Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

The Specifics of Closed-Loop Impulse Control ⋆

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Abstract: This paper deals with the problem of designing feedback solutions for systems with impulsive control inputs which meets increasing demand. It emphasizes the specifics of such solutions. Their description opens routes for effective calculation and application.

Keywords: impulse control, dynamic programming, viscosity solutions

1. INTRODUCTION

Problems of feedback impulse control and their solution are the object of increasing demand. Their formulation and some solutions approaches were indicated by Bensoussan and Lions [1982], Motta and Rampazzo [1996], Kurzhanski and Daryin [2008], El Farouq et al. [2010], Daryin et al. [2011], Daryin and Kurzhanski [2013].

The design of such solutions through Hamiltonian techniques and related nonlinear analysis faces specifics which differs them from usual dynamic programming schemes for standard problems of control. The present paper emphasizes such specifics indicating analytical schemes and solution properties for both feedback and feed-forward robust impulse control. Such information is important for effective calculation and realistic application.

2. NOTATION

A *modulus* is a scalar continuous non-decreasing function $\omega(\cdot)$ such that $\omega(0) = 0$. Let $\mathscr{D}$ be a subset of a normed space. Function $f : \mathscr{D} \to \mathbb{R}$ is *uniformly continuous* on $\mathscr{D}$, if there exists a modulus $\omega_f(\cdot)$ such that

$$|f(x) - f(y)| \leq \omega_f(\|x - y\|), \quad \forall x, y \in \mathscr{D}.$$

It is always possible to select a modulus of sublinear growth, i.e. there exists a constant C_f such that

$$|f(x) - f(y)| \leq \omega_f(\|x - y\|) \leq C_f(1 + \|x - y\|).$$

The set of uniformly continuous functions bounded from below by a constant on $\mathscr{D}$ is denoted by $UC_{bb}(\mathscr{D})$. If function f is bounded, its supremum is denoted by M_f. If $\omega_f(t) = L_f t$, then function f is Lipschitz-continuous with constant L_f. $BV([t_0, t_1 + 0); \mathbb{R}^m)$ is the set of m-vector functions of bounded variation on $[t_0, t_1]$; such functions are assumed left-continuous. $\chi(t)$ is the step function, equal to 0 for $t \leq 0$ and 1 otherwise.

3. THE IMPULSE CONTROL PROBLEM

Consider an impulse control system for $s \in [t, t_1]$:

$$dx(s) = f(s, x(s))ds + B(s)dU(s), \quad x(t) = x, \tag{1}$$

and the functional of Mayer–Bolza type to be minimized

$$J(U(\cdot) \mid t, x) = \mathrm{Var}_{[t,t_1+0)} U(\cdot) + \varphi(x(t_1 + 0)) \tag{2}$$

where $x(s) = x(s \mid U(\cdot), t, x)$ is the trajectory of (1) under control $U(\cdot)$ emanating from $x(t) = x$.

Here $x \in \mathbb{R}^n$ is the state vector. Control $U(\cdot)$ is a m-vector function of bounded variation. Terminal functional $\varphi(\cdot)$ is continuous and bounded from below. Function $f(t, x)$ is continuous in t and Lipschitz-continuous in x. Matrix function $B(s) \in \mathbb{R}^{n\times m}$ is taken to be continuous.

For linear systems (when $f(t, x) = A(t)x$), a well-known result by Krasovskii [1959] is that there exists an optimal control of form

$$U^*(s) = \sum_{i=1}^{N} h_i \chi(s - \tau_i), \quad t \leq \tau_1 < \cdots < \tau_N \leq t_1. \tag{3}$$

Furthermore, Neustadt [1964] indicated that $N \leq n$. For nonlinear systems this may not be the case. However, since functions of form (3) are weakly* dense in $BV[t, t_1 + 0)$ and the functional J is weakly* lower semicontinuous, we may take the minimum over controls of such type.

Definition 1. The *value function* for impulse control system (1) under functional (2) is

$$V(t, x) = \inf_{U(\cdot)} J(U(\cdot) \mid t, x) \tag{4}$$

where the infimum is taken over controls of form (3).

Note that unlike initial time t which is varied, the terminal time t_1 is still fixed in advance. Where necessary, we shall use extended notation [1] $V(t, x) = V(t, x; t_1, \varphi(\cdot))$ in order to emphasize dependence of optimal value on terminal time t_1 and terminal function $\varphi(\cdot)$. The corresponding extended notation for functional J is

$$J(U(\cdot) \mid t, x) = J(U(\cdot) \mid t, x; t_1, \varphi(\cdot)).$$

3.1 Properties of the Value Function

Theorem 1. The value function $V(t, x; t_1, \varphi(\cdot))$ satisfies the **Principle of Optimality**. For all $\tau \in [t, t_1]$

$$V(t, x; t_1, \varphi(\cdot)) = V(t, x; \tau, V(\tau, \cdot; t_1, \varphi(\cdot))). \tag{5}$$

⋆ This work is supported by Russian Foundation for Basic Research (grants 12-01-00261-a, 13-01-90419-Ukr-f-a) and by the program "State Support of the Leading Scientific Schools" (NS-2692.2014.1).

[1] This should not be confused with notation for directional derivative $V'(t, x \mid \vartheta, \xi)$ of $V(t, x)$ along direction (ϑ, ξ).

Proof. Here a special attention should be paid to the instant τ, since the minimum over possible impulses at time τ appears twice at the right-hand side of (5). Unlike proof for problems with bounded inputs, control may change the state at this very instant. The following proof covers both cases (when there is an impulse at time τ or there is none).

For any $\varepsilon > 0$ there exists a control $U_\varepsilon(\cdot)$ such that

$$\alpha = V(t,x) \geq J(U_\varepsilon(\cdot) \mid t, x; t_1, \varphi(\cdot)) - \varepsilon.$$

We split $U_\varepsilon(\cdot) = U_\varepsilon^{(1)}(s) + U_\varepsilon^{(2)}(s)$ into

$$U_\varepsilon^{(1)}(s) = \begin{cases} U_\varepsilon(s), & s \leq \tau; \\ U_\varepsilon(\tau+0), & s > \tau; \end{cases}$$

$$U_\varepsilon^{(2)}(s) = \begin{cases} U_\varepsilon(\tau+0), & s \leq \tau; \\ U_\varepsilon(s), & s > \tau. \end{cases}$$

Note that $\operatorname{Var} U_\varepsilon(\cdot) = \operatorname{Var} U_\varepsilon^{(1)}(\cdot) + \operatorname{Var} U_\varepsilon^{(2)}(\cdot)$. We have

$$\begin{aligned} \beta &= V(t,x;\tau,V(\tau,\cdot)) \\ &= \inf_{BV[t,\tau+0]} J(U^{(1)}(\cdot) \mid t,x;\tau,V(\tau,\cdot)) \\ &\leq J(U_\varepsilon^{(1)}(\cdot) \mid t,x;\tau,V(\tau,\cdot)) \\ &= \operatorname{Var} U_\varepsilon^{(1)}(\cdot) + \inf_{BV[\tau,t_1+0)} J(U^{(2)}(\cdot) \mid \tau, x(\tau+0); t_1, \varphi(\cdot)) \\ &\leq \operatorname{Var} U_\varepsilon^{(1)}(\cdot) + J(U_\varepsilon^{(2)}(\cdot) \mid \tau, x(\tau+0); t_1, \varphi(\cdot)) \\ &= \operatorname{Var} U_\varepsilon^{(1)}(\cdot) + \operatorname{Var} U_\varepsilon^{(2)}(\cdot) + \varphi(x(t_1+0)) \\ &= J(U_\varepsilon(\cdot) \mid t,x) \leq \alpha + \varepsilon. \end{aligned}$$

Here $x(s)$ is the corresponding state trajectory. Since $\varepsilon > 0$ is arbitrary, we get $\alpha \geq \beta$.

Now we prove the opposite inequality. For any $\varepsilon > 0$ there exists a control $U_\varepsilon^{(1)}(\cdot)$ such that

$$\beta = V(t,x;\tau,V(\tau,\cdot)) \geq J(U_\varepsilon^{(1)}(\cdot) \mid t,x;\tau,V(\tau,\cdot)) - \tfrac{\varepsilon}{2},$$

and furthermore there exists a control $U_\varepsilon^{(2)}(\cdot)$ such that

$$\begin{aligned} &V(\tau, x^{(1)}(\tau+0); t_1, \varphi(\cdot)) \\ &\qquad \geq J(U_\varepsilon^{(2)}(\cdot) \mid \tau, x^{(1)}(\tau+0); t_1, \varphi(\cdot)) - \tfrac{\varepsilon}{2}. \end{aligned}$$

Here $x^{(1)}(s)$ is the trajectory that corresponds to $U_\varepsilon^{(1)}(\cdot)$ emanating from $x^{(1)}(t) = x$, $x^{(2)}(s)$ is the trajectory that corresponds to $U_\varepsilon^{(2)}(\cdot)$ emanating from $x^{(2)}(\tau) = x^{(1)}(\tau+0)$. We need separate notations for two parts of the trajectory since both $U_\varepsilon^{(1)}(\cdot)$ and $U_\varepsilon^{(2)}(\cdot)$ may include the impulse at time τ.

We define control

$$U_\varepsilon(s) = \begin{cases} U_\varepsilon^{(1)}(s) + U_\varepsilon^{(2)}(\tau), & s \leq \tau; \\ U_\varepsilon^{(1)}(\tau+0) + U_\varepsilon^{(2)}(s), & s > \tau. \end{cases}$$

We have $\operatorname{Var} U_\varepsilon(\cdot) \leq \operatorname{Var} U_\varepsilon^{(1)}(\cdot) + \operatorname{Var} U_\varepsilon^{(2)}(\cdot)$, and if $x(s)$ corresponds to $U_\varepsilon(\cdot)$ and starts from $x(t) = x$, then $x(t_1 + 0) = x^{(2)}(t_1+0)$. Thus

$$\begin{aligned} \beta &\geq \operatorname{Var} U_\varepsilon^{(1)}(\cdot) + \operatorname{Var} U_\varepsilon^{(2)}(\cdot) + \varphi(x^{(2)}(t_1+0)) - \varepsilon \geq \\ &\operatorname{Var} U_\varepsilon(\cdot) + \varphi(x(t_1+0)) - \varepsilon \geq V(t,x) - \varepsilon = \alpha - \varepsilon, \end{aligned}$$

and again since ε is arbitrary, we get $\beta \geq \alpha$.

Remark 1. The proof does not exclude cases $\tau = t$, $\tau = t_1$, or both $\tau = t = t_1$.

Remark 2. The Principle of Optimality is of crucial importance in what follows. In particular, it implies that the pair (t,x) is the *state* of the system which contains all the information required to solve the problem within the remaining time interval $[t, t_1]$.

Now we derive some infinitesimal properties of $V(t,x)$ implied by the Principle of Optimality. We need to make a technical assumptions on the value function.

Assumption 1. Value function is directionally differentiable at (t,x). The mapping $g(\tau,\xi) = V'(t,x \mid \tau,\xi)$ is continuous in (τ,ξ).

This assumption is typical for linear systems. More generally it holds if V is directionally differentiable at (t,x) and Lipschitz-continuous in the neighborhood of (t,x).

Lemma 1. For all $x \in \mathbb{R}^n$ function $f(s) = V(s, x_0(s))$, $s \in [t, t_1]$, is non-decreasing, where $x_0(s)$ is the trajectory under zero control emanating from $x(t) = x$.

Proof. By Principle of Optimality

$$V(t,x) = \inf_{U(\cdot)} J(U(\cdot) \mid t,x;s,V(s,\cdot)).$$

A choice of $U(\cdot) \equiv 0$ yields

$$f(t) = V(t,x) \leq V(s, x_0(s)) = f(s).$$

Corollary 1. The right directional derivative of f is non-negative:

$$f'_+(t) = V'(t,x \mid 1, f(t,x)) \geq 0,$$

or, if V is differentiable at (t,x), then

$$V_t + \langle V_x, f(t,x) \rangle \geq 0.$$

Lemma 2. For all $h \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $t \leq t_1$ the following inequality is true:

$$V(t,x) \leq V(t, x + B(t)h) + \|h\|. \tag{6}$$

Proof. By principle of Optimality

$$V(t,x) = \inf_{U(\cdot)} J(U(\cdot) \mid t,x;t,V(t,\cdot)).$$

Taking a specific control $U(s) = h\chi(s-t)$, we immediately come to (6).

Corollary 2. Setting $h = \varepsilon\xi$ and passing to the limit as $\varepsilon \downarrow 0$, we have

$$|V'(t,x \mid 0, B(t)\xi)| \leq \|\xi\|,$$

or, if $V(t,x)$ is x-differentiable at (t,x), then

$$|\langle V_x, B(t)\xi \rangle| \leq \|\xi\|.$$

3.2 The Hamilton–Jacobi–Bellman Equation

The Principle of Optimality may now be used for deriving an analogy of the HJB equation. At each state (t,x) there are two possibilities. Either we may choose an optimal control without an impulse at time t, or all optimal controls will have an impulse.

In the **first case** (impulse at t,x is not required), the value function should remain constant under zero control input. Then its total right derivative

$$dV/dt|^+_{dU=0} = V_t + \langle V_x, f(t,x) \rangle = 0.$$

Note that due to Corollary **??** for an arbitrary state (t,x) there is the inequality

$$dV/dt|^+_{dU=0} \geq 0. \tag{7}$$

In the **second case** (mandatory impulse at t, x) there should exist a vector $h \in \mathbb{R}^m$ such that

$$V(t,x) = V(t, x + B(t)h) + \|h\|.$$

That is, variation of $dU(\tau) = h\delta(\tau - t)$ should be compensated by an equal decrease of the value function. Since $V(t,x)$ is convex, from Lemma 2 we have

$$V(t,x) = V(t, x + \varepsilon B(t)h) + \varepsilon \|h\|, \quad \varepsilon \in [0,1].$$

Hence $V'(t, x \mid 0, B(t)h) = -\|h\|$, while for an arbitrary state (t,x) and arbitrary h due to Lemma 2 one has

$$V'(t, x \mid 0, B(t)h) \geq -\|h\|. \tag{8}$$

Summarizing, we observe that at each state (t,x) the value function satisfies two inequalities (7) and (8), with at least one of them turning into an equality. Introducing two Hamiltonians

$$H_1 = V'(t, x \mid 1, f(t,x)) \quad \text{and} \tag{9}$$

$$H_2 = \min_{\|h\|=1} \{V'(t, x \mid 0, B(t)h) + \|h\|\}, \tag{10}$$

this condition may be presented

$$\min\{H_1,\ H_2\} = 0. \tag{11}$$

Relation (11) is called the Hamilton–Jacobi–Bellman equation (HJB in abbreviated form). We have therefore proven

Theorem 2. Under Assumption 1 the value function $V(t,x)$ satisfies the HJB equation (11) with initial condition

$$V(t_1, x) = V(t_1, x; t_1, \varphi(\cdot)). \tag{12}$$

The Hamiltonian H_1 describes a motion with zero control input, while H_2 describes impulses. At points of differentiability for V they are expressed in more conventional form as

$$\begin{aligned} H_1 &= H_1(t, x, V_t, V_x) = V_t + \langle V_x, f(t,x)\rangle, \\ H_2 &= H_2(t, x, V_t, V_x) = \min_{\|h\|=1} \{\langle V_x, B(t)h\rangle + \|h\|\} \\ &= 1 - \left\|B^T(t)V_x\right\|. \end{aligned}$$

The solution to (11) is understood in the sense of directional derivatives, and $V(t,x)$ satisfies it everywhere on $[t_0, t_1] \times \mathbb{R}^n$. However, such a solution is not necessarily unique, as shown in the following example.

Example 1. Consider a linear impulse control problem with $A(t) \equiv 0$, $B(t) \equiv I$ and affine terminal function: $\varphi(x) = \langle c, x\rangle + d$, $\|c\| = 1$. The value function $V(t,x) = \varphi(x)$ satisfies the HJB equation

$$\min\{V_t,\ 1 - \|V_x\|\} = 0, \quad V(t_1, x) = \varphi(x),$$

in the classical sense. But so does any function of form

$$W(t,x) = f(t) + \varphi(x), \quad f'(t) \geq 0, \quad f(t_1) = 0.$$

We now proceed to characterize the value function as the *only* solution to HJB equation under additional assumptions. See Crandall et al. [1984], Bardi and Capuzzo-Dolcetta [1997] for the definition and properties of viscosity solutions.

Theorem 3. Value function $V(t,x)$ is a viscosity solution of HJB equation

$$\begin{aligned} &\max\{-H_1,\ -H_2\} \\ &= \max\left\{-V_t - \langle V_x, f(t,x)\rangle,\ \left\|B^T(t)V_x\right\| - 1\right\} = 0 \end{aligned} \tag{13}$$

with initial condition (12).

Proof. Let (t,x) be a maximum of $V - \phi$ and $V(t,x) = \phi(t,x)$. We shall prove that $-H_1(t, x, \phi_t, \phi_x) \leq 0$ and $-H_2(t, x, \phi_t, \phi_x) \leq 0$. By Lemma 1

$$\begin{aligned} \phi(t+\varepsilon, x_0(t+\varepsilon)) &\geq V(t+\varepsilon, x_0(t+\varepsilon)) \\ &\geq V(t,x) = \phi(t,x), \end{aligned}$$

and hence $d\phi/dt = \phi_t + \langle \phi_x, f(t,x)\rangle \geq 0$, leading to $-H_1(t, x, \phi_t, \phi_x) \leq 0$.

Next we take a unit vector h and by Lemma 2 we get

$$\begin{aligned} \phi(t, x + \varepsilon B(t)h) &\geq V(t, x + \varepsilon B(t)h) \\ &\geq V(t,x) - \varepsilon = \phi(t,x) - \varepsilon. \end{aligned}$$

Dividing by ε and passing to the limit as $\varepsilon \to 0$, this yields

$$-\langle \phi_x, B(t)h\rangle \leq 1.$$

Since this holds for all h from unit sphere, we maximize over h and get

$$\left\|B^T(t)\phi_x\right\| \leq 1 \quad \Longleftrightarrow \quad -H_2(t, x, \phi_t, \phi_x) \leq 0.$$

Now let (t,x) be a minimum of $V - \phi$ and $V(t,x) = \phi(t,x)$. We shall prove that either $-H_1(t, x, \phi_t, \phi_x) \geq 0$ or $-H_2(T, x, \phi_t, \phi_x) \geq 0$.

There exists control $U^*(\cdot)$ of type (3) such that

$$U^*(s) = \sum\nolimits_{i=1}^{N} h_i \chi(s - \tau_i), \quad t \leq \tau_1 < \cdots < \tau_N \leq t + \varepsilon.$$

with $N \leq n$, $h_i \neq 0$ and

$$V(t,x) = J(U^*(t,x) \mid t, x; t_1, \varphi(\cdot)).$$

If $N = 0$ or $\tau_1 > t$, then for sufficiently small ε

$$\phi(t+\varepsilon, x) \leq V(t+\varepsilon, x_0(t+\varepsilon)) = V(t,x) = \phi(t,x)$$

so that

$$d\phi/dt = \phi_t + \langle \phi_x, f(t,x)\rangle \leq 0$$

and thus $-H_1(t, x, \phi_t, \phi_x) \geq 0$. Otherwise if $\tau_1 = t$, then set $h = h_1 / \|h_1\|$ and for $0 \leq \varepsilon < \|h_1\|$ we have

$$\begin{aligned} \phi(t, x + \varepsilon B(t)h) &\leq V(t, x + \varepsilon B(t)h) \\ &= V(t,x) - \varepsilon = \phi(t,x) - \varepsilon. \end{aligned}$$

Dividing by ε we get $\langle \phi_x, B(t)h\rangle \leq -1$, thus $\left\|B^T(t)\phi_x\right\| \geq 1$ and $-H_2(t, x, \phi_t, \phi_x) \geq 0$.

3.3 The Feedback Control Law

Due to (11), in any position (t,x) there are following possibilities for the control. Firstly, if $H_2 > 0$, then $H_1 = 0$, and the control should be chosen locally as $dU = 0$.

Secondly, if $H_1 > 0$, in which case it is necessary that $H_2 = 0$, and the control has a jump in direction $h = -B^T(t)V_x$ (or, if V is not differentiable at (t,x), in a direction h such that $V'(t, x \mid 0, h) = -\|h\|$). The magnitude $\lambda > 0$ of the jump is to be selected such that after the jump we again have $H_1 = 0$. Then (locally) the control will be

$$dU(\tau) = -\lambda h\, d\chi(\tau - t). \tag{14}$$

Finally, if both $H_1 = 0$ and $H_2 = 0$, then additional analysis is necessary. One of the following cases (or both at the same time) are possible.

- If for a sufficiently small $\varepsilon > 0$ one has $V(t+\varepsilon, x_0(t+\varepsilon)) = V(t,x)$, the control may be chosen as zero.
- If there exist a direction of jump h with positive magnitude λ, then the control may be chosen as (14).

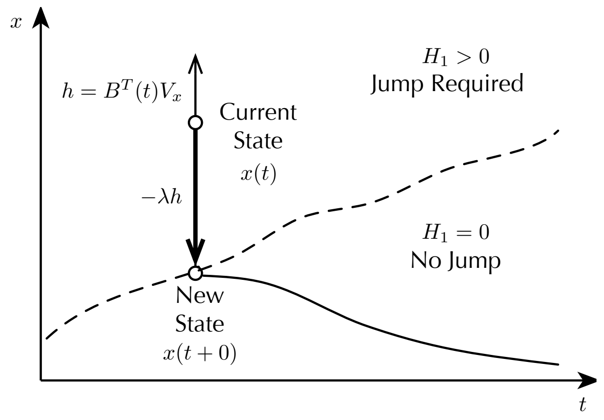

Fig. 1. Structure of impulse feedback law

Control law described above is illustrated in Figure 1. See Daryin and Kurzhanski [2013] for rigorous definition of impulse feedback control.

3.4 The Robust Number of Impulses

As indicated earlier, for linear systems ($f(t,x) = A(t)x$) in the class of open-loop controls there exists a control of form (3) with the number of impulses less or equal to state dimension n. Now we show that under a certain condition the realized open-loop trajectories under optimal feedback control possess the same property.

Namely, suppose that when an impulse is possible (there exists direction of jump h), its amplitude is chosen as either $\lambda = 0$ or $\lambda = \lambda_{\max}$. Here $\lambda_{\max}$ is the maximum possible amplitude (so that impulses of amplitude $\lambda > \lambda_{\max}$ are not optimal according to HJB.) Then the realized control trajectory will have at most n impulses.

To prove this fact, we recall the proof for open-loop controls. Consider a control of form

$$U(t) = \sum_{i=1}^{N} \lambda_i h_i \chi(t - \tau_i), \quad \|h_i\| = 1, \quad \lambda_i > 0 \quad (15)$$

with $N \geq n+1$. Its total variation is $\boldsymbol{\lambda} = \sum_{i=1}^{N} \lambda_i$. Form a vector

$$c = \sum_{i=1}^{N} \lambda_i B_i h_i, \quad B_i = X(t_1, \tau_i)B(\tau_i).$$

(Recall that $X(t,\tau)$ is the fundamental matrix corresponding to linear dynamics $A(t)$.) We then construct a control of the same form with fewer impulses with variation less or equal to $\boldsymbol{\lambda}$ yielding the same vector c. Vectors $B_i h_i \in \mathbb{R}^n$, $i = \overline{1,N}$, are linearly dependent, so that there exists a non-trivial linear combination

$$\sum_{i=1}^{N} \alpha_i B_i h_i = 0, \quad \sum_{i=1}^{N} \alpha_i = \boldsymbol{\alpha} \geq 0.$$

If $\boldsymbol{\alpha} < 0$, just change the signs of all α_i. Note that there exists $\alpha_i > 0$. Consider control

$$U_\varkappa(t) = \sum_{i=1}^{N} (\lambda_i - \varkappa \alpha_i) h_i, \quad \varkappa = \min\{\lambda_i / \alpha_i \mid \alpha_i > 0\}.$$

At least one of numbers λ'_i is zero. We further have

$$\operatorname{Var} U_\varkappa(\cdot) = \sum_{i=1}^{N} \lambda'_i = \boldsymbol{\lambda} - \varkappa \boldsymbol{\alpha} = \boldsymbol{\lambda}' \leq \boldsymbol{\lambda},$$

$$\sum_{i=1}^{N} \lambda'_i B_i h_i = \sum_{i=1}^{N} \lambda_i B_i h_i - \varkappa \sum_{i=1}^{N} \alpha_i B_i h_i = c.$$

We proceed this way repeatedly until the number of impulses N is not greater than n.

Note that if $\boldsymbol{\lambda}$ was the minimum variation, then necessarily $\boldsymbol{\lambda}' = \boldsymbol{\lambda}$, $\boldsymbol{\alpha} = 0$ and there exists $\alpha_i < 0$.

Suppose that control (15) was generated by a feedback rule choosing maximum possible values of λ_i. We may assume that $\alpha_1 = \cdots = \alpha_{j-1} = 0$, $\alpha_j < 0$ (otherwise change sings of all α_i). Then $\lambda'_1 = \lambda_1, \ldots, \lambda'_{j-1} = \lambda_{j-1}$, $\lambda'_j > \lambda_j$. This is contradiction with the choice of maximum possible λ_j.

3.5 The Feedforward Control Law

Here we describe the feedforward scheme for calculating realized control and state trajectories.

(1) Set $\sigma_1 = t$.
(2) At time σ_i calculate the optimal open-loop control $U_i^*(\cdot)$ on interval $[\sigma_i, t_1 + 0)$ given state $(\sigma_i, x(\sigma_i))$.
(3) Choose $\sigma_{i+1} \in (\sigma_i + \Delta\sigma_{\min}, t_1]$ (or $\sigma_{i+1} = t_1$ if this interval is empty).
(4) Use $U_i^*(\cdot)$ as control input on $[\sigma_i, \sigma_{i+1} + 0)$.
(5) If $\sigma_{i+1} < t_1$, proceed to step 2.

The proof of robust number of impulses for feedback control from Subsection 3.4 also applies to feedforward controls.

4. BOUNDED SYSTEM DYNAMICS

Consider the following Hamilton–Jacobi equation for function $V(t,x) : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$:

$$\max\{H_1, H_2\} = 0, \quad t \in [t_0, t_1], \quad x \in \mathbb{R}^n, \quad (16)$$
$$H_1 = -V_t - \langle V_x, f(t,x) \rangle, \quad H_2 = \left\| B^T(t) V_x \right\| - 1,$$
$$V(t_1, x) = \boldsymbol{\varphi}(x) = \inf_{h \in \mathbb{R}^m} \{\varphi(x + B(t_1)h) + \|h\|\}.$$

We make the following assumptions.

Assumption 2. Function $\boldsymbol{\varphi}(\cdot) \geq 0$.

Assumption 3. Function $\boldsymbol{\varphi}(\cdot)$ is uniformly continuous.

Assumption 4. Function $f(t,x)$ is Lipschitz-continuous on $[t_0, t_1] \times \mathbb{R}^n$ and satisfies $|\langle x, f(t,x) \rangle| \leq C_f \|x\|$.

Assumption 5. Function $B(t)$ is Lipschitz-continuous on $[t_0, t_1]$ with constant L_B.

Assumption 2 is equivalent to $\varphi(x) \geq 0$. Assumption 3 holds if $\varphi(\cdot)$ is uniformly continuous, but this is not a necessary condition.

An example of Assumption 4 is $f(t,x) = A(t)x + f_0(t,x)$, where $A^T(t) = -A(t)$ and $f_0(t,x)$ is bounded and Lipschitz-continuous.

Theorem 4. Let assumptions 2, 3, 4, 5 be satisfied. If functions V and W from $UC_{bb}([t_0, t_1] \times \mathbb{R}^n)$ are viscosity subsolution and supersolution to (16), respectively, then $V \leq W$.

Proof. Suppose that the opposite holds, i.e. there is a point $(\bar{t}, \bar{x}) \in [t_0, t_1) \times \mathbb{R}^n$ such that

$$V(\bar{t}, \bar{x}) - W(\bar{t}, \bar{x}) \geq 2\delta > 0.$$

Then there exists $\gamma \in (0,1)$ such that

$$\gamma V(\bar{t}, \bar{x}) - W(\bar{t}, \bar{x}) \geq \delta > 0.$$

Denote by $\omega(t)$ the common modulus of continuity of functions V, W and $\boldsymbol{\varphi}$.

Lemma 3. The following estimate holds:
$\gamma V(t,x) - W(s,y) \le \omega(t_1 - t) + \omega(t_1 - s) + \omega(\|x-y\|)$,
or in simplified form, there exists constant C_1 such that
$$\gamma V(t,x) - W(s,y) \le C_1 (1 + \|x-y\|).$$

Define an auxiliary function
$$\Phi(t,x,s,y) = \gamma V(t,x) - W(s,y) - \frac{(t-s)^2 + \|x-y\|^2}{\varepsilon} - \alpha\left(\|x\|^2 + \|y\|^2\right) + \sigma(s+t).$$
Here $\varepsilon \in (0,1)$ and $\alpha > 0$ are sufficiently small. Parameter $\sigma = \delta/4(1+|t_0|+|t_1|) > 0$ is such that $|\sigma t| \le \delta/4$ for all $t \in [t_0, t_1]$. By Lemma 3, $\Phi \to -\infty$ when $\|x\|, \|y\| \to \infty$ and it attains its maximum value at some point (t^*, x^*, s^*, y^*).

Lemma 4. There exists constant C_2 (independent of ε and α) such that
$$(t^* - s^*)^2 + \|x^* - y^*\|^2 \le C_2 \varepsilon\, \omega\left(\sqrt{\varepsilon}\right).$$
Lemma 5. There exists constant C_3 (independent of ε and α) such that $\sqrt{\alpha}\,\|x^*\| \le C_3$, $\sqrt{\alpha}\,\|y^*\| \le C_3$.

Lemma 6. If ε is sufficiently small, then $t^*, s^* < t_1$.

We choose specific test functions
$$\begin{aligned}\phi(t,x) &= W(s^*,y^*) + (\|x-y^*\|^2 + |t-s^*|^2)/\varepsilon \\ &\quad + \alpha(\|x\|^2 + \|y^*\|^2) - \sigma(t+s^*),\\ \psi(s,y) &= \gamma V(t^*,x^*) - (\|x^*-y\|^2 + |t^*-s|^2)/\varepsilon \\ &\quad - \alpha(\|x^*\|^2 + \|y\|^2) + \sigma(t^*+s).\end{aligned}$$
Their derivatives are
$$\begin{aligned}\phi_t(t^*,x^*) &= 2(t^*-s^*)/\varepsilon - \sigma,\\ \psi_s(s^*,y^*) &= 2(t^*-s^*)/\varepsilon + \sigma,\\ \phi_x(t^*,x^*) &= 2(x^*-y^*)/\varepsilon + 2\alpha x^*,\\ \psi_y(s^*,y^*) &= 2(x^*-y^*)/\varepsilon - 2\alpha y^*.\end{aligned}$$
We have
$$\begin{aligned}\Phi(t,x,s^*,y^*) &= \gamma V(t,x) - \phi(t,x), \\ \Phi(t^*,x^*,s,y) &= \psi(s,y) - W(s,y).\end{aligned} \tag{17}$$
Therefore $\gamma V - \phi$ attains its maximum at (t^*,x^*) and $W - \psi$ attains its minimum at (s^*,y^*). Since V is a viscosity subsolution, test function ϕ/γ satisfies at point (t^*,x^*)
$$\begin{cases}\phi_t + \langle \phi_x, f(t^*,x^*)\rangle \ge 0,\\ \left\|B^T(t^*)\phi_x\right\| \le \gamma.\end{cases} \tag{18}$$
W is a viscosity supersolution, and thus ψ satisfies
$$\left[\begin{array}{l}\psi_s + \langle \psi_y, f(s^*,y^*)\rangle \le 0,\\ \left\|B^T(s^*)\psi_y\right\| \ge 1,\end{array}\right. \tag{19}$$
at point (s^*,y^*). We show that neither of the latter two conditions can be satisfied.

In the **first case**,
$$\psi_s - \phi_t + \langle \psi_y, f(s^*,y^*)\rangle - \langle \phi_x, f(t^*,x^*)\rangle \le 0.$$
We have $\psi_s - \phi_t = 2\sigma$ and
$$\begin{aligned}&\langle \phi_x, f(t^*,x^*)\rangle - \langle \psi_y, f(s^*,y^*)\rangle \\ &\quad = 2\langle x^*-y^*, f(x^*) - f(y^*)\rangle/\varepsilon \\ &\quad + 2\alpha\langle x^*, f(x^*)\rangle + 2\alpha\langle y^*, f(y^*)\rangle \\ &\le 2L_f \|x^*-y^*\|^2/\varepsilon + 2\alpha C_f(\|x^*\| + \|y^*\|) \\ &\quad \le 2C_2\omega\left(\sqrt{\varepsilon}\right) + 4\sqrt{\alpha} C_3 C_f \underset{\varepsilon,\alpha\to 0}{\longrightarrow} 0.\end{aligned}$$
Thus for sufficiently small ε, α we have $2\sigma \le 0$ which contradicts the fact that $\sigma > 0$.

In the **second case** $\left\|B^T(s^*)\psi_y\right\| \ge 1$. But at the same time $\left\|B^T(t^*)\phi_x\right\| \le \gamma < 1$. We have
$$\begin{aligned}0 < 1 - \gamma &\le \left\|B^T(s^*)\psi_y\right\| - \left\|B^T(t^*)\phi_x\right\| \\ &\le \left\|B^T(s^*)\psi_y - B^T(t^*)\phi_x\right\| \\ &\le 2\left\|B^T(t^*) - B^T(s^*)\right\| \|x^*-y^*\|/\varepsilon \\ &+ 2\alpha\left\|B^T(t^*)x^*\right\| + 2\alpha\left\|B^T(s^*)y^*\right\| \\ &\le 2L_B C_2 \omega(\sqrt{\varepsilon}) + 4\sqrt{\alpha} C_3 M_B \underset{\varepsilon,\alpha\to 0}{\longrightarrow} 0.\end{aligned}$$
A contradiction since left-hand side is a positive constant.

Thus neither of two cases may take place, and we have arrived at a contradiction, which proves that $V \le W$.

5. UNBOUNDED HJB SOLUTIONS

Requiring solutions to be uniformly continuous effectively means that they are bounded, as well as the terminal function. In order to allow for unbounded solutions, we introduce a change of dependent variable $V = h(\hat{V})$ given by function $h(r)$ such that $h \in C^1(\mathscr{I})$, $h'(r) > 0$, $h(\mathscr{I}) = \mathbb{R}$, $\mathscr{I} = (p,q)$, $-\infty \le p < q \le +\infty$. The HJB equation (16) then rewrites as
$$\max\{H_1, H_2\} = 0, \quad t \in [t_0,t_1], \quad x \in \mathbb{R}^n, \tag{20}$$
$$H_1 = -\hat{V}_t - \langle \hat{V}_x, f(t,x)\rangle, \quad H_2 = \|B^T(t)\hat{V}_x\| - 1/h_r(\hat{V}),$$
$$\hat{V}(t_1,x) = \hat{\boldsymbol{\varphi}}(x) = h^{-1}(\boldsymbol{\varphi}(x)).$$
It is straightforward to check that V is a subsolution (supersolution) to (16) if and only if $\hat{V}$ is a subsolution (supersolution) to (20).

Theorem 5. Suppose that

(1) functions $\hat{\boldsymbol{\varphi}}$, f, B satisfy assumptions 2–5;
(2) $0 < q < \infty$;
(3) functions $\hat{V}$ and $\hat{W}$ from $UC([t_0,t_1]\times\mathbb{R}^n)$ take values in $(p_0,q) \subseteq (p,q)$;
(4) $h_r(r)$ is non-decreasing on (p_0,q);
(5) $\hat{V}$ and $\hat{W}$ are viscosity subsolution and supersolution to (20), respectively.

Then $\hat{V} \le \hat{W}$.

Proof. The proof is similar to Theorem 4. Relations (18) and (19) take form (respectively)
$$\begin{cases}\phi_t + \langle \phi_x, f(t^*,x^*)\rangle \ge 0,\\ \left\|B^T(t^*)\phi_x\right\| \le \gamma/h_r(V);\end{cases} \quad \left[\begin{array}{l}\psi_s + \langle \psi_y, f(s^*,y^*)\rangle \le 0,\\ \left\|B^T(s^*)\psi_y\right\| \ge 1/h_r(W).\end{array}\right.$$

For sufficiently small α and σ we have $\gamma q > \gamma\hat{V}(t^*,x^*) > \hat{W}(s^*,y^*)$. It follows from conditions of the theorem that
$$\frac{1}{h_r(\hat{W})} - \frac{\gamma}{h_r(\hat{V})} > \frac{1-\gamma}{h_r(\hat{W})} > \frac{1-\gamma}{h_r(\gamma q)} = \text{const} > 0.$$
This inequality is used instead of $1 - \gamma > 0$ to prove that the second case is not possible.

Assumption 6. There exists a constant C_h such that
$$\left(h^{-1}\right)_r(r) \le C_h/(1+r^2).$$

This assumption holds for a particular transformation function $V = \tan(\hat{V} - \hat{V}_0)$, $\hat{V} = \arctan V + \hat{V}_0$.

Assumption 7. There exist constants C_1, C_2 such that
$$V(t,x) \geq C_1 \|x\|^{1/2} + C_2.$$
Assumption 8. Function $V(t,x)$ satisfies
$$|V(t,x) - V(t,y)| \leq \omega(\|x-y\|),$$
$$|V(t,x) - V(s,x)| \leq (1+\|x\|)\omega(|t-s|).$$
Lemma 7. Suppose that function $h_r(r)$ satisfies Assumption 6 and is non-decreasing for $r \geq \hat{V}_0$, where $\hat{V}(t,x) \geq \hat{V}_0$; function $V(t,x)$ satisfies Assumptions 7, 8. Then function $\hat{V} = h^{-1}(V)$ is uniformly continuous on $[t_0,t_1] \times \mathbb{R}^n$.

Proof. Note that $\hat{V}_0 > -\infty$ due to Assumption 7. We have from our assumptions for some r^*
$$|W(t,x) - W(t,y)| = \left|h^{-1}(V(t,x)) - h^{-1}(V(t,y))\right|$$
$$= \frac{1}{h_r(r^*)}|V(t,x) - V(t,y)| \leq \frac{\omega(\|x-y\|)}{h_r(p_0)}.$$
Suppose that $V_1 = V(t,x) \geq V_2 = V(s,x)$. Then for some $r^\sharp \in [V_2, V_1]$
$$|W(t,x) - W(s,x)| = \left|h^{-1}(V_1) - h^{-1}(V_2)\right|$$
$$= |V_1 - V_2|/h_r(r^\sharp) \leq |V_1 - V_2|/h_r(V_2)$$
$$\leq C_h \frac{V_1 - V_2}{1+V_2^2} \leq C\omega(|t-s|).$$

6. UNBOUNDED SYSTEM DYNAMICS

Now we relax Assumption 4 to allow for arbitrary linear dynamics.

Assumption 9. Function $f(t,x) = A(t)x + f_0(t,x)$, where $A(t)$ is a continuous matrix function and $f_0(t,x)$ is a Lipschitz- continuous vector function on $[t_0,t_1] \times \mathbb{R}^n$ satisfying $|\langle x, f_0(t,x)\rangle| \leq C_f \|x\|$.

We introduce a change of variables $\hat{x} = X(t_1,t)x$, where $X(t,s)$ is the fundamental matrix corresponding to linear system with matrix $A(t)$: $X_t(t,s) = A(t)X(t,s)$, $X(s,s) = I$. Then HJB equation (20) takes form
$$\max\{H_1, H_2\} = 0, \quad t \in [t_0,t_1], \quad x \in \mathbb{R}^n, \tag{21}$$
$$H_1 = -\hat{V}_t - \langle \hat{V}_{\hat{x}}, f_0(t,x)\rangle, \quad H_2 = \|\hat{B}^T(t)\hat{V}_{\hat{x}}\| - 1/h_r(\hat{V}),$$
$$\hat{V}(t_1,x) = \hat{\varphi}(x).$$
Here matrix function $\hat{B}(t) = X(t_1,t)B(t)$ is Lipschitz-continuous. Since mapping $(t,x) \to (t, X(t_1,t)x)$ is a diffeomorphism, thus if $\hat{V}(t,x)$ is a sub- or supersolution to (20), then the transformed function $\hat{V}(t,\hat{x})$ is a sub- or supersolution to (21). This mapping also preserves Assumptions 7 and 8:
$$|V(t, X(t,t_1)\hat{x}) - V(s, X(s,t_1)\hat{x})|$$
$$\leq |V(t, X(t,t_1)\hat{x}) - V(s, X(t,t_1)\hat{x})|$$
$$+ |V(s, X(t,t_1)\hat{x}) - V(s, X(s,t_1)\hat{x})|$$
$$\leq (1 + \|X(t,t_1)\hat{x}\|)\omega(|t-s|)$$
$$+ \omega(\|X(t,t_1)\hat{x} - X(s,t_1)\hat{x}\|) \leq C(1+\|\hat{x}\|)\omega(|t-s|)$$
since X is bounded and Lipschitz-continuous.

We have arrived at final result.

Theorem 6. Suppose that

(1) assumptions 2, 3, 5, and 9 are satisfied;
(2) functions V and W satisfy Assumptions 7 and 8;
(3) functions V and W are viscosity subsolution and supersolution to (16), respectively.

Then $V \leq W$.

Corollary 3. (Verification Theorem). If a function $V(t,x)$ satisfies HJB equation (13) in the viscosity sense and the initial condition (12), then it is the value function (4).

7. CONCLUSION

We have studied the basic properties of feedback solutions to impulse control problems. The approach described here is applicable to impulse systems with unknown but bounded disturbances (see Daryin et al. [2011]), which will be subject of future work.

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