Cramer-Rao Lower Bound for a Nonlinear Filtering Problem with Multiplicative Measurement Errors and Forcing Noise

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Abstract: The recurrence algorithms for the Cramer-Rao lower bound for a discrete-time nonlinear filtering problem in the conditions when a forcing noise, measurement errors and initial covariance matrix depend on the state vector to be estimated are derived. It is assumed that the state vector being estimated includes a subvector of time-invariant unknown parameters. Some examples are given to illustrate the applicability of the algorithms obtained.

Keywords: nonlinear filtering, multiplicative measurement errors, multiplicative forcing noise unknown parameters, accuracy, Cramer lower bound.

1. INTRODUCTION

When efficient algorithms for processing of measurement data are developed in the context of the Bayesian filtering theory, it is a common practice for researchers to solve two problems: the problem of the analysis of the potential accuracy obtained using the algorithm, optimal in the sense of the chosen criterion, and the problem of design of a computationally economical algorithm that provides accuracy close to potential. Such an approach is widely used, in particular, for the development of algorithms for navigation data processing and tracking problems (Dmitriev and Stepanov 1998, Bergman 2001, Ristic et al. 2004). The covariance matrix of estimation errors of the optimal algorithm is conventionally used as a characteristic of the potential accuracy. This matrix is determined by simulation which involves the procedure for calculating the optimal estimate. It is well known that, generally, it is impossible to design a universal and computationally convenient optimal algorithm for the problems of nonlinear filtering. Despite the fact that researchers have advanced in designing such algorithms recently due to, in particular, the application of various modifications of the Monte Carlo method (Doucet 2001, Gustafsson et al. 2002, Ristic 2004) the calculation of optimal estimates by these methods is computationally intensive (Snyder 2008, Stepanov and Berkovskiy 2011). In this regard, the development of approximate procedures for the analysis of potential accuracy of estimation is vitally important for the solution of applied problems. One of such procedures is based on the calculation of the Cramer-Rao lower bound (CRLB) (Van Trees 1968).

The methods of obtaining algorithms for CRLB calculation and their application in nonlinear filtering problems have been the subject matter of many publications (Galdos 1980, Van Trees and Bell 2007). For example, in (Koshaev and Stepanov 1997, Tichavsky et al. 1998, Simandl et al. 2001), the authors obtained convenient recurrence algorithms for CRLB calculation for discrete-time nonlinear filtering problems with additive measurement errors and forcing (process) noise in the equations for the state vector. These algorithms have been successfully used to solve a wide range of problems related in particular to the processing of navigation data (Dmitriev and Stepanov 1998, Bergman 1999, 2001, Batista et al. 2013). However, in practice, there is often a need to solve problems in which the properties of forcing noise and measurement errors depend on the unknown state vector to be estimated, thus endowing them multiplicative nature. It is to this problem that the paper is devoted. Actually, we continue the research reported in (Stepanov et al. 2013). Here we suppose that not only properties of a forcing noise depend on the unknown state vector, but such dependency is also valid for measurement noise and the initial covariance matrix. These generalizations are very important in estimating the parameters of Markov random processes, widely used in the problems of navigation and tracking data filtering.

2. PROBLEM STATEMENT

Let us assume that we have composite \( n + r \) -dimensional vector \( \bar{x}_i = (x_i^T, \theta^T)^T \), which includes \( n \) -dimensional Markov sequence \( x_i = (x_{i1}, x_{i2}, ..., x_{in})^T \) and \( r \) -dimensional vector \( \theta = (\theta_1, ..., \theta_r)^T \) of unknown time-invariant parameters described by the following equations:

\[
x_i = \Phi_i \{x_{i-1}, \theta\} + \Gamma_i \{x_{i-1}, \theta\} \omega_i,
\]

and we also have \( m \) -dimensional measurements

\[
y_i = s_i \{x_i, \theta\} + \psi_i \{\theta\} \nu_i.
\]
Here $\Phi_i(x_i, \theta)$ are the known nonlinear vector-functions of $n$ and $m$ dimensions; $\Gamma_i(x_i, \theta), \Psi_i(\theta)$ are the known matrices of $n \times p$ and $m \times m$ dimensions, the elements of which are nonlinear functions of their arguments; $w_i$ and $v_i$ are white-noise zero-mean Gaussian sequences of $p$ and $m$ dimensions, for which the relations $E[w_i w_i^T] = \delta_{ik} Q_i, E[v_i v_i^T] = \delta_{ik} R_i$ hold; $Q_i$ and $R_i$ are covariance matrices; $\delta_{ik}$ is the Kronecker operator; $0, x_0$ are random vectors with the known probability density function (PDF) $f(x_0, \theta) = f(x_0 f(\theta))$, where $f(x_0 f(\theta))$ is Gaussian, i.e., $f(x_0 f(\theta)) = N(x_0; 0, P_0(\theta))$, with $E_{x_0}\{x_0\} = 0, E_{x_0}\{x_0 x_0^T\} = P_0(\theta)$. Along with $\hat{x}_i = (x_i^T, \theta^T)^T$, we introduce composite vectors $X_i = (x_i^T, \theta^T)^T, \hat{X}_i = (x_i^T, \theta^T)^T, Y_i = (y_{i1}^T, ..., y_{in}^T)^T$ of $(i+n)$, $(i+1)n+r$, and $im$ dimensions. We can write the following the Cramer-Rao inequality, for the vector $\hat{X}_i = (x_i^T, \theta^T)^T$ (Galdos 1980):

$$J_i^{-1} \leq G_i = E_{\hat{X}_i, \theta} \left\{ \left( \hat{X}_i - \hat{\hat{X}}_i, Y_i - \hat{\hat{X}}_i, Y_i \right) \right\},$$

where

$$J_i = E_{\hat{X}_i, \theta} \left[ d \ln f(\hat{X}_i, Y_i) \left( d \ln f(\hat{X}_i, Y_i) \right)^T \right],$$

and $f(\hat{X}_i, Y_i)$ is the PDF for vectors $\hat{X}_i$ and $Y_i$. Let us separate the lower $(n+r) \times (n+r)$ diagonal block in $J_i^{-1}$

$$\begin{bmatrix} Y_i^0 & Y_i^0 \\ Y_i^{\theta 0} & Y_i^0 \end{bmatrix} = \begin{bmatrix} 0_{(n+r) \times (n+r)} & I_{(n+r)} \end{bmatrix} \begin{bmatrix} J_i^{-1} & (0_{(n+r) \times (n+r)}) \end{bmatrix},$$

where $0_{(n+r) \times (n+r)} - (n+r) \times (n+r)$ is a zero matrix and $I_{(n+r)}$ is a unit $n+r$ matrix. The matrices $Y_i^0, Y_i^{\theta 0}$ determine CRLB for vectors $x_i$ and $\theta$.

The purpose of this work is to obtain a recurrence algorithm for $Y_i^0, Y_i^{\theta 0}$.

3. ALGORITHM FOR CRLB

Doing mathematical operations in the way similar to that of (Stepanov et al. 2013), we can show that for $Y_i^0$ and $Y_i^{\theta 0}$ the following relations hold good:

$$Y_i^0 = \left( \frac{F_i^0 - \bar{L}_i}{\hat{\bar{Q}}_i} \right)^{-1},$$

$$Y_i^{\theta 0} = \hat{\bar{Q}}_i^{-1} + \hat{\bar{Q}}_i^{-1} \bar{L}_i Y_i^0 \frac{\bar{E}_i}{\hat{\bar{Q}}_i^{-1}}.$$
\[
\delta \hat{R}_i = \delta \hat{R}_i^T (l, \mu) = \frac{1}{2} E_{i, \theta} \left[ T \left( \frac{\partial \tilde{R}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{R}_i(x_i, \theta) \frac{\partial \tilde{R}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{R}_i(x_i, \theta) \right) \right],
\]

where \( l, \mu = I, n \).

\[
\delta \hat{Q}_i = \delta \hat{Q}_i^T (l, \mu) = \frac{1}{2} E_{i, \theta} \left[ T \left( \frac{\partial \tilde{Q}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{Q}_i(x_i, \theta) \frac{\partial \tilde{Q}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{Q}_i(x_i, \theta) \right) \right],
\]

\[
\delta \hat{Q}_i^{x_i = 1} (l, \mu) = \frac{1}{2} E_{i, \theta} \frac{1}{T} \left[ \frac{\partial \tilde{Q}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{Q}_i(x_i, \theta) \frac{\partial \tilde{Q}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{Q}_i(x_i, \theta) \right],
\]

\[
\delta \hat{Q}_i^0 (l, \mu) = \frac{1}{2} E_{i, \theta} \frac{1}{T} \left[ \frac{\partial \tilde{Q}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{Q}_i(x_i, \theta) \frac{\partial \tilde{Q}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{Q}_i(x_i, \theta) \right],
\]

\[
\delta \hat{Q}_i^{x_i = 1, \theta - 1} (l, \mu) = \frac{1}{2} E_{i, \theta} \frac{1}{T} \left[ \frac{\partial \tilde{Q}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{Q}_i(x_i, \theta) \frac{\partial \tilde{Q}_i^{-1}(x_i, \theta)}{\partial x_i} \tilde{Q}_i(x_i, \theta) \right],
\]

In these equations the following notations are used:

\[
\frac{ds(x)}{dx} = \begin{bmatrix}
\frac{\partial s_1(x)}{\partial x_1} & ... & \frac{\partial s_n(x)}{\partial x_n}
\end{bmatrix},
\]

\[
\frac{ds'(x)}{dx} = \left[ \frac{ds(x)}{dx} \right]^T.
\]

Note that when \( f(\theta) = N(\theta; 0, R_0) \) and \( P_0(\theta) = P_0 \) we can write \( \tilde{F}_0 = P_0^{-1} \). Let us consider some specific cases.

**Case 1.** The subvector \( \theta \) is absent \((r = 0)\), the measurement errors and forcing noise are additive; moreover, \( \Gamma_i(x_i, \theta) = \Gamma \), \( \Psi_i(\theta) = I \) is a unit matrix, \( f(x_i, \theta) = f(\theta), P_0(\theta) = P_0 \), \( f(\theta) = N(\theta; 0, R_0) \). For these assumptions \( \delta \tilde{R}_i = 0, \delta \tilde{Q}_i^{x_i = 1} = 0, \delta \tilde{Q}_i^0 = 0, \delta \tilde{Q}_i^{x_i = 1, \theta - 1} = 0 \), \( \tilde{Q}_i = \tilde{Q}_i = \Gamma_i Q_i T_i \), \( \delta \tilde{Y}_i^{x_i = 1} = 0, \delta \tilde{Y}_i^0 = 0, \delta \tilde{Y}_i^{x_i = 1, \theta - 1} = 0, \), \( \tilde{L}_i = \tilde{L}_i = 0, \), \( j = 0, i - 1 \) and therefore

\[
\tilde{Y}_i^{x_i = 1} = \tilde{Y}_i^0 = \tilde{Y}_i^{x_i = 1, \theta - 1} = \tilde{Y}_i^{x_i = 1, \theta - 1} = 0, \tilde{L}_i = \tilde{L}_i = 0, \tilde{\Xi}_i = \tilde{\Xi}_i = \tilde{\Xi}_i = \tilde{\Xi}_i = 0, \tilde{\Xi}_i = \tilde{\Xi}_i = \tilde{\Xi}_i = \tilde{\Xi}_i = P_0^{-1} .
\]

From (12) it does not depend on the state vector, we can write:

\[
\tilde{F}_i = E_{i, \theta} \left( \frac{\partial \Phi_i(x_i, \theta)}{\partial x_i} \right) \tilde{Q}_i^{-1} = \tilde{\Phi}_i^{x_i = 1} \tilde{Q}_i^{-1}, \]
If, in addition, \( \delta \xi_{j,\theta} = 0 \) and \( \delta \Phi_{j-1,\theta} = 0 \), then all \( \tilde{L}_j = 0 \), \( j = 0, \ldots, J - 1 \); therefore,

\[
Y^0_i = \left( F^0_i \right)^{-1} \text{, } \gamma^0_{ij} = \Xi^{-1}_{ij} \text{,} \tag{16}
\]

\[
\Xi_i = \delta \gamma^0_i + \dot{\Xi}^{-1}_{i-1} = \left( \Xi^{-1}_{i-1} + \delta \Phi_i \right)^{-1} \Xi_i, \tag{17}
\]

or

\[
\Xi_0 = \delta \gamma^0_0 + \dot{\Xi}^{-1}_{J-1} + \delta \Phi_0 = P_0^{-1} \text{,} \tag{18}
\]

Below, we give some simplest examples to illustrate the application of the relations obtained.

### 4. EXAMPLE

Assume that we need to estimate an unknown parameter \( \theta = q \) of a random walk (Wiener process) \( z(t) \) by its discrete measurements with additive measurement errors. Let us consider different variants of the problem solution.

#### Model 1.

We use the following model for discrete time:

\[
x_i = x_{i-1} + \sqrt{\Delta} w_i \text{, } \quad \dot{q} = 0, \tag{20}
\]

\[
y_i = z_i + v_i = s_i(x_i, q) + q x_i + v_i. \tag{21}
\]

where \( \Phi_i(x_i, q) = x_i \), \( s_i(x_i, q) = q x_i \), \( \Gamma_i = \sqrt{\Delta} \sigma_w \), \( \Delta t \) is the sampling interval; \( v_i \) and \( w_i \) are zero-mean Gaussian white noise with variances \( \sigma_v^2 \) and \( \sigma_w^2 \), respectively; \( x_0 \) and \( q \) are independent random values with PDF \( f(x_0) = N(x_0; 0, \sigma_x^2) \) and \( f(q) \) is a PDF, for which the \( \bar{q} = E(q) \) and \( \sigma_q^2 = E(q - \bar{q})^2 \) are known. A feature of this model is that the shaping filter for \( z_i \) does not depend on \( \theta \) and nonlinearity is only due to nonlinearity in measurements. Using the above relations, we can write:

\[
\dot{\Xi}^{-1}_{i-1} = \frac{1}{\sigma_n^2 \Delta t} \text{, } \dot{\Phi}_i = \frac{1}{\sigma_n^2 \Delta t} \text{, } \delta \gamma^0_i = E_q \left( \frac{\dot{q}^2}{\sigma_v^2} \right) = \frac{\sigma_q^2 + \sigma_v^2}{\sigma_v^2}, \tag{22}
\]

\[
\delta \Phi_{i+1,\theta} = 0, \quad \delta \Phi_{i,\theta} = 0, \quad \delta \Phi_{i-1,\theta} = 0, \quad \delta \Phi_{i,\theta} = 0, \quad \delta \Phi_{i-1,\theta} = 0, \quad \delta \Phi_{i,\theta} = 0, \quad \delta \Phi_{i,\theta} = 0, \quad \delta \Phi_{i-1,\theta} = 0, \quad \delta \Phi_{i,\theta} = 0, \quad \delta \Phi_{i,\theta} = 0, \tag{23}
\]

Taking into consideration the fact that this example corresponds to case 2, and, in addition, all \( L_j = 0, \quad j = 0, J \), we can use (16), (17), and (19). Thus:

\[
\hat{F}^q_i = \hat{F}^q_i + \delta \gamma^0_i + \dot{\Phi}_i \left( \Xi^{-1}_{i-1} + \dot{\Xi}_i \right)^{-1} = \hat{F}_0 + \hat{F}_0^{-1}, \tag{24}
\]

\[
\Xi^0 = \frac{\sigma_q^2 + \sigma_v^2}{\sigma_v^2} + \left( \Xi^{-1}_{i-1} + \dot{\Xi}_i \right)^{-1}, \quad \Xi^0 = 1/\sigma_v^2. \tag{25}
\]

Finally,

\[
Y^q_i = \left( \frac{\sigma_q^2 + \sigma_v^2}{\sigma_v^2} + \left( \Xi^{-1}_{i-1} + \dot{\Xi}_i \right)^{-1} \right)^{-1}, \quad Y^q = \sigma_v^2. \tag{26}
\]

It is also easy to see that for \( f(q) = N(q; \bar{q}, \sigma_q^2) \), then \( F_0^{\bar{q}} = \frac{1}{\sigma_q^2} \). Thus, we can state the fact that in the case under consideration, the type of \( f(q) \) at fixed values of \( \bar{q} = E(q) \) and \( \sigma_q^2 = E(q - \bar{q})^2 \) does not practically affect the final result.

It should be noted that the CRLB for the model (20), (21) is equivalent to the covariance in the linear estimation problem of vector (20) by measurements

\[
Y^m_{11} = \left( \frac{\sigma^2_q + \sigma^2_v}{\sigma^2_v} \right) x_i + Y^m_{12}, \tag{27}
\]

\[
Y^m_{12} = \left( \frac{\sigma^2 + \sigma^2_q}{\sigma^2_v} \right) q + Y^m_{22}, \tag{28}
\]

where \( Y^m_{11} \) are independent of \( x_0 \) and \( q \) zero-mean Gaussian white-noise sequences with variances \( \sigma^2_v \), whereas \( x_0 \) and \( q \) are independent of each other. Gaussian random values with variances \( \sigma^2_q \) and \( \left( \hat{F}^q_0 \right)^{-1} \). In other words, the value that determines the CRLB for \( q \) corresponds to the case of \( q \) estimation from measurements of the form (21) under the assumption that \( x_i \) is replaced by the known coefficient \( \sqrt{\sigma^2_q + \sigma^2_v} \). In turn, the CRLB for \( x_i \) corresponds to the case of \( x_i \) estimation from the same measurements, but under another assumption, namely, that \( q \) is replaced by the known coefficient \( \sqrt{\sigma^2_q + \sigma^2_v} \).

#### Model 2.

Let us include the unknown parameters \( \theta \) in the state vector \( x_i = (x_i, q) \) and use the same model (20), (21), but in so doing, our aim is to find the recurrence relation for the CRLB for vector \( x_i \). In this case, taking into consideration the fact that \( s_i(x_i, q) = q x_i \), \( \Phi_i(x_i, q) = \left[ \Phi_i(x_i, q) \right] \) \( q \) and \( \dot{F}_i \) \( \left[ \dot{F}_i \right] \), we can use (15) which does not require nonsingularity of
matrix $\tilde{Q}_i$. Since
\[ \frac{d\tilde{s}_i(x'_i)}{d(x'_i)^T} = \begin{pmatrix} \frac{\partial s_i(x'_i)}{\partial x_{i1}} & \frac{\partial s_i(x'_i)}{\partial x_{i2}} \end{pmatrix} = (q, x_i) , \]
then
\[ \tilde{s}_{x'_i} = \tilde{E}_i \frac{1}{\sigma_v^2} \begin{bmatrix} q^2 & x_i q \\ x_i q & x_i^2 \end{bmatrix} = \begin{bmatrix} \sigma_v^2 + q^2 & 0 \\ 0 & \sigma^2 + i\Delta\sigma_w \end{bmatrix} . \]
By virtue of the fact that $\Delta\tilde{\Phi}_i = 0$ and $\tilde{G}_i = E$ , (19) takes the form:
\[ \tilde{y}^*_{i} = \left( \tilde{y}^*_{i-1} + \tilde{G}_{i} \tilde{f}_{i}^{T} \right)^{-1} = \left( \frac{1}{\sigma_v^2} \begin{bmatrix} \sigma_v^2 + q^2 & 0 \\ 0 & \sigma^2 + i\Delta\sigma_w \end{bmatrix} + \begin{bmatrix} \Delta\tau_{x'_i} & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} , \]
where
\[ \tilde{y}^*_{0} = P_0 = \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \left( \tilde{F}_0 \right)^{-1} \end{bmatrix} . \]

Model 3. We can use another shaping filter for $z_i = x_i$:
\begin{align*}
    x_i &= x_{i-1} + q\sqrt{\Delta w_i} , \\
    \dot{q} &= 0 , \\
    y_i &= x_i + v_i ,
\end{align*}
where $v_i$ and $w_i$ are the same as in the previous case. As in the first two models, we assume that for $f(q)$, the first two moments $\bar{q} = E(q)$ and $\sigma_q^2 = E(q - \bar{q})^2$ are known, and, besides, the value of $a = E \left( \frac{1}{q^2} \right) = \int \frac{1}{q^2} f(q) dq$ is also determined. It should be noted that Gaussian PDF does not satisfy the latter requirement because such integral diverges. The feature of this statement is that the model for the measurements are linear, whereas equation for the state vector is nonlinear, since the coefficient of the forcing noise depends on the unknown parameter. From (25)-(26) it follows that
\[ \Phi_i \left( x_{i-1}, 0 \right) = x_{i-1} , \\ s_i \left( x_i, q \right) = x_i , \\ \Gamma_i \left( x_{i-1}, q \right) = q . \]
In this case:
\[ \tilde{Q}_i^{-1} \left( x_{i-1}, q \right) = \frac{1}{q^2 \sigma_v^2 \Delta t} , \quad \tilde{f}_i = E \left( \frac{1}{q^2} \right) = a , \quad \tilde{s}_{x'_i} = \frac{1}{\sigma_v^2} , \]
\[ \delta s_{x'_i} = 0 , \quad \delta s_{x'_i} = 0 , \quad \delta \Phi_i = E \left( \frac{1}{q^2 \sigma_v^2 \Delta t} \right) = a , \quad \delta \Phi_i = 0 , \]
\[ \delta \Phi_i = 0 , \quad \delta \Phi_i^{-1} = 0 , \quad \tilde{Q}_i \left( x_{i-1}, q \right) = \Gamma_i \left( x_{i-1}, q \right) \Gamma_i^T \left( x_{i-1}, q \right) = q^2 \Delta t \sigma_w^2 , \]
\[ \delta \tilde{Q}_i = 0 , \quad \delta \tilde{Q}_i^{-1} = 0 , \quad \tilde{Q}_i = \frac{1}{2} E \left( \frac{1}{q^2 \sigma_v^2 \Delta t} \right) = a , \quad \Phi_i = 1 , \quad L_j = 0 , \]
\[ j = 0 \tilde{F}_0 = E \left( \frac{d}{dq} \ln f(q) \left( \frac{d}{dq} \ln f(q) \right)^T \right) . \]
Since here, too, all $L_j = 0$, $j = 0 \tilde{D}_j$ , using (19) we can write
\[ \tilde{F}_i = \tilde{F}_0 + a \tilde{F}_i \tilde{G}_{i} \tilde{f}_{i}^{T} \tilde{F}_0 \tilde{G}_{i} \tilde{f}_{i}^{T} = E \left( \frac{d}{dq} \ln f(q) \left( \frac{d}{dq} \ln f(q) \right)^T \right) , \]
\[ \tilde{z}_i = \frac{1}{\sigma_v^2} + a - \frac{a^2}{\tilde{z}_{i-1} + a} = \frac{\left( \tilde{z}_{i-1} (a\sigma_v^2 + 1) + a \right)}{\sigma_v^2 (\tilde{z}_{i-1} + a)} , \tilde{z}_0 = P_0 . \]
Therefore, we have
\begin{align*}
    \tilde{y}^*_{i} &= \left( \tilde{F}_0 + \frac{ia}{2} \right)^{-1} , \\
    \tilde{y}^*_{i} &= \left( \tilde{F}_0 + \frac{ia}{2} \right)^{-1} \left( \tilde{y}^*_{i-1} \right)^{-1} + a , \\
    \tilde{y}^*_{0} &= P_0 .
\end{align*}
In this example we calculate the CRLB for $q$, which determines the properties of the Wiener process. It is interesting to compare (22) and (27) for the same values of $q$ and $\sigma_q^2$. Let assume that $f(q) = \frac{3q^2 e^{-0.5q^2 \sigma_q^3}}{\sigma_q^3}$ (Weibull PDF). In this case, $\sigma_q = \frac{3\sqrt{3} \left( \frac{2}{3} \right)}{2\pi} \bar{q}$, $a = \frac{2\pi}{\sqrt{3} \sigma_q^2 \Gamma \left( \frac{2}{3} \right)}$, $\tilde{F}_0 = \frac{4a}{3}$, where $\Gamma(\beta)$ is the Gamma function. Fig. 1 presents the results of the CRLB calculations obtained using (22) and (27), with $\sigma_q = 0.0019$, $\bar{q} = 0.12$, $\sigma_v = 0.3$, $\sigma_w = 1$, and $\Delta t = 1$.
Here, we also give the values of the root-mean-square (RMS) error for the optimal estimate computed using Monte-Carlo simulation as $\sigma_q^{MC} \left( j \right) = \frac{1}{L} \sum_{j=1}^{L} \left( q_j - \tilde{q}_j \right)^2$, where $L$-number of samples in Monte-Carlo simulation, $q_j$ and $\tilde{q}_j$ are the samples and optimal estimates calculated using the algorithm described, for example, in (Ivanov et al., 2000).
5. CONCLUSIONS

Recurrence relations have been obtained for the calculation of CRLB in the discrete-time nonlinear filtering problem in the conditions when the forcing noise, measurement errors and initial covariance matrix depend on the state vector to be estimated which also includes the subvector of unknown time-invariant parameters.

Some specific cases have been considered. The relation between the derived recurrence algorithm for the CRLB calculation and the known algorithms corresponding to the case of additive forcing and measurement noise has been established.

An example of CRLB calculation in the estimation problem of the parameters of the random walk process has been considered. The results obtained allowed the conclusion that there is an obvious dependence of CRLB on the type of the model used to describe the process under study for nonlinear filtering problem. This dependence is worth further study.

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