

From the above, one can see for a chosen nominal transfer function $G_2^*(s)$, the condition (17) provides useful inequalities for determining $G_1(s)$ by numerical methods so that the criteria (14) are satisfied.

3. DESIGN OF UNCERTAIN NONLINEAR SYSTEM

This section develops the design criteria in the form of inequalities that can be solved by numerical methods for the uncertain nonlinear system (1) to satisfy (6). Indeed, Theorems 2 and 3 are the main results of the paper.

Assumption 1. *For every input $f \in \mathcal{P}$, there exists at least a solution (e, u) that satisfies (1), where $e : \mathbb{R}_+ \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$. Assume that all initial conditions are zero for $t \leq 0$.*

Assumption 2. *Let the backlash $\psi(\cdot)$ be represented by the uncertain band model (3).*

By using the decomposition technique (Oldak et al. 1994), the backlash is replaced by a constant gain and a bounded disturbance as follows.

$$\psi(u) = Ku + d, \tag{18}$$

where $d \in \mathcal{D}$ and the set $\mathcal{D}$ is defined by

$$\mathcal{D} \triangleq \{d \in \mathbb{L}_\infty | \ ||d||_\infty \leq h\}. \tag{19}$$

Thus, we obtain the auxiliary system displayed in Fig. 6 and described by

$$\left.\begin{array}{l} u' = g_c * e' \\ e' = f - g_p * [Ku' + d], \ G_p(s) \in \mathcal{G}_p \end{array}\right\}, \tag{20}$$

where $f \in \mathcal{P}$. Note that the replacement (18) is valid when $\hat{u}$ is finite. See more details on this in Nguyen and Arunsawatwong (2013).

Next, define the peak values of e' and u' for each $G_p(s) \in \mathcal{G}_p$ as follows.

$$\hat{e}' \triangleq \sup_{f \in \mathcal{P},\, d \in \mathcal{D}} ||e'||_\infty, \quad \text{and} \quad \hat{u}' \triangleq \sup_{f \in \mathcal{P},\, d \in \mathcal{D}} ||u'||_\infty. \tag{21}$$

In connection with the system (20), let g be the impulse response of the transfer function from d to u'; that is,

$$G(s) \triangleq G_p(s)G_c(s, \mathbf{p})[1 + KG_p(s)G_c(s, \mathbf{p})]^{-1}. \tag{22}$$

Let $\mathcal{A}$ denote the set of the impulse responses of all BIBO stable transfer functions, which has the form

$$g(t) = \begin{cases} g_a(t) + \sum\limits_{i=0}^{\infty} g_i \delta(t - t_i), & t \geq 0 \\ 0, & t < 0 \end{cases}, \tag{23}$$

where $\delta(\cdot)$ denotes the Dirac delta function, $0 \leq t_0 < t_1 < t_2 ...$ are constants, $\sum_{i=0}^{\infty} |g_i| < \infty$ and $\int_0^\infty |g_a(t)| dt < \infty$. See Desoer and Vidyasagar (1975) for more details on $\mathcal{A}$.

Assumption 3. *For $G(s)$ defined by (22), the impulse response g satisfies the conditions that $g \in \mathcal{A}$ and $\dot{g} \in \mathcal{A}$ for every $G_p(s) \in \mathcal{G}_p$.*

It is worth noting that, by virtue of the convolution representation, $G_p(s)$ can be either a lumped- or distributed-parameter plant provided that Assumption 3 holds.

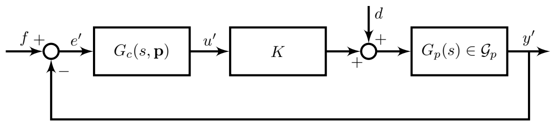

Fig. 6. The auxiliary linear system for the system (1).

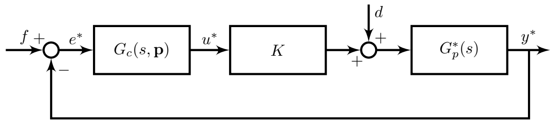

Fig. 7. The nominal system for the uncertain system (20).

The following theorem reveals that a design solution associated with the auxiliary system (20) is also a solution to the original design problem (6).

Theorem 2. *Consider the system* (1) *where* $\hat{u} < \infty$. *Let Assumptions 1, 2 and 3 hold. The original design criteria* (6) *are satisfied if, for the auxiliary system* (20), *the following conditions hold:*

$$\sup_{G_p \in \mathcal{G}_p} \hat{e}' \le E_{\max}, \quad \sup_{G_p \in \mathcal{G}_p} \hat{u}' \le U_{\max}. \tag{24}$$

Proof By the application of Kakutani's theorem (see, e.g., Barreiro and Baños 2006), the theorem follows. The detail can be found in Nguyen's (2014) thesis. □

From Theorem 2, it readily follows that a solution to the inequalities (24), which is associated with the uncertain linear system (20), is also a solution to the inequalities (6). However, the inequalities (24) are not suitable for solution by numerical methods because the computation of $\sup_{G_p \in \mathcal{G}_p} \hat{e}'$ and $\sup_{G_p \in \mathcal{G}_p} \hat{u}'$ is intractable. Therefore, (24) will be replaced by readily computable inequalities to be derived by using Theorem 1.

Consider the nominal system shown in Fig. 7 where $f \in \mathcal{P}, d \in \mathcal{D}$ and $G_p^*(s)$ denotes the nominal transfer function for $G_p(s) \in \mathcal{G}_p$.

Assume that the nominal system is BIBO stable. Consequently, the following limits exist

$$\sigma_1 \triangleq \lim_{t\to\infty} e^*(t, \mathbf{1}), \quad \sigma_2 \triangleq \lim_{t\to\infty} u^*(t, \mathbf{1}), \tag{25}$$

where $e^*(t, \mathbf{1})$ and $u^*(t, \mathbf{1})$ are the values of e^* and u^* at the time t in response to the inputs $f = \mathbf{1}(t)$ and $d(t) = 0$. Define

$$\left.\begin{array}{l} \tilde{\mu}_1 \triangleq A|\sigma_1| + B||e^*(\mathbf{1}) - \sigma_1||_1 \\ \tilde{\mu}_2 \triangleq A|\sigma_2| + B||u^*(\mathbf{1}) - \sigma_2||_1 \end{array}\right\}, \tag{26}$$

where

$$\left.\begin{array}{l} A = \sup\{||z||_1 : G_p \in \mathcal{G}_p, \quad z = g_p - g_p^*\} \\ B = \sup\{|z(0)| + ||\dot{z}||_1 : G_p \in \mathcal{G}_p\}. \end{array}\right\}. \tag{27}$$

Let $\hat{e}^*$ and $\hat{u}^*$ denote the peak values of e^* and u^* and be given by

$$\hat{e}^* \triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} ||e^*||_\infty, \quad \hat{u}^* \triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} ||u^*||_\infty. \tag{28}$$

Sufficient conditions for ensuring the satisfaction of the inequalities (24) are stated below in terms of $\hat{e}^*$ and $\hat{u}^*$.

Theorem 3. *Consider the system* (1) *where* $\hat{u} < \infty$. *Let Assumptions 1, 2 and 3 hold. Assume that the nominal system in Fig. 7 is BIBO stable and that* $\tilde{\mu}_1$ *and* $\tilde{\mu}_2$ *defined in* (26) *are finite. The inequalities* (24) *for the auxiliary system* (20), *and hence the criteria* (6), *are satisfied if* $\tilde{\mu}_2 < 1$ *and if*

$$\left.\begin{array}{l} \hat{\phi}_e \le E_{\max}, \quad \text{where } \hat{\phi}_e \triangleq \dfrac{\hat{e}^* + \tilde{\mu}_1 h}{1 - \tilde{\mu}_2} \\ \hat{\phi}_u \le U_{\max}, \quad \text{where } \hat{\phi}_u \triangleq \dfrac{\hat{u}^* + \tilde{\mu}_2 h}{K(1 - \tilde{\mu}_2)} \end{array}\right\}. \tag{29}$$

Proof By using Theorem 1, the theorem readily follows. The detail can be found in Nguyen's (2014) thesis. □

For the possible set $\mathcal{P}$ in (5), the formulae for computing $\hat{\phi}_e$ and $\hat{\phi}_u$ are given in the following. Since $\hat{e}^*$ and $\hat{u}^*$ are the peak outputs of the nominal linear system, which has no uncertainty, one can easily deduce by the principle of superposition that

$$\hat{e}^* = \phi_{ef} + h||e_d^*(\delta)||_1, \quad \hat{u}^* = \phi_{uf} + h||u_d^*(\delta)||_1, \tag{30}$$

where $e_d^*(\delta)$ and $u_d^*(\delta)$ are the responses of e^* and u^*, respectively, subject to the inputs $d(t) = \delta(t)$ and $f(t) = 0$, and

$$\phi_{ef} \triangleq \sup_{f\in\mathcal{P},\, d=0} ||e^*||_\infty, \quad \phi_{uf} \triangleq \sup_{f\in\mathcal{P},\, d=0} ||u^*||_\infty. \tag{31}$$

It should be noted that ϕ_{ef} and ϕ_{uf} can be computed by using method developed in Silpsrikul and Arunsawatwong (2010). Therefore, the peak values $\hat{e}^*$ and $\hat{u}^*$ can be readily obtained in practice.

From the above, it is easy to deduce that

$$\left.\begin{array}{l} \hat{\phi}_e = (h||e_d^*(\delta))||_1 + \phi_{ef} + h\tilde{\mu}_1)/(1 - \tilde{\mu}_2) \\ \hat{\phi}_u = (hK||u_d^*(\delta))||_1 + K\phi_{uf} + h\tilde{\mu}_2)/K(1 - \tilde{\mu}_2) \end{array}\right\}. \tag{32}$$

Therefore, the design problem now becomes the problem of determining a design parameter $\mathbf{p}$ satisfying

$$\left.\begin{array}{l} \hat{\phi}_e(\mathbf{p}) \le E_{\max} \\ \hat{\phi}_u(\mathbf{p}) \le U_{\max} \end{array}\right\}, \quad \text{subject to } \tilde{\mu}_2(\mathbf{p}) < 1. \tag{33}$$

It should be noted that, if there is no pole-zero cancellation between $G_c(s, \mathbf{p})$ and $G_p^*(s)$, $\tilde{\mu}_2(\mathbf{p}) < 1$ implies that $\tilde{\mu}_1(\mathbf{p})$ is finite.

From the above discussion, it is easy to see that $\hat{\phi}_e(\mathbf{p})$, $\hat{\phi}_u(\mathbf{p})$ and $\tilde{\mu}_i(\mathbf{p})$ can be readily computed in practice. Thus, the inequalities (33) are to be used instead of (6) and hence are called the *surrogate design criteria*.

4. STABILITY CONDITIONS

In solving the design inequalities (33) by numerical methods, a search algorithm needs to start from a *stability point* (that is, a point $\mathbf{p}$ for which the associated performance measures $\hat{\phi}_e$ and $\hat{\phi}_u$ are finite).

It is important to note that Theorems 2 and 3 require the assumption that $\hat{u}$ is finite for all $G_p(s) \in \mathcal{G}_p$ in order to guarantee the validation of the decomposition of ψ. Hence, a stability condition that ensures the finiteness of $\sup_{G_p \in \mathcal{G}_p} \hat{e}$ and $\sup_{G_p \in \mathcal{G}_p} \hat{u}$ is needed.

The following reveals that the BIBO stability of the auxiliary linear system (20) implies that of the original nonlinear system (1).

Theorem 4. *Consider the nonlinear system* (1). *Let Assumptions 1, 2 and 3 hold. Define the composite transfer* *function* $\hat{G}(s) \triangleq G_c(s,\mathbf{p})G_p(s)$. *For every* $G_p(s) \in \mathcal{G}_p$, *let* $\hat{G}(s)$ *be strictly proper and suppose that* $g_c \in \mathcal{A}$ *where* g_c *is the impulse response of* $G_c(s,\mathbf{p})$. *Then it follows that the responses* u *and* e *are bounded for any* $f \in \mathcal{P}$.

Proof Note that Assumption 3 implies that the transfer function $G(s) = G_p(s)G_c(s,\mathbf{p})[1 + KG_p(s)G_c(s,\mathbf{p})]^{-1}$ is BIBO stable for every $G_p(s) \in \mathcal{G}_p$. Then by applying a Lemma in Mai (2010) and Barreiro and Baños's (2006) result to the uncertain system (1), the theorem is obtained. The detail can be found in Nguyen's (2014) thesis. □

From Theorem 4, it readily follows that to ensure the finiteness of $\sup_{G_p\in\mathcal{G}_p} \hat{e}$ and $\sup_{G_p\in\mathcal{G}_p} \hat{u}$ for the system (1), one needs to determine the controller $G_c(s,\mathbf{p})$ that makes the transfer function $G(s) = G_p(s)G_c(s,\mathbf{p})[1 + KG_p(s)G_c(s,\mathbf{p})]^{-1}$ BIBO stable for all $G_p(s) \in \mathcal{G}_p$.

For retarded delay differential systems (which of course includes rational systems), it is well known (see, e.g., Arunsawatwong (1996) and the references therein) that the system (or alternatively its transfer function) is BIBO stable if and only if all the characteristic roots (or alternatively all the poles of the transfer function) have negative real parts. Let $f(s)$ be the characteristic function of a retarded delay differential system. Let ϕ_0 denote the abscissa of stability of $f(s)$ defined by

$$\phi_0 \triangleq \sup\{\mathrm{Re}(s) : \ f(s) = 0\}.$$

Then it follows that the system is BIBO stable if

$$\phi_0(\mathbf{p}) \leq -\varepsilon \quad \text{where} \quad 0 < \varepsilon \ll 1. \tag{34}$$

It should be noted that the inequality (34) can be used in practice to determine a value of $\mathbf{p}$ for which the system is BIBO stable, by numerical methods. For further details, see Zakian and Al-Naib (1973); Arunsawatwong (1994, 1996); Zakian (2005). In this work, the abscissa of stability for retarded delay differential systems is computed by using the method developed in Arunsawatwong (1994, 1996).

5. NUMERICAL EXAMPLE

In this section, we carry out a case study in process control system where the plant has time delay.

As shown in Hägglund (2007); Choudhury et al. (2005), in process control systems, wear (or erosion) leads to the appearance of the backlash phenomenon in the linkage mechanism in the positioner and actuator of valves. According to Hägglund (2007), it is reported that a backlash of 10% increases the peak error at load disturbance with 50%. When the backlash becomes too large, the valve needs to be replaced. However, replacing the valve cannot be done without interrupting the process. For this and for economical reasons, it may be better to take into account the backlash phenomenon in the controller design process.

Consider an uncertain plant with time-delay whose transfer function is described by

$$G_p(s) = \frac{ae^{-0.2s}}{s^2 + bs + 2}, \quad a \in [2.5, 3.5], \quad b \in [2.9, 3.1]. \tag{35}$$

The backlash is assumed to have a unity slope and its bandwidth is only known to the extent that it is in the interval (0,0.1]. Assume that the control objective is to ensure that both $|e(t,f)|$ and $|u(t,f)|$ stay within the respective bounds 0.2 and 10 for all time and for all inputs $f \in \mathcal{P}$ where

$$M = 1 \quad \text{and} \quad D = 0.1. \tag{36}$$

Accordingly, the design criteria are expressed as

$$\hat{\phi}_0(\mathbf{p}) \leq -0.01, \ \tilde{\mu}_2(\mathbf{p}) \leq 0.5, \ \hat{\phi}_e(\mathbf{p}) \leq 0.2, \ \hat{\phi}_u(\mathbf{p}) \leq 10, \tag{37}$$

where

$$\hat{\phi}_0 = \sup\{\phi_0 : a \in [2.5, 3.5], b \in [2.9, 3.1]\}. \tag{38}$$

From (32), one can see that if $\tilde{\mu}_2(\mathbf{p})$ is close to 1, then the values of $\hat{\phi}_e(\mathbf{p})$ and $\hat{\phi}_u(\mathbf{p})$ become very large. Therefore, any $\mathbf{p}$ for which $\tilde{\mu}_2(\mathbf{p})$ is close to 1 is not likely to yield a design solution.

The nominal plant transfer function $G_p^*(s)$ is chosen with

$$a = 3 \text{ and } b = 3. \tag{39}$$

According to Assumption 3 and Theorem 4, it is required for every $a \in [2.5, 3.5]$ and every $b \in [2.9, 3.1]$ that $G(s)$ (see (22)) be strictly proper and BIBO stable and that the composite transfer function $\hat{G}(s)$ be strictly proper. All these requirements can be fulfilled for this example when $G_c(s,\mathbf{p})$ is chosen to be a proper transfer function.

In order to compute $\hat{\phi}_e$ and $\hat{\phi}_u$, the impulse responses of the system need to be computed repeatedly for many different values of $\mathbf{p}$ (see Section 3). In this example, we use the efficient and reliable algorithms described in Arunsawatwong (1998, 1994), which are based on Zakian I_{MN} approximations, to evaluate such responses. Furthermore, the design inequalities are solved by using a numerical search algorithm called the moving boundaries process (MBP) (Zakian and Al-Naib 1973; Zakian 1996, 2005).

After exhaustive searches with first order controllers, no solution was found. Thus, a second-order controller which has the form

$$G_c(s,\mathbf{p}) = \frac{p_1(s^2 + p_2 s + p_3)}{s^2 + p_4 s + p_5} \tag{40}$$

is to be tried where $\mathbf{p} = [p_1, p_2, p_3, p_4, p_5]^T \in \mathbb{R}^5$ is the design parameter to be determined.

After a number of iterations, a design solution $\mathbf{p}$ is found where

$$\mathbf{p} = [6.770, 2.860, 1.901, 12.127, 0.010]^T \tag{41}$$

and the corresponding performance measures are

$$\begin{gathered} \hat{\phi}_0(\mathbf{p}) = -0.969, \ \tilde{\mu}_2(\mathbf{p}) = 0.273, \\ \hat{\phi}_e(\mathbf{p}) = 0.199, \ \hat{\phi}_u(\mathbf{p}) = 1.191. \end{gathered} \tag{42}$$

To verify the performance of the obtained controller, a test input $\hat{f}$ is generated randomly so that its magnitude and its slope do not exceed the bounds given in (36). The waveform of $\hat{f}$ and the corresponding responses of the system are shown in Fig. 8. The backlash model used in the simulation is the friction-driven hysteresis model, which is one particular case included in the uncertainty model (see Barreiro and Baños 2006). To illustrate the uncertainty of the backlash, sufficiently many different values of the bandwidth of the backlash is used in the range $(0, 0.1]$. The simulation results clearly show that the error and the controller output responses stay within the specified bounds.

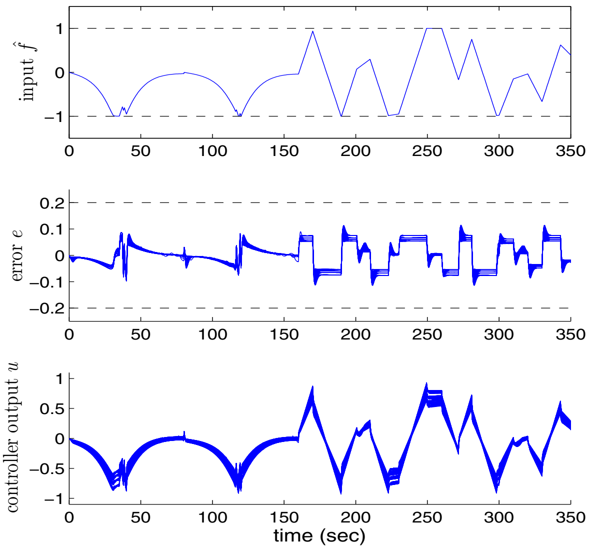

Fig. 8. Simulation results with the controller (40) and the backlash with sufficiently many different values of bandwidth in the range $(0, 0.1]$ and the parameters $a = 2.5(0.1)3.5$, $b = 2.90(0.02)3.10$

6. DISCUSSION AND CONCLUSIONS

This paper has developed a practical method for designing the feedback control system (1). The control objective is to ensure that the error e and the controller output u stay within the prescribed ranges $\pm E_{\max}$ and $\pm U_{\max}$, respectively, for all time and for all inputs $f \in \mathcal{P}$ in the presence of uncertainties appearing in both the backlash and the plant models. The backlash is decomposed using the technique due to Oldak et al. (1994). By using Kakutani's theorem (e.g., Barreiro and Baños 2006) in conjunction with the extension of the theory of majorants in Mai et al. (2011), the paper has developed computationally tractable design inequalities (33) associated with a chosen nominal system. Since the nominal system is linear and has no uncertainty, the performance measures are readily computed by known methods. The method developed in this work is applicable to both lumped- and distributed-parameter plants as long as Assumption 3 holds.

The method developed in the paper is applied to the design of a robust controller for process control systems where backlash characteristics appear in the linkage mechanism in the actuator of valves. Taking the uncertain backlash into account is an advantageous alternative solution since replacing the valve in which the backlash becomes too large is not feasible all the time, especially when the system is in operation. The simulation results have illustrated the usefulness of the proposed method.

It is interesting to note (Barreiro and Baños 2006) that, unlike design methods based on a direct cancellation that requires the backlash to be at the plant input, the approach used in developing the design method in the paper can be applied to the cases where the backlash is at the output of the plant (see Mai et al. 2010).

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