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Multifunctional transformation method in flow modeling

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Abstract: In the paper a new approach for solution of PDE's generating by transient gas flow networks of pipelines is considered. In particular, the fundamental solution representation for the considered partial differential problem is based on exploiting the so-called canonical system formed by the collection of the eigenfunctions for the underlying operator and its adjoint. This canonical system is the base of the multifunctional integral transformation for spatial variables which is the core of the developed operational calculus for PDE's. The main objective is to present a new approach to construct then the parallel numerical method which meets the suitable accuracy and relatively small computation time.

Keywords: Gas pipelines, Partial differential equations, Integral spatial transformation, Operational calculus, Spectral analysis.

1. INTRODUCTION

Most real physical processes are governed by partial differential equations (PDEs). Hence, the development of the adequate methods for solutions of such PDEs is actual task. In the engineering practice it is widely used the Laplace transformation with respect to the time variable in the model formulated by ODEs. This paper presents the so-called multifunctional transformation (MFT) method for solution of partial differential equations proposed by Dymkou V. (2006). The application of this idea to some cases of multidimensional partial differential equations is given by Dymkou V., Rabenstein R. and Stefen P. (2006).

The introduced the multifunctional transformation (MFT) for the space variable presents some new elements of operational calculus. One of the reasons for the development of the multi functional transformation was that its application together with Laplace transformation turns the initial boundary value problems into the block diagram model representations which are used widely in engineering practice and which are available for the well known computationally efficient parallel numerical methods. This approach shown the good results for solution of some actual engineering problems of liquid metal magnetohydrodynamic (MHD) flows (see Dymkou V. et al.(2009), Potherat A. et al. (2010)).

The gas transportation network (GTN) is a well known example of a complex and large scale distributed parameter system of great practical interest Mohring J et al. (2004), Dymkou S. (2006). Although in the last decades a number of papers devoted to this theme were published (see, for example, Simone Research Group, 2000), the gas networks still remains an actual problem. A general model of the gas transportation network includes a number of nonlinear elements such as pipelines, gasholders, compressor stations and others. A detailed description of dynamic processes in gas pipeline units based on partial differential momentum and continuity equations is rather complex, and is sometimes used in theoretical studies. Usually the proposed equations involve a number of variables and can become quite cumbersome. The wide practical application of such models is blocked by their complexity to implement in a reasonable manner and time.

At the first stage the mathematical model of gas transport pipe units can be introduced on the basis of their linearization that leads to some linear PDE's (Osiadacz A., 1987). Also, some approximation (Dymkov M et al., 2012) can be introduced by exploiting $2-D$ and repetitive models (Rogers E. et al., 2007, Dymkov M. et al., 2008). We propose to use at this stage the MFT-approach by Dymkou V. (2006). In particular, the fundamental solution representation for the PDEs problem under consideration was given on a strong mathematical basis by exploiting the canonical system of the underlying operator and its adjoint.

The obtained solution indicates the natural way to introduce the multifunctional transformation with respect to the space variables. Both transformations, the Laplace transformation for the time variable and multifunctional transformation for the space variables turn the initial boundary value problem into an algebraic equation which leads to their effective numerical implementation. This, in turn, gives a good way to design then the parallel numerical method which meets the suitable accuracy and relatively small computation time.

2. GAS FLOW MODEL IN A PIPELINE UNIT

The aim of this section is to introduce the linear PDE's and operator setting for studying gas flow in pipeline units. The state space parameters are gas pressure p and mass flow Q at the points of the pipe. All other physical parameters of the pipe and gas used here are constant at the moment of calculation. It is known that some important dynamic characteristics of the processes can be evaluated from the linearized model of the processes. The most accurate linear model can be realized in some neighborhood of the known basic regime $(\overline{Q}, \overline{p})$ of the considered process. In particular, the following system of linear differential equations can be used for description of the disturbed state space parameters for the turbulent, isothermal gas flow in the unit pipeline (see, for example, Osiadacz A., 1987)

$$\frac{\partial Q(t,x)}{\partial t} = -s\frac{\partial p(t,x)}{\partial x} - \rho Q(t,x) - \beta p(t,x),$$

$$\frac{\partial p(t,x)}{\partial t} = \alpha \frac{\partial Q(t,x)}{\partial x}, \tag{1}$$

where x denotes the space variable, t the time variable, s the cross sectional area, d the pipeline diameter, c the isothermal speed of sound and ν the friction factor. Here we denote

$$\alpha = -\frac{c^2}{s},\ \gamma = \frac{\nu c^2}{2ds} = -\alpha\frac{\nu}{2d},\quad \beta = \gamma\frac{\bar{Q}^2}{\bar{p}^2},\quad \rho = 2\gamma\frac{\bar{Q}^2}{\bar{p}\bar{Q}},$$

and $x \in [0,1],\ t > 0$.

Rewrite now the given equations in operator form as

$$D_t y(t,x) = Ly(t,x), \tag{2}$$

where $y(t,x) = \begin{bmatrix} y_1(t,x) \\ y_2(t,x) \end{bmatrix} \doteq \begin{bmatrix} Q(t,x) \\ p(t,x) \end{bmatrix}$ and the operator L is given as

$$L = A + BD_x = \begin{bmatrix} -\rho & -sD_x - \beta \\ \alpha D_x & 0 \end{bmatrix} \tag{3}$$

and

$$A = \begin{bmatrix} -\rho & -\beta \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -s \\ \alpha & 0 \end{bmatrix}. \tag{4}$$

Here D_t and D_x describe the partial derivatives with respect to the time and space variables, respectively. The initial and boundary conditions are given by

$$y(0,x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} q_0(x) \\ p_0(x) \end{bmatrix}. \tag{5}$$

and

$$U_1(y) = \int_0^1 y_2(t,x)(t,x)dx = \int_0^1 p(t,x)dx = const, \tag{6}$$

$$U_2(y) = y_2(t,0) - y_2(t,1) = p(t,0) - p(t,1) = 0, \forall t \geq 0,$$

respectively.

The given initial condition images the arisen disturbances $q(x), p(x)$ at the initial moment $t = 0$ for the preassigned pumping regime $\overline{Q}(t,x), \overline{p}(t,x)$ in the pipeline, and the boundary data images the conservation of the gas pressure in the pipe. Note, that the given pressure conservation conditions lead to the pipeline storage condition. Indeed, the condition (6) and integration the second equation of (1) on $x \in [0,1]$ give

$$\frac{d}{dt}\Big(\int_0^1 p(t,x)dx\Big) = 0, \frac{d}{dt}\Big(\int_0^1 p(t,x)dx\Big) = \alpha \int_0^1 \frac{\partial Q(t,x)}{\partial x}dx,$$

Since

$$\int_0^1 \frac{\partial Q(t,x)}{\partial x}dx = Q(t,1) - Q(t,0)$$

then the last is equivalent to the following pipeline storage condition

$$y_1(t,1) - y_1(t,0) = Q(t,1) - Q(t,0) = 0 \forall t \geq 0. \tag{7}$$

3. BASIC NOTIONS OF MULTIFUNCTIONAL TRANSFORMATION METHOD

In this section we give a short overview of the basic elements of MFT-method needed for this paper. The details and strong description of this approach can be found in Dymkou V. (2006). In particular, we show how the fundamental solution representation for the considered PDEs problem can be written on a strong mathematical basis by exploiting the canonical system of the underlying operator and its adjoint. The obtained solution indicates the natural way to introduce the multifunctional transformation (MFT) with respect to the space variables. Then, applying both transformations, the Laplace transformation for the time variable and multifunctional transformation for the space variables, turn the initial boundary value problem into an algebraic equation.

We consider the following nonhomogeneous initial boundary value problem

$$\frac{\partial}{\partial t}y(t,x) = Ly(t,x) + v(t,x), \quad t > 0, \quad x \in \Omega,$$

$$y(t,x)|_{t=0} = y_a(x),\ x \in \Omega, \tag{8}$$

$$U_\mu\big(y(t,x)\big) = 0,\ t > 0,\ x \in \partial\Omega,\ \mu = 1, \cdots, m.$$

The set Ω denotes a domain in R^N and $\partial\Omega$ its boundary. The functions $v(t,\cdot)$ and $y(t,\cdot)$ are elements of H (or $H(\Omega)$) for each fixed value of $t \geq 0$, where H is a Hilbert space with the scalar product $(\cdot,\cdot)$. The operator L is assumed to be a differential operator with respect to the space variable x which acts on properly elected elements of H. We presuppose that $(-L)$ is a sectorial operator on the Hilbert space H such that L has a compact resolvent and the spectrum $\sigma(L)$ is not empty. The linear differential form of $U_\mu\big(y(t,x)\big) = 0$ represents the collection of all homogeneous boundary conditions which are necessary to yield a unique solution of the initial-boundary-value problem.

The corresponding eigenproblem for the prime operator L can be summarized in the form

$$L: \quad \begin{cases} \sigma(L) = \{\lambda_i\}_i - \text{ the set of eigenvalues} \\ E = \{e_{p,m}(x,\lambda_i)\}_{i,p,m} - \text{ the canonical system.} \end{cases}$$

The associated eigenproblem for the adjoint operator $L^\dagger$ is summarized as follows

$$L^\dagger: \quad \begin{cases} \sigma(L^\dagger) = \{\lambda_i^*\}_i - \text{ the set of eigenvalues} \\ E^\dagger = \{\epsilon_{p,m}(x,\lambda_i^*)\}_{i,p,m} - \text{ the canonical system.} \end{cases}$$

Note that all elements $e_{p,m}(x,\lambda_i)$ from the canonical system E should satisfy the homogeneous boundary conditions $U_\mu\big(e_{p,m}(x,\lambda_i)\big) = 0,\ \mu = 1,\dots,m$. And all elements $\epsilon_{p,m}(x,\lambda_i^*)$ from the adjoint canonical system $E^\dagger$ should satisfy the corresponding adjoint homogeneous boundary conditions $V_\mu\big(\epsilon_{p,m}(x,\lambda_i^*)\big) = 0$.

The canonical system of the adjoint operator $L^\dagger$ is used as kernels of the required spatial transformation. Namely, the multi-functional transformation $\mathcal{T}$ is the set of transformations $\{\mathcal{T}_{p,m}(\lambda_i)\}$, which are constructed by the canonical system $E^\dagger$ of the adjoint spatial operator $L^\dagger$ and act on elements $f \in H$ as

$$\mathcal{T}_{p,m}(\lambda_i)\{f(\cdot)\} = \overline{f}_{p,m}(\lambda_i) = (f(\cdot), \epsilon_{p,m}(\cdot,\lambda_i^*)), \qquad (9)$$

where $p = 1,\dots,P_i = P(\lambda_i),\ \ m = 0,\dots,M_p$ are defined for each $\lambda_i \in \sigma(L)$ by the canonical system. Here $P(\lambda)$ is the dimension of the eigenvector space for the eigenvalue λ, M_p is the multiplicity of the eigenvector $\epsilon_{p,m}(\cdot,\lambda)$, and $(\cdot,\cdot)$ denotes the inner product in the Hilbert space H.

From the definitions of the MFT and the adjoint operator it follows that

$$\mathcal{T}_{p,m}(\lambda_i)\{Lf(\cdot)\} = (Lf(\cdot), \epsilon_{p,m}(\cdot,\lambda_i^*)) \qquad (10)$$
$$= (f(\cdot), L^\dagger \epsilon_{p,m}(\cdot,\lambda_i^*)).$$

It can be shown that

$$\mathcal{T}_{p,m}(\lambda_i)\{Lf(\cdot)\} = \lambda_i \overline{f}_{p,m}(\lambda_i) + \overline{f}_{p,m-1}(\lambda_i),$$
$$m = 0,\dots,M_p, \qquad (11)$$

where we use the agreement $\overline{f}_{p,\mu}(\lambda_i) \equiv 0,\ \mu < 0$. This property of the introduced transformation is a generalization of the so-called "differentiation theorem" well known from Laplace-, Fourier-, and other functional transformations. In the image domain, the effect of the operator L is expressed in the transformed domain by a multiplication with λ_i, which takes the role of the discrete frequency variable. Therefore, application of the MFT to $Lf(\cdot)$ replaces the image of the operator L on the element $f \in H$ by its direct transformations in the frequency domain.

To define the inverse transformation we use the canonical systems E of the operator L. Namely, the inverse transformation is given by the following series

$$\mathcal{T}^{-1}\{\overline{f}_{p,m}(\lambda_i)\} = f(\cdot) = \qquad (12)$$
$$\sum_{\lambda_i \in \sigma(L)} \sum_{p=1}^{P_i} \sum_{m=0}^{M_p} \overline{f}_{p,m}(\lambda_i) e_{p,M_p-m}(\cdot,\lambda_i).$$

Then, applying the Laplace transform with respect to the variable t directly to the problem (8) gives

$$sY(s,x) = LY(s,x) + V(s,x) + y_a(x),$$
$$U_\mu\big(Y(s,x)\big) = 0,\ s \in \mathbf{C},\ x \in \partial\Omega,\ \mu = 1,\cdots,m. \qquad (13)$$

Applying now the MFT spatial transformation for the obtained Laplace output $Y(s,x)$ of (8) in the $s-$ domain we have the following solution representation

$$\overline{Y}_{p,m}(s,\lambda_i) = \frac{\overline{V}_{p,m}(s,\lambda_i)}{s-\lambda_i} + \frac{\overline{y}_{a,p,m}(\lambda_i)}{s-\lambda_i} + \frac{\overline{Y}_{p,m-1}(s,\lambda_i)}{s-\lambda_i}$$
$$= \sum_{\mu=0}^{m} \left(\frac{\overline{V}_{p,m-\mu}(s,\lambda_i)}{(s-\lambda_i)^{\mu+1}} + \frac{\overline{y}_{a,p,m-\mu}(\lambda_i)}{(s-\lambda_i)^{\mu+1}} \right) \qquad (14)$$

The first two terms of the expression (14) represent the transformed excitation function $V_{p,m}(s,\lambda_i)$ and the transformed boundary conditions $y_{a,p,m}(\lambda_i)$.

Using now the inverse MFT transformation we can obtain the solution of the problem in the Laplace domain as

$$\mathcal{T}^{-1}\{\overline{Y}_{p,m}(s,\lambda_i)\} = Y(s,x) \qquad (15)$$
$$= \sum_{\lambda_i \in \sigma(L)} \sum_{p=1}^{P_i} \sum_{m=0}^{M_p} \overline{Y}_{p,m}(s,\lambda_i) e_{p,M_p-m}(x,\lambda_i).$$

Hence

$$Y(s,x) = \sum_{\lambda_i \in \sigma(L)} \sum_{p=1}^{P_i} \sum_{m=0}^{M_p} \left(\sum_{\mu=0}^{m} \frac{\overline{V}_{p,m-\mu}(s,\lambda_i)}{(s-\lambda_i)^{\mu+1}} \right.$$
$$\left. + \frac{\overline{y}_{a,p,m-\mu}(\lambda_i)}{(s-\lambda_i)^{\mu+1}} \right) e_{p,M_p-m}(x,\lambda_i). \qquad (16)$$

Thus, the MFT-method reduces the differential operators of boundary value problems to algebraic equations, similar to the Laplace transformation approach used for the time variable.

4. MFT METHOD IN GAS PIPELINES MODEL

To apply the MFT-method for the problem (2) - (6) we have to find the corresponding canonical systems for the given prime and adjoint operators L and $L^\dagger$ introduced in Section 2.

4.1 Adjoint operator

The considered operators act on the space $AC^1([0,1])$ of absolutely continuous functions together with its first derivatives on interval $[0,1]$ that is dense in the Hilbert space $L_2[0,1]$ of square integrable functions with the standard scalar product $(f,g) = \int_0^1 f^T(x)g(x)dx,\ \ f,g \in L_2$, where superscript f^T denotes transposed of f.

To construct the MFT-transformation, the operator L has to be a sectorial operator on the considered space, with compact resolvent operator $R(s,L)$ and non-empty spectrum $\sigma(L)$.

First, using the so-called Green's formula, we determine the adjoint operator:

$$(Ly,z)=\int_0^1 z^T Lydx=\int_0^1 z^T(A+BD_x)ydx=$$

$$(y,(A^T-B^TD_x)z)+z^TBy\Big|_{x=0}^{x=1}, \quad (17)$$

where $z(t,x)=\begin{bmatrix} z_1(t,x) \\ z_2(t,x) \end{bmatrix}$. Equation (17) complies with the definition of the adjoint operator only if $z^TBy\big|_{x=0}^{x=1}=0$ vanishes,

$$z^T(t,x)By(t,x)\big|_{x=0}^{x=1}=z^T(t,1)By(t,1)-z^T(t,0)By(t,0)$$

$$=\alpha[z_2(t,1)y_1(t,1)-z_2(t,0)y_1(t,0]+$$

$$s[z_1(t,0)y_2(t,0)-z_1(t,1)y_2(t,1)]=0.$$

Using the boundary conditions this equation can be rewritten as

$$\alpha\big(z_2(t,1)-z_2(t,0)\big)y_1(t,0)= s\big(z_1(t,1)-z_1(t,0)\big)y_2(t,0). \quad (18)$$

The conditions for $z_1(t,x)$ and $z_2(t,x)$ at the boundary $x=1$ and $x=0$ have to be chosen such that this expression becomes zero: $z_2(t,0)-z_2(t,1)=0$, and $z_1(t,1)-z_1(t,0)=0$.

Thus, the adjoint operator $L^\dagger$ is given by

$$L^\dagger=A^H-B^HD_x=\begin{bmatrix} -\rho & -\alpha D_x \\ -\beta+sD_x & 0 \end{bmatrix}, \quad (19)$$

subject to the boundary conditions of the form

$$V_1(z)=0 \Leftrightarrow\ z_1(t,1)-z_1(t,0)=0,$$
$$V_2(z)=0 \Leftrightarrow\ z_2(t,1)-z_2(t,0)=0. \quad (20)$$

4.2 Eigenvalue problems

Prime operator The eigenvalue problem of the operator L is

$$[\lambda I-L]e_{p,0}(x,\lambda)=0,\ p=1,\dots,P(\lambda), \quad (21)$$

where $e_{p,0}(x,\lambda)=[f^1_{p,0}(x,\lambda),f^2_{p,0}(x,\lambda)]^T$ and $P(\lambda)$ is the unknown number of all linear independent eigenvectors corresponding to the eigenvalue λ. Let us rewrite equation (21) in the following form

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} f^1_{p,0} \\ f^2_{p,0} \end{bmatrix}+\begin{bmatrix} \rho & sD_x+\beta \\ -\alpha D_x & 0 \end{bmatrix}\begin{bmatrix} f^1_{p,0} \\ f^2_{p,0} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} (22)$$

then we have the following linear system for $f^1_{p,0}$ and $f^2_{p,0}$

$$\begin{cases} \lambda f^1_{p,0}+\rho f^1_{p,0}+sf^{2'}_{p,0}+\beta f^2_{p,0}=0 \\ \lambda f^2_{p,0}-\alpha f^{1'}_{p,0}=0 \end{cases} \xrightarrow[\lambda\neq 0]{}$$

$$\begin{cases} s\alpha f^{1''}_{p,0}+\alpha\beta f^{1'}_{p,0}+\lambda(\lambda+\rho)f^1_{p,0}=0 \\ f^2_{p,0}=\dfrac{\alpha}{\lambda}f^{1'}_{p,0} \end{cases}. \quad (23)$$

By the superprimes we have denoted the derivatives with respect to the spatial variable x. Note that for the case $\lambda_0=0$ the differential equations (23) are represented as follows

$$\begin{cases} \rho f^1_{p,0}+sf^{2'}_{p,0}+\beta f^2_{p,0}=0 \\ \alpha f^{1'}_{p,0}=0 \end{cases} \quad (24)$$

and, hence, in this case the general solution of these equations is

$$f^1_{p,0}(x,\lambda_0)=c_0,\ f^2_{p,0}(x,\lambda_0)=c_1e^{-\frac{\beta}{s}x}-2c_0.$$

The boundary conditions (6) are satisfied for these functions iff $c_1=0$ such that the function

$$e_{1,0}(x,\lambda_0)=\begin{bmatrix} f^1_{p,0}(x,\lambda_0) \\ f^2_{p,0}(x,\lambda_0) \end{bmatrix}=c_0\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

is the eigenfunction of the operator L corresponding the trivial eigenvalue $\lambda_0=0$.

The corresponding characteristic polynomial for the first differential equation of (23) is:

$$s\alpha K^2+\alpha\beta K+\lambda(\lambda+\rho)=0 \quad (25)$$

where $\alpha<0, s>0, \beta>0, \rho>0$ and the roots are:

$$K_{1,2}=\frac{-\alpha\beta\pm\sqrt{(\alpha\beta)^2-4s\alpha\lambda(\lambda+\rho)}}{2s\alpha}=$$

$$\frac{-\beta}{2s}\pm\sqrt{\frac{\beta^2}{4s^2}-\frac{\lambda(\lambda+\rho)}{s\alpha}}=a\pm jb, \quad (26)$$

where j is the imaginary unit $j^2=-1$, and

$$a=\frac{-\beta}{2s},\ b=\sqrt{-\Big(\frac{\beta^2}{4s^2}-\frac{\lambda(\lambda+\rho)}{s\alpha}\Big)}. \quad (27)$$

From the theory of linear differential equations follows that (23) in accordance with the roots $K_{1,2}$ of characteristic polynomial (26) have the following solutions $f^1_{p,0}(x,\lambda)$ and $f^2_{p,0}(x,\lambda)$:

$$\begin{cases} f^1_{p,0}(x,\lambda)=c_1e^{(a+jb)x}+c_2e^{(a-jb)x}) \\ f^2_{p,0}(x,\lambda)=\dfrac{\alpha}{\lambda}[c_1(a+jb)e^{(a+jb)x}+c_2(a-jb)e^{(a-jb)x}]. \end{cases}$$

In this case due to the boundary conditions (6) and (7), we obtain the following algebraic system

$$\begin{cases} c_1+c_2=c_1e^{a+jb}+c_2e^{a-jb} \\ c_1(a+jb)+c_2(a-jb)= \\ c_1(a+jb)e^{a+jb}+c_2(a-jb)e^{a-jb} \end{cases} \quad (28)$$

This system has a nontrivial solution $c_1\neq 0,\ c_2\neq 0$ iff

$$\det\begin{vmatrix} 1-e^{a+jb} & 1-e^{a-jb} \\ (a+jb)(1-e^{a+jb}) & (a-jb)(1-e^{a-jb}) \end{vmatrix}=0. \ (29)$$

And finally the system (28) has a nontrivial solution $c_1\neq 0,\ c_2\neq 0$ iff

$$2(1-e^{a+jb})(1-e^{a-jb})jb=0$$

that is equivalent to

$$1-e^{a+jb}=0,\ \ 1-e^{a-jb}=0,\ \text{and}\ b=0. \quad (30)$$

The condition $b=0$ and (27) lead to the relation

$$\frac{\beta^2}{4s^2} - \frac{\lambda(\lambda+\rho)}{s\alpha} = 0 \tag{31}$$

Hence, in this case we have

$$\lambda_{1,2} = -\beta \pm \beta\sqrt{1+\frac{4\alpha}{s}}. \tag{32}$$

The first and second conditions of (30) can be rewritten as $e^{a+jb} = e^0$ and $e^{a-jb} = e^0$, respectively. Since the exponential function $e^z, z \in C$ has imaginary period, we have $a + jb = 2\pi nj,\ n = 0, \pm 1, \pm 2, ...$ and $a - jb = 2\pi kj,\ k = 0, \pm 1, \pm 2, ...$

Due to (26) the corresponding eigenvalues $\lambda^{(n)}$ satisfy the following relations

$$\pm\sqrt{\frac{\beta^2}{4s^2} - \frac{\lambda(\lambda+\rho)}{s\alpha}} = 2\pi nj + \frac{\beta}{2s},\ n = 0, \pm 1, ... \tag{33}$$

and hence we have for $n = 0, \pm 1, \pm 2, ...$

$$\lambda^{(n)}_{1,2} = -\beta + \sqrt{\beta^2 - 2\alpha\beta\pi nj + 4s\alpha\pi^2 n^2}. \tag{34}$$

(Here we write + instead ± since the complex valued square root are two values.)

After carrying out some lengthly but routine calculation. it can be shown that the corresponding eigenfunction are given by the following formulas:

i) for the case $K_{1,2} = a \pm jb$ where $b = 0$ and the corresponding eigenvalues λ_1, λ_2 are given by (32) the needed eigenfunctions are:

$$e_{1,0}(x,\lambda_1) = \begin{bmatrix} f^1_{p,0}(x,\lambda_1) \\ f^2_{p,0}(x,\lambda_1) \end{bmatrix} = c_1 e^{ax} \begin{bmatrix} 1 \\ \dfrac{a\alpha}{\lambda_1} \end{bmatrix},$$

$$e_{1,0}(x,\lambda_2) = \begin{bmatrix} f^1_{p,0}(x,\lambda_2) \\ f^2_{p,0}(x,\lambda_2) \end{bmatrix} = c_2 e^{ax} \begin{bmatrix} 1 \\ \dfrac{a\alpha}{\lambda_2} \end{bmatrix}.$$

ii) for the case $K_{1,2} = a \pm jb = 2\pi nj$, where $n = 0, \pm 1, \pm 2, ...$ and the corresponding eigenvalues $\lambda^{(n)}_{1,2}$ are given by (32) we have the following eigenfunctions

$$e_{1,0}(x,\lambda^{(n)}_i) = \begin{bmatrix} f^1_{p,0}(x,\lambda^{(n)}_i) \\ f^2_{p,0}(x,\lambda^{(n)}_i) \end{bmatrix} =$$

$$c^{(n)}_i e^{2\pi njx} \begin{bmatrix} 1 \\ \dfrac{2a\alpha\pi nj}{\lambda^{(n)}_i} \end{bmatrix}, i = 1, 2,\ n = 0, \pm 1, ... \tag{35}$$

Adjoint operator. The eigenvalue problem of the operator $L^\dagger$ is

$$[\hat{\lambda}I - L^\dagger]\epsilon_{p,0}(x,\hat{\lambda}) = 0,\ p = 1, \dots, P(\lambda), \tag{36}$$

where $\epsilon_{p,0}(x,\hat{\lambda}) = [z^1_{p,0}(x,\hat{\lambda}), z^2_{p,0}(x,\hat{\lambda})]^T$ and $P(\hat{\lambda})$ is the unknown number of all linear independent eigenvectors corresponding to the eigenvalue $\hat{\lambda}$. Let us rewrite equation (36) in the following form

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\lambda} & 0 \\ 0 & \hat{\lambda} \end{bmatrix} \begin{bmatrix} z^1_{p,0} \\ z^2_{p,0} \end{bmatrix} +$$

$$\begin{bmatrix} \rho & \alpha D_x \\ \beta - sD_x & 0 \end{bmatrix} \begin{bmatrix} z^1_{p,0} \\ z^2_{p,0} \end{bmatrix} \tag{37}$$

then we have the following linear system for $z^1_{p,0}$ and $z^2_{p,0}$

$$\begin{cases} \hat{\lambda} z^1_{p,0} + \rho z^1_{p,0} + \alpha z^{2'}_{p,0} = 0 \\ \hat{\lambda} z^2_{p,0} + \beta z^1_{p,0} - s z^{1'}_{p,0} = 0 \end{cases} \xrightarrow{\hat{\lambda} \neq 0}$$

$$\begin{cases} s\alpha z^{1''}_{p,0} - \alpha\beta z^{1'}_{p,0} + \hat{\lambda}(\hat{\lambda}+\rho) z^1_{p,0} = 0 \\ z^2_{p,0} = -\dfrac{\beta}{\hat{\lambda}} z^1_{p,0} + \dfrac{s}{\hat{\lambda}} z^{1'}_{p,0}. \end{cases} \tag{38}$$

Again, by the superprimes we have denoted the derivatives with respect to the spatial variable x.

It is easy to check that for the case $\hat{\lambda}_0 = 0$ the corresponding eigenfunction is

$$\epsilon_{1,0}(x,\hat{\lambda_0}) = \begin{bmatrix} f^1_{p,0}(x,\hat{\lambda_0}) \\ f^2_{p,0}(x,\hat{\lambda_0}) \end{bmatrix} = c_0 \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The characteristic polynomial for the first equation (38) is given as

$$s\alpha K^2 - \alpha\beta K + \hat{\lambda}(\hat{\lambda}+\rho) = 0 \tag{39}$$

and the roots are:

$$K_{1,2} = \frac{\alpha\beta \pm \sqrt{(\alpha\beta)^2 - 4s\alpha\hat{\lambda}(\hat{\lambda}+\rho)}}{2s\alpha} =$$

$$= \frac{\beta}{2s} \pm \sqrt{\frac{\beta^2}{4s^2} - \frac{\hat{\lambda}(\hat{\lambda}+\rho)}{s\alpha}} = -a \pm jb, \tag{40}$$

where again

$$a = \frac{-\beta}{2s}, b = \sqrt{-\left(\frac{\beta^2}{4s^2} - \frac{\hat{\lambda}(\hat{\lambda}+\rho)}{s\alpha}\right)}.$$

Hence, the linear differential equations of (38) have the general solutions $z^1_{p,0}(x,\hat{\lambda})$ and $z^2_{p,0}(x,\hat{\lambda})$ of the following form:

$$\begin{cases} z^1_{p,0}(x,\hat{\lambda}) = c_1 e^{(-a+jb)x} + c_2 e^{(-a-jb)x}) \\ z^2_{p,0}(x,\hat{\lambda}) = -\dfrac{1}{\hat{\lambda}}\Big[c_1 e^{(-a+jb)x}(as - jbs + \beta) + \\ \qquad + c_2 e^{(-a-jb)x}(as + jbs + \beta)\Big]. \end{cases} \tag{41}$$

In this case due to the boundary conditions (20), we obtain the following algebraic equations

$$\begin{cases} c_1 + c_2 = c_1 e^{-a+jb} + c_2 e^{-a-jb}, \\ c_1((a - jb)s + \beta) + c_2((a + jb)s + \beta) = \\ c_1((a - jb)s + \beta)e^{-a+jb} + c_2((-a - jb)s + \beta)e^{-a-jb} \end{cases}$$

This system has a nontrivial solution $c_1 \neq 0,\ c_2 \neq 0$ iff

$$\det \begin{vmatrix} 1-e^{-a+jb} & 1-e^{-a-jb} \\ A(1-e^{a+jb}) & B(1-e^{-a-jb}) \end{vmatrix} = 0. \quad (42)$$

where $A = ((a+jb)s+\beta), B = ((a-jb)s+\beta)$. And finally the system (42) has a nontrivial solution $c_1 \neq 0,\ c_2 \neq 0$ iff

$$2(1-e^{-a+jb})(1-e^{-a-jb})jb = 0$$

that is equivalent to

$$1-e^{-a+jb} = 0,\ \ 1-e^{-a-jb} = 0,\ \text{and } b = 0. \quad (43)$$

The condition $b = 0$ and (40) give

$$\hat{\lambda}_{1,2} = -\beta \pm \beta\sqrt{1+\frac{4\alpha}{s}}. \quad (44)$$

The first and second conditions of (43) can be rewritten as $e^{-a+jb} = e^0$ and $e^{-a-jb} = e^0$, respectively. Again, since the exponential function $e^z, z \in C$ has imaginary period, we have $-a+jb = 2\pi nj,\ n = 0, \pm1, \pm2, ...$ and $-a-jb = 2\pi kj,\ k = 0, \pm1, \pm2, ...$ Due to (40) the corresponding eigenvalues $\hat{\lambda}^{(n)}$ are given

$$\hat{\lambda}^{(n)}_{1,2} = -\beta + \sqrt{\beta^2 - 2\alpha\beta\pi nj + 4s\alpha\pi^2 n^2}. \quad (45)$$

After carrying out some lengthly but routine calculation. it can be shown that the corresponding eigenfunction are given by the following formulas:

i) for the case $K_{1,2} = -a \pm jb$ where $b = 0$ and the corresponding eigenvalues $\hat{\lambda}_i,\ i-1,2$ are given by (44) the needed eigenfunctions are:

$$\epsilon_{1,0}(x,\hat{\lambda}_i) = \begin{bmatrix} z^1_{p,0}(x,\hat{\lambda}_i) \\ z^2_{p,0}(x,\hat{\lambda}_i) \end{bmatrix} = c_i e^{-ax} \begin{bmatrix} 1 \\ -\dfrac{2(as+\beta)\alpha}{\hat{\lambda}_i} \end{bmatrix}.$$

ii) for the case $K_{1,2} = -a \pm jb = 2\pi nj$ where $n = 0, \pm1, \pm2, ...$ and the corresponding eigenvalues $\hat{\lambda}^{(n)}_{1,2},\ n = 0, \pm1, \pm2, ...$ are given by (44) we have the following eigenfunctions

$$\epsilon_{1,0}(x,\hat{\lambda}^{(n)}_i) = c^{(n)}_i e^{2\pi njx} \begin{bmatrix} 1 \\ -\dfrac{2(as+\beta)\alpha\pi nj}{\hat{\lambda}^{(n)}_i} \end{bmatrix} \quad (46)$$

Thus, the needed eigenfunctions for the prime L and the adjoint $L^\dagger$ operators are obtained. Hence, in accordance with Section 3, the solution of the problem (2) - (6) in the Laplace domain is given by formula (16).

5. CONCLUSION

This paper presents the multifunctional transformation (MFT) method for solution of PDE's arisen in modeling transient gas flow networks of pipelines. The subject of ongoing work is also the development of numerical algorithms and experiments for models considered here applied to gas transportation networks based on the real data. Note that some elements of the functional transformation method was used in digital sound modeling (Trautmann L., Rabenstein R., 2003). It is necessary to add that this paper covers only first attempts to investigate the pipeline units on the base of new calculus operational approach developed for partial differential equations in Dymkou V. (2006) and which are available for the well known computationally efficient parallel numerical methods, and which was successfully applied in Dymkou V., Potherat A. (2009) and Potherat A., Dymkou V. (2010). These results constitute a very promising base for further research towards applications to the other real models, for example, to Timoshenko S. (1921) beam equation that is used now in nanotechnology to design atomic microscope where the so-called cantilever beam of the size $200 \times 35 \times 2mkm$ is a key element.

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