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# Stability of Kalman Filtering with Multiple Sensors Involving Lossy Communications

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**Abstract:** We study a networked state estimation problem for a linear system with multiple sensors, each of which transmits its measurements to a central estimator via a lossy communication network for computing the minimum mean-square-error (MMSE) state estimate. Under a general Markov packet loss process, we establish necessary and sufficient conditions for the stability of the estimator for any diagonalizable system in the sense that the mean of the state estimation error covariance matrix is uniformly bounded. For the second-order systems under an i.i.d. packet loss model, the stability condition is expressed as a simple inequality in terms of open-loop poles and the packet loss rate.

Keywords: Networked state estimation, distributed sensing, packet loss, minimum mean square error, Kalman filtering.

## 1. INTRODUCTION

With the rapid development of the sensor network and communication technologies, the problem of networked state estimation has received significant attention in the recent years [Sinopoli et al., 2004, Schenato et al., 2007, Hespanha et al., 2007]. One of the major difficulties is due to packet loss in transmitting the sensor measurements. This work focuses on an estimation framework where *multiple* sensors are deployed to observe a large linear system and send their measurement to a remote estimator through a lossy network. In particular, each sensor uses an independent channel for communicating with the central estimator where the minimum mean-square-error (MMSE) state estimate is computed.

Under a single sensor case, the estimation framework was initially studied in Sinopoli et al. [2004]. By treating the received measurements as intermittent measurements, the Kalman filter technique is applied to compute the networked MMSE state estimate with single sensor [Sinopoli et al., 2004]. However, the stability of the state estimator is known to be seriously influenced by the packet loss model and the algebraic structure of the system in a coupled and complicated manner [Huang and Dey, 2007, You et al., 2011, Mo and Sinopoli, 2010]. Strictly speaking, it is still not well understood how they jointly affect the stability of the networked MMSE state estimator.

Two frameworks for the networked state estimation are proposed in the literature, by transmitting either the raw measurements directly, or the state estimate instead. The former approach is easy to implement but the associated stability condition is difficult to derive, whereas the latter one yields simpler stability conditions [Schenato, 2008] but adds the processing burden to the transmitters. The latter one may not be possible when considering the constraints of the hardware and power in sensor networks, and tends to transmit more data through the network. Under our distributed sensing setting, pre-computing the state estimate in each sensor might not be sensible due to the use of only partial state measurements. In Sun and Deng [2004], each sensor locally computes a state estimate and the central estimator aggregates these local estimates. Such an estimate is typically not optimal, and requires the stability of local estimators. This is an unnecessarily strong assumption for the distributed setting. For these reasons, we will adopt the former approach(each sensor transmits its raw measurements to the estimator) in this paper.

To quantify the effect of packet loss, two channel models have been widely adopted: 1) the independent and identically distributed (i.i.d.) model where the packet loss process is modeled as an i.i.d. Bernoulli process [Sinopoli et al., 2004]; 2) the Markovian model where the packet process is described by a binary Markov chain [Huang and Dey, 2007], which is inspired by the so-called Gilbert-Elliott (GE) channel. Under the i.i.d. model, references [Sinopoli et al., 2004, Mo and Sinopoli, 2010, Plarre and Bullo, 2009, Mo and Sinopoli, 2008] focused on the stability of the intermittent Kalman filter with only one sensor transmitting its raw measurements, and there exists a critical packet loss rate, above which the mean of the state estimation error covariance matrix will diverge to infinity [Sinopoli et al., 2004]. An upper bound and lower bound

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for the critical packet loss rate are also given in Sinopoli et al. [2004]. For a general vector system, it is known to be difficult to express the critical packet loss rate. Motivated but also inspired by the limitation of Sinopoli et al. [2004], the lower bound is shown to be tight in Plarre and Bullo [2009] for the system with one-step observable, which continues to hold for the so-called non-degenerate systems [Mo and Sinopoli, 2010]. However, a counterexample was found in You et al. [2011] that the critical packet loss rate strictly lies between the lower and upper bounds. For the GE packet loss model, the necessary and sufficient stability condition for the second-order systems and certain classes of higher-order systems are explicitly given in You et al. [2011]. In Rohr et al. [2013], they studied a wider class of Markovian network model.

In comparison, this paper studies the networked state estimation problem with *multiple* sensors. This is motivated by many real-world scenarios where the system covers a large spatial domain and distributed sensing is needed, with each sensor measuring partial state information. The stability analysis of the resulting networked MMSE estimator is challenging, and has not been studied since the system structure with multiple sensors is rather complicated. Note that the approaches in Huang and Dey [2007], You et al. [2011], Mo and Sinopoli [2010], Sun and Deng [2004], Plarre and Bullo [2009], Mo and Sinopoli [2008] are no longer applicable to this setting. We establish a necessary and sufficient condition for the stability of the state estimator for diagonalizable systems under multiple sensors. An efficient algorithm is also designed to check the condition. We demonstrate, through a second-order system under the i.i.d. packet loss model, that the stability condition reduces to a simple inequality. Thus, how the unstable open-loop poles and packet loss rates jointly affect the stability of the MMSE estimator is clearly revealed. From this perspective, our results substantially advance the existing literature, which only consider the case with a single sensor.

The rest of the paper is organized as follows. The problem formulation is described and the MMSE estimate for the system with multiple sensors over a lossy channel is derived in Section 2. In Section 3, the stability condition for the MMSE estimator of a diagonalizable system is given. For second-order systems, stability conditions are given by a simple inequality in Section 4. Concluding remarks are drawn in Section 5.

## 2. PROBLEM FORMULATION

Consider a discrete-time stochastic system

$$x_{k+1} = Ax_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state and $w_k$ is a white Gaussian noise with covariance matrix $Q > 0$. The initial state $x_0$ is a Gaussian random vector with mean $\bar{x}_0$ and covariance matrix $P_0 > 0$. To remotely estimate the system state, we use a sensor network with $d \geq 2$ sensors to take noisy measurements, i.e.,

$$y_k^i = C_i x_k + v_k^i, \quad i \in \{1, 2, \ldots, d\}, \tag{2}$$

where $v_k^i \in \mathbb{R}^{m_i}$ is a white Gaussian noise of sensor $i$ with covariance matrix $R_i > 0$ and $\sum_{i=1}^d m_i = m$. In addition, $x_0$, $w_k$ and $v_k^i$ are mutually independent. All the random variables in this paper are assumed to be defined on a common probability space $(\Omega, \mathbb{P}, \mathcal{F})$, where $\Omega$ is the space of elementary events, $\mathcal{F}$ is the underlying $\sigma$-field on $\Omega$, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. Throughout the paper, we denote

$$y_k = \text{col}\{y_k^1, y_k^2, \ldots, y_k^d\}, C = \text{col}\{C_1, C_2, \ldots, C_d\}, \tag{3}$$

where $\text{col}\{\cdot\}$ is a column operator, i.e., $\text{col}\{C_1, C_2\} = [C_1^T, C_2^T]^T$, and assume that $(A, C)$ is observable.

Each sensor and the central estimator are linked through a communication network. Due to the channel unreliability, the transmitted packets may be randomly lost. We use a binary random process $\gamma_k^i$ to describe the packet loss process. That is, $\gamma_k^i = 1$ indicates that the packet transmitted from sensor $i$ is successfully delivered to the estimator at time $k$, or $\gamma_k^i = 0$ if the packet is lost.

The implication of packet loss is that the estimator may fail to generate a stable state estimator. To study how the packet loss will affect the stability of the MMSE estimator, we denote

$$\Upsilon_k = \text{diag}\{\gamma_k^1 I_1, \ldots, \gamma_k^d I_d\}, \tag{4}$$

where $I_i \in \mathbb{R}^{m_i \times m_i}$ is an identity matrix, and define the packet receival matrix as

$$S_k = \text{diag}\{\Upsilon_0, \Upsilon_1, \ldots, \Upsilon_{k-1}\}. \tag{5}$$

The set of all possible $S_k$ will be denoted by $\mathbb{S}_k$ which consists of $2^{kd}$ elements. The information available to the estimator at time $k$ is given as follows:

$$\mathcal{F}_k = \{(\Upsilon_0, \Upsilon_0 y_0), (\Upsilon_1, \Upsilon_1 y_1) \ldots, (\Upsilon_k, \Upsilon_k y_k)\}. \tag{6}$$

Denote the MMSE (one-step-ahead) predictor and the MMSE estimator by

$$\hat{x}_{k|k-1} = \mathbb{E}[x_k|\mathcal{F}_{k-1}] \text{ and } \hat{x}_{k|k} = \mathbb{E}[x_k|\mathcal{F}_k]$$

respectively. Their corresponding estimation error covariance matrices are then given by

$$P_{k|k-1} = \mathbb{E}[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})'|\mathcal{F}_{k-1}]$$

and

$$P_{k|k} = \mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})'|\mathcal{F}_k].$$

Then, a Kalman like algorithm is developed to recursively compute the MMSE estimate and we establish the packet loss condition under which the mean of the state estimation error covariance matrix is uniformly bounded, i.e.,

$$\limsup_{k\to\infty} \mathbb{E}[P_{k|k}] < \infty, \tag{7}$$

where the mathematical expectation is taken with respect to the random process $\{\Upsilon_k\}$. Here (7) is interpreted that there exists a positive-definite matrix $\bar{P} > 0$ such that for all $k \geq 0$,

$$\mathbb{E}[P_{k|k}] < \bar{P} \text{ or } \mathbb{E}[P_{k|k-1}] < \bar{P}.$$

Similar to that of Sinopoli et al. [2004], the Kalman filter is still optimal under multiple sensors as shown below.

*Theorem 1.* The MMSE estimate for the networked system in (1)-(2) is recursively computed by

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \Upsilon_k (y_k - C\hat{x}_{k|k-1}); \tag{8}$$

$$P_{k|k} = P_{k|k-1} - K_k \Upsilon_k C P_{k|k-1}, \tag{9}$$

where the Kalman gain

$$K_k = P_{k|k-1} C^* \Upsilon_k (\Upsilon_k C P_{k|k-1} C^* \Upsilon_k + R)^{-1}$$

and $R = \text{diag}\{R_1, \ldots, R_d\}$.

*Proof:* By following [Sinopoli et al., 2004], the measurement noise distribution with packet loss can be described by

$$p(v_k^i|\gamma_k^i) \sim \begin{cases} \mathcal{N}(0, R_i), & \text{if } \gamma_k^i = 1; \\ \mathcal{N}(0, \sigma^2 I_i), & \text{if } \gamma_k^i = 0, \end{cases} \tag{10}$$

where $\sigma$ is arbitrarily large. Then, the MMSE estimate is computed by

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + P_{k|k-1}C^*(CP_{k|k-1}C^* \\ &\quad + \Upsilon_k R + (I - \Upsilon_k)\sigma^2)^{-1}(y_k - C\hat{x}_{k|k-1}), \quad (11) \\ P_{k|k} &= P_{k|k-1} - P_{k|k-1}C^*(CP_{k|k-1}C^* \\ &\quad + \Upsilon_k R + (I - \Upsilon_k)\sigma^2)^{-1}CP_{k|k-1}. \quad (12) \end{aligned}$$

Without loss of generality, suppose that $\gamma_k^1 = 1, \ldots, \gamma_k^i = 1$ and $\gamma_k^{i+1} = 0, \ldots, \gamma_k^d = 0$. Denote

$$\bar{C}_1 = \mathrm{col}\{C_1, C_2, \ldots, C_i\}; \ \bar{C}_2 = \{C_{i+1}, C_{i+2}, \ \ldots C_d\};$$
$$\bar{R}_1 = \mathrm{diag}\{R_1, \ldots, R_i\}; \ \ \bar{R}_2 = \mathrm{diag}\{R_{i+1}, \ldots, R_d\}.$$

It follows that

$$\begin{aligned} &CP_{k|k-1}C^* + \Upsilon_k R + (I - \Upsilon_k)\sigma^2 \\ &= \begin{bmatrix} \bar{C}_1 P_{k|k-1}\bar{C}_1^* + \bar{R}_1 & \bar{C}_1 P_{k|k-1}\bar{C}_2^* \\ \bar{C}_2 P_{k|k-1}\bar{C}_1^* & \bar{C}_2 P_{k|k-1}\bar{C}_2^* + \sigma^2 I_2 \end{bmatrix}. \end{aligned}$$

One can easily derive that

$$\begin{aligned} &\lim_{\sigma\to\infty} (CP_{k|k-1}C^* + \Upsilon_k R + (I - \Upsilon_k)\sigma^2)^{-1} \\ &= \begin{bmatrix} (\bar{C}_1 P_{k|k-1}\bar{C}_1^* + \bar{R}_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \Upsilon_k(\Upsilon_k C P_{k|k-1} C^* \Upsilon_k + R)^{-1}\Upsilon_k. \end{aligned} \tag{13}$$

The rest of proof is complete by taking $\sigma \to \infty$ in (11) and (12). ■

In the sequel, we shall study the stability of the above MMSE estimator.

## 3. STABILITY ANALYSIS OF THE MMSE ESTIMATOR

To establish the stability condition for the MMSE estimator (8)-(9), we define the $N$-step *regression matrix*

$$O_N = S_N \mathrm{col}\{C, CA, \ldots, CA^{N-1}\}. \tag{14}$$

Note that $O_N$ is closely related to the observability of the system under packet loss, and is central to stability analysis. Specifically, the higher the packet loss rate is, the easier $O_N$ becomes column rank deficient, which may result in the instability of the MMSE estimator. Thus, we extensively explore the rank condition of $O_N$. Denote $P_k := P_{k|k-1}$, and we have the following result, whose proof is not difficult, and is omitted due to page limitation.

*Lemma 2.* Suppose that $A$ is invertible and $O_N$ has full column rank. There exists a positive definite matrix $\bar{P}_N$, independent of $P_0$, such that

$$P_N \le \bar{P}_N. \tag{15}$$

We define the set of all $S_N$ leading to column rank deficient $O_N$ (i.e., not having full column rank) by

$$\mathcal{R}_N = \{S_N | O_N \text{ is column rank deficient}\}, \tag{16}$$

and the probability of this set is

$$\mathbb{P}(\mathcal{R}_N) \triangleq \mathbb{P}(S_N \in \mathcal{R}_N) = \sum_{S_N \in \mathcal{R}_N} \mathbb{P}(S_N). \tag{17}$$

This quantity is important to stability analysis as it characterizes the probability of the regression matrix $O_N$ losing observability. Two cases under different system structures are discussed in the sequel.

### 3.1 Single Eigen-Block

All the open-loop poles of this class of systems are with the same magnitude, i.e., it is characterized by the following assumption.

*Assumption 1.* $A = \alpha \ \mathrm{diag}(e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2}, \ldots, e^{\mathrm{i}\theta_n})$ for some common magnitude $\alpha > 0$.

We define the *period* of $A$ as the minimum positive integer $\tau$ such that $A^\tau = \alpha^\tau I$. We say that $A$ is *periodic* with a period of $\tau$ if $\tau < \infty$. If such a $\tau$ does not exist, we say that $A$ is aperiodic, and set $\tau = \infty$. The packet loss process is modeled as a general Markov process in Assumption 2.

*Assumption 2.* The packet loss process $\Upsilon_k$ is a Markov process with finite order $\nu \le \tau$ satisfying

$$\mathbb{P}\{\Upsilon_k | \Upsilon_{k-1}, \ldots, \Upsilon_{k-\nu}\} = \mathbb{P}\{\Upsilon_k | \Upsilon_{k-1}, \ldots, \Upsilon_{k-\nu-s}\} > 0$$

for any $s \in \mathbb{N}$.

*Remark 3.* If $\nu > \tau$, we select an integer $k_0$ such that $k_0\tau \ge \nu$, and let the period of $A$ be $k_0\tau$. Then, the following results continue to hold.

Now, we are in the position to deliver our main result on the single eigen-block case.

*Theorem 4.* Under Assumptions 1 and 2, the necessary and sufficient condition for $\limsup_{N\to\infty} \mathbb{E}[P_N] < \infty$ is that

$$\alpha^2 \limsup_{N\to\infty} (\mathbb{P}(\mathcal{R}_N))^{1/N} < 1. \tag{18}$$

Two lemmas below are needed to prove Theorem 4.

*Lemma 5.* Under Assumption 1 and $O_N$ is column rank deficient. Given any $\underline{p} > 0$ satisfying $\underline{p}I \le \min\{P_0, Q\}$, it holds that $\mathrm{Tr}(P_N) \ge \underline{p}\alpha^{2N}$.

*Proof*: Consider a special case that $w_k = 0$ and $v_k^i = 0$ for all $k \in \mathbb{N}$ and $i \in \{1, 2, \ldots, d\}$. Then the estimation error covariance matrix of the MMSE estimate is denoted by $\underline{P}_N$, and is computed as

$$\underline{P}_N = A^N P_0 (A^N)^* - A^N P_0 O_N^* (O_N P_0 O_N^*)^\dagger O_N P_0 (A^N)^*. \tag{19}$$

Obviously, $\underline{P}_N \le P_N$. Since the right hand side (RHS) of (19) is monotonically increasing in $P_0$[Sinopoli et al., 2004], it follows that

$$\underline{P}_N \ge \underline{p} A^N (I - O_N^*(O_N O_N^*)^\dagger O_N)(A^N)^*. \tag{20}$$

By the singular value decomposition [Horn and Johnson, 1985], it holds that $O_N = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V$, where $U$ and $V$ are unitary matrices, and $D$ is an $a \times a$ invertible matrix with $a = \mathrm{rank}(O_N) < n$. Subsequently, $O_N^*(O_N O_N^*)^\dagger O_N = V^* \begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix} V$. Together with (20), it follows that

$$\underline{P}_N \ge \underline{p} A^N V^* \begin{bmatrix} 0 & 0 \\ 0 & I_{n-a} \end{bmatrix} V (A^N)^*. \tag{21}$$

Using Assumption 1 and that $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$ for any compatible matrices $X$ and $Y$, the above implies $\mathrm{Tr}(\underline{P}_N) \geq \underline{p}\alpha^{2N}\mathrm{Tr}(I_{n-a}) \geq \underline{p}\alpha^{2N}$. The proof is complete by noting that $\mathrm{Tr}(P_N) \geq \mathrm{Tr}(\underline{P}_N)$. ■

To explicitly express the dependence of $P_N$ on $S_N$ and $P_0$ (the covariance matrix of $x_0$), we denote it by $\phi(P_0, S_N)$. Let $P_0 = xI$ for $x > 0$, the associated $\mathbb{E}[P_N]$ is

$$\xi_N(x) = \sum_{S_N \in \mathbb{S}_N} \phi(xI, S_N)\mathbb{P}(S_N). \tag{22}$$

Then, we have the following result.

*Lemma 6.* For any $P_0 > 0$, if there exists $N_0 > 0$ and $\tilde{x} > 0$ such that $\tilde{x}I \geq \xi_{N_0}(\tilde{x})$ and $\tilde{x}I \geq P_0$, then $\limsup_{N\to\infty} \mathbb{E}[P_N] < \infty$.

*Proof*: By the monotonicity of $\phi(\cdot, S_N)$[Sinopoli et al., 2004], we have $\phi(P_0, S_N) \leq \phi(\tilde{x}I, S_N)$, which further implies that

$$\begin{aligned}\mathbb{E}[P_N] &= \sum_{S_N \in \mathbb{S}_N} \phi(P_0, S_N)\mathbb{P}(S_N)\\ &\leq \sum_{S_N \in \mathbb{S}_N} \phi(\tilde{x}I, S_N)\mathbb{P}(S_N)\\ &= \xi_N(\tilde{x}).\end{aligned} \tag{23}$$

By the concavity of $\xi_N(x)$[Sinopoli et al., 2004], we obtain that $\xi_{kN}(x) \leq \xi_N \circ \xi_N \circ \cdots \circ \xi_N(x) \triangleq \xi_N^{(k)}(x)$. This implies that

$$\limsup_{k\to\infty} \xi_{kN_0}(\tilde{x}) \leq \limsup_{k\to\infty} \xi_{N_0}^{(k)}(\tilde{x}) \leq \tilde{x}I. \tag{24}$$

For any integers $0 \leq k_1 < k_2$, we define the notation (generalized from $S_k$)

$$S_{k_1,k_2} = \mathrm{diag}\{\Upsilon_{k_1}, \Upsilon_{k_1+1}, \ldots, \Upsilon_{k_2-1}\}. \tag{25}$$

Given any integer $0 < l < N_0$, it obtains that

$$\xi_{N+l}(x) = \sum_{S_N \in \mathbb{S}_N, S_{N,N+l} \in \mathbb{S}_l} \phi(\phi(xI, S_N), S_{N,N+l}) \mathbb{P}(S_N)\mathbb{P}(S_{N,N+l}|S_N). \tag{26}$$

Note that, for any $S_N$ and $P > 0$, we have

$$\begin{aligned}&\sum_{S_{N,N+l} \in \mathbb{S}_l} \phi(P, S_{N,N+l})\mathbb{P}(S_{N,N+l}|S_N)\\ &\leq (A^l P (A^l)^* + G_l \Sigma_{Q,l} G_l^*),\end{aligned}$$

where $G_l = \begin{bmatrix} A^{l-1} & \ldots & A & I\end{bmatrix}$ and $\Sigma_{Q,l} = \mathrm{diag}\{\underbrace{Q, \ldots, Q}_{l}\}$.

Then, it follows that

$$\begin{aligned}\xi_{N+l}(x) &\leq \sum_{S_N \in \mathbb{S}_N} (A^l \phi(xI, S_N)(A^l)^* + G_l \Sigma_{Q,l} G_l^*)\mathbb{P}(S_N)\\ &= A^l \xi_N(x)(A^l)^* + G_l \Sigma_{Q,l} G_l^*.\end{aligned}$$

Since $l$ is finite, it holds that $\limsup_{k\to\infty} \xi_{kN_0}(x) < \infty \Leftrightarrow \limsup_{k\to\infty} \xi_{kN_0+l}(x) < \infty$. Together with (24), we obtain that $\limsup_{N\to\infty} \xi_N(\tilde{x}) < \infty$. By (23), it finally yields that $\limsup_{N\to\infty} \mathbb{E}[P_N] < \infty$. ■

*Proof of Theorem 4*:

**Necessity**: Denote the complement of set $\mathcal{R}_N$ by $\mathcal{R}_N^c$, which contains all $S_N$ leading to a full column rank $O_N$.

Let $m_0$ be the minimum integer such that $\mathbb{P}(\mathcal{R}_{m_0}^c) > 0$. Note that such a finite $m_0$ must exist. Indeed, since $(A, C)$ is observable, $O_{m_0}$ must be possible to have full column rank for some $m_0 \leq n$, e.g., $\Upsilon_k = I$ for all $k \in \{0, 1, \cdots, m_0 - 1\}$, which in turn implies that $\mathbb{P}(\mathcal{R}_{m_0}^c) \geq \mathbb{P}\{\Upsilon_k = I, 0 \leq k \leq m_0 - 1\} > 0$.

Since $P_N \geq Q > 0$[Anderson and Moore, 1979], let $\underline{p}$ be the minimum of minimum eigenvalues of matrices $Q$ and $P_0$. Then, $P_N \geq \underline{p}I$ for all $N \in \mathbb{N}$. Given a sufficiently large integer $N$, consider a time horizon from 0 to $Nm_0$. We shall use $\mathrm{Tr}(\mathbb{E}[P_{Nm_0}])$ to derive the necessary condition for stability.

Base on (25), the associated regression matrix is

$$O_{k_1,k_2} = S_{k_1,k_2}\mathrm{col}\{C, CA, \ldots, CA^{k_2-k_1-1}\}.$$

Let $l$ be any integer with $0 \leq l < m_0$, the set of all possible $S_{l,Nm_0}$ is divided into the following disjoint subsets.

- Subset 1: $O_{l,Nm_0}$ is column rank deficient;
- Subset 2: $O_{l,Nm_0}$ has full column rank but $O_{m_0+l,Nm_0}$ is column rank deficient;
- Subset 3: $O_{m_0+l,Nm_0}$ has full column rank but $O_{2m_0+l,Nm_0}$ is column rank deficient;
- ……
- Subset $N$: $O_{(N-2)m_0+l,Nm_0}$ has full column rank but $O_{(N-1)m_0+l,Nm_0}$ is column rank deficient;

Then the probability of each Subset is studied: The probability of Subset 1 is given by $\mathbb{P}(S_{l,Nm_0} \in \mathcal{R}_{Nm_0-l})$. Similarly, the probability of Subset $j+1$ is given by $\mathbb{P}(S_{jm_0+l,Nm_0} \in \mathcal{R}_{(N-j)m_0-l})\mathbb{P}(S_{(j-1)m_0+l,Nm_0} \in \mathcal{R}^c_{(N-j+1)m_0-l}|S_{jm_0+l,Nm_0} \in \mathcal{R}_{(N-j)m_0-l})$, $j = 1, \ldots, N-1$.

Under Assumption 2, for any $t \in \mathbb{N}$, there exists positive $\beta_\nu$ and $\varepsilon_\nu$ such that

$$\begin{aligned}\min\nolimits_{p\in\mathbb{N}}\mathbb{P}(S_{p,p+t} \in \mathcal{R}_t) \geq \beta_\nu\mathbb{P}(\mathcal{R}_t);\\ \min\nolimits_{p\in\mathbb{N}}\mathbb{P}(S_{p,p+t} \in \mathcal{R}_t^c) \geq \varepsilon_\nu\mathbb{P}(\mathcal{R}_t^c).\end{aligned} \tag{27}$$

Using the above decomposition, Lemma 5 and (27), it can be shown that

$$\begin{aligned}\mathrm{Tr}(\mathbb{E}[P_{Nm_0}]) &= \sum_{S_{Nm_0} \in \text{Subset 1}} \mathrm{Tr}(\phi(P_0, S_{Nm_0}))\mathbb{P}(S_{Nm_0}) + \\ &\cdots + \sum_{S_{Nm_0} \in \text{Subset N}} \mathrm{Tr}(\phi(P_0, S_{Nm_0}))\mathbb{P}(S_{Nm_0})\\ &\geq \underline{p}\beta_\nu\varepsilon_\nu\mathbb{P}(\mathcal{R}_{m_0}^c)\sum_{j=2}^{N}\alpha^{2(jm_0-l)}\mathbb{P}(\mathcal{R}_{jm_0-l}).\end{aligned} \tag{28}$$

Note that $\beta_\nu$ and $\varepsilon_\nu$ are independent of $N$, and strictly positive. By $\limsup_{N\to\infty}\mathrm{Tr}(\mathbb{E}[P_{Nm_0}]) < \infty$, it is necessary that $\alpha^{2(jm_0-l)}\mathbb{P}(\mathcal{R}_{jm_0-l}) < 1$ as $j \to \infty$, or equivalently, $\alpha^2(\mathbb{P}(\mathcal{R}_{jm_0-l}))^{1/(jm_0-l)} < 1$ as $j \to \infty$. Since $l$ is arbitrarily selected from the set $\{0, \ldots, m_0 - 1\}$, we conclude that

$$\limsup_{N\to\infty}\mathbb{E}[P_N] < \infty \Rightarrow \alpha^2\limsup_{N\to\infty}(\mathbb{P}(\mathcal{R}_N))^{1/N} < 1. \tag{29}$$

**Sufficiency**: Let $S_N^0$ be the event that there is no packet received up to time $N$, i.e., $\Upsilon_k = 0$ for all $0 \le k < N$. Take any scalar $q$ with $qI > Q$. Given any $x > 0$, it follows that

$$\begin{aligned}\phi(xI, S_N^0) &= A^N(A^N)^*x + \sum_{j=0}^{N-1} A^j Q (A^j)^* \\ &\le (\alpha^{2N}x + \sum_{j=0}^{N-1}\alpha^{2j}q)I \\ &= (\alpha^{2N}x + \frac{\alpha^{2N}-1}{\alpha^2-1}q)I \\ &\le \alpha^{2N}(x + \frac{q}{\alpha^2-1})I. \end{aligned} \tag{30}$$

It is known that without using any measurement, the estimation error covariance matrix cannot decrease. This implies that for any $S_N \in \mathcal{R}_N$,

$$\phi(xI, S_N) \le \phi(xI, S_N^0). \tag{31}$$

Splitting the set $\mathbb{S}_N$ into $\mathcal{R}_N$ and its complement $\mathcal{R}_N^c$, it follows from Lemma 2 that

$$\begin{aligned}\xi_N(x) &= \sum_{S_N \in \mathcal{R}_N \cup \mathcal{R}_N^c} \phi(xI, S_N)\mathbb{P}(S_N) \\ &\le \mathbb{P}(\mathcal{R}_N^c)\bar{P}_N + \sum_{S_N \in \mathcal{R}_N} \phi(xI, S_N)\mathbb{P}(S_N) \\ &\le \mathbb{P}(\mathcal{R}_N^c)\bar{P}_N + \mathbb{P}(\mathcal{R}_N)\phi(xI, S_N^0) \\ &\le \mathbb{P}(\mathcal{R}_N^c)\bar{P}_N + \mathbb{P}(\mathcal{R}_N)\alpha^{2N}(x + \frac{q}{\alpha^2-1})I. \end{aligned}$$

Since $\limsup_{N\to\infty} \alpha^{2N}\mathbb{P}(\mathcal{R}_N) < 1$, it is clear that for sufficiently large $x > 0$ and $xI > P_0$, we have $\xi_N(x) \le xI$ as $N \to \infty$. By Lemma 6, it follows that $\limsup_{N\to\infty} \mathbb{E}[P_N] < \infty$ for any $P_0 > 0$. ∎

*3.2 Extension to Multiple Eigen-Blocks*

We now generalize the result on single eigen-block to the multiple eigen-blocks.

*Assumption 3.* $A = \mathrm{diag}\{A_1, A_2, \ldots, A_g\}$, where $A_i = \alpha_i \,\mathrm{diag}\{e^{\mathrm{i}\theta_{i1}}, e^{\mathrm{i}\theta_{i2}}, \ldots, e^{\mathrm{i}\theta_{in_i}}\}$ is a $n_i \times n_i$ matrix with $\alpha_i > 0$, $\alpha_i \ne \alpha_j$ for any $i \ne j$ and $\sum_{i=1}^g n_i = n$.

In light of the structure of $A$ with multiple eigen-blocks, we decompose $O_N$ into

$$O_N = \begin{bmatrix} O_N^1 & O_N^2 & \ldots & O_N^g \end{bmatrix},$$

where $O_N^i$ is a $mN \times n_i$ matrix.

The main result for the multiple eigen-block case is given below.

*Theorem 7.* Under Assumptions 2 and 3, the necessary and sufficient condition for $\limsup_{N\to\infty} \mathbb{E}[P_N] < \infty$ is that

$$\alpha_i^2 \limsup_{N\to\infty} (\mathbb{P}(\mathcal{R}_N(i)))^{1/N} < 1, \forall\, i \in \{1, 2, \ldots, g\}, \tag{32}$$

where $\mathcal{R}_N(i) = \{S_N | O_N^i \text{ is column rank deficient}\}$.

*Remark 8.* It is clear that Theorem 7 covers the result in Theorem 4. The proof of Theorem 7 is very complicated and technical, although the idea is the same as that of Theorem 4.

*3.3 Computation of* $\mathbb{P}(\mathcal{R}_N)$

Using Theorem 4, the key factor in determining the stability condition for systems satisfying Assumption 1 is to compute the probability $\mathbb{P}(\mathcal{R}_N)$. We show how to compute this in this subsection. Note that the computation of $\mathbb{P}(\mathcal{R}_N(i))$ for systems satisfying Assumption 3 can be done similarly.

If $A^\tau = \alpha^\tau I$, then the packet received at time $k$ is equivalent to the packet received at time $k + \tau$ as far as the rank deficiency of $O_N$ is concerned for a large $N$. Thus the sequence $S_N$ can be projected onto a shorter sequence $\widetilde{S}_N$ using the algorithm below, and we will have $\mathbb{P}(S_N \in \mathcal{R}_N) = \mathbb{P}(\widetilde{S}_N \in \mathcal{R}_\tau)$.

**Algorithm 1 Projection Algorithm**

**Step 1:** For any $k \in \{1, 2, \ldots, \tau\}$, define $\widetilde{\gamma}_k^i = \gamma_k^i \vee \gamma_{k+\tau}^i \vee, \ldots, \vee \gamma_{k+\lceil N/\tau \rceil \tau}$ and $\widetilde{\Upsilon}_k = \mathrm{diag}(\widetilde{\gamma}_k^1 I_1, \ldots, \widetilde{\gamma}_k^d I_d)$, where $\vee$ is Boolean OR operator, and $\lceil \cdot \rceil$ is the ceiling function.

**Step 2:** Define $\widetilde{S}_N = \mathrm{diag}(\widetilde{\Upsilon}_1, \ldots, \widetilde{\Upsilon}_\tau)$.

Because $\mathcal{R}_\tau$ is a finite set for a periodic $A$, we denote its elements by $\{s_1, s_2, \ldots, s_r\}$ and rank them by $\mathbb{P}(\widetilde{S}_{N+\tau} = s_i | \widetilde{S}_N = s_j) = 0$ if $j > i$. In particular, $s_1$ designates the event that no packets have arrived.

Define $\Gamma_i$ as the set of all combinations of $S_{N-\nu,N}$ conditioned on $\widetilde{S}_N = s_i$, and we further assume that the elements in $\Gamma_i$ are $\{\Gamma_i(1), \Gamma_i(2), \ldots, \Gamma_i(t_i)\}$.

Under Assumption 2, for $\mathbb{P}(\widetilde{S}_{(N+1)\tau} = s_i | \widetilde{S}_{N\tau} = s_j) = 0, j > i$, we define a lower triangular transition matrix $M$ with $r \times r$ blocks as

$$M = \begin{bmatrix} E_{11} & 0 & \ldots & 0 \\ E_{21} & E_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{r1} & E_{r2} & \ldots & E_{rr} \end{bmatrix}. \tag{33}$$

Note that each block $E_{ij} \in \mathbb{R}^{t_i \times t_j}, j \le i$ is a transition matrix with its $(p, q)$-th element

$$\begin{aligned}&\mathbb{P}(\widetilde{S}_{(N+1)\tau} = s_i, S_{(N+1)\tau-\nu,(N+1)\tau} = \Gamma_i(p) \\ &\quad | \widetilde{S}_{N\tau} = s_j, S_{N\tau-\nu,N\tau} = \Gamma_j(q)). \end{aligned} \tag{34}$$

Based on Assumption 2, $M$ is independent of $N$.

It is interesting that the stability condition can be characterized by the maximum eigenvalue of $M$ in the following result, whose proof is omitted to save space.

*Theorem 9.* Let $m_0$ be the minimum positive integer that $\mathbb{P}(\mathcal{R}_{m_0}^c) > 0$. Suppose Assumption 2 holds and that $A$ is periodic with a period $\tau \ge m_0$. Then,

$$\mathbb{P}(\mathcal{R}_{N\tau}) = uM^N v \tag{35}$$

for any integer $N \ge 0$, where $u = [1\ 1\ \ldots\ 1]$, $v = [1\ 0\ \ldots\ 0]'$. Moreover,

$$\begin{aligned}\limsup_{N\to\infty} (\mathbb{P}(\mathcal{R}_N))^{1/N} &= (\lambda_{\max}(M))^{1/\tau} \\ &= \max_{1 \le i \le r} \{\lambda_{\max}(E_{ii})^{1/\tau}\}, \end{aligned} \tag{36}$$

where $\lambda_{\max}(M)$ and $\lambda_{max}(E_{ii})$ are the largest eigenvalues of $M$ and $E_{ii}$ in magnitude, respectively.

*Remark 10.* If $\tau < m_0$, we could find a minimum integer $k_0$ such that $k_0\tau \geq m_0$, and replace $\tau$ by $k_0\tau$ in Theorem 9.

## 4. SECOND-ORDER SYSTEMS WITH MULTIPLE SENSORS

In this section, the necessary and sufficient stability condition for the MMSE estimator of a second-order system(as defined in Assumption 5) with multiple sensors is explicitly expressed as simple inequalities under i.i.d. packet loss model. The system is also given in (1)-(2), but with some additional assumptions.

*Assumption 4.* $\{\gamma_k^i\}$ is an i.i.d. process with the packet arrival rate $p_i = \mathbb{E}\{\gamma_k^i = 1\}$, and $\{\gamma_k^i\}$ and $\{\gamma_k^j\}$ are two independent processes for any $1 \leq i \neq j \leq d$.

*Assumption 5.* $A = \alpha \text{ diag}\{e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2}\}$ and $\tau$ is the minimum integer that $A^\tau = \alpha^\tau I (\tau = \infty$ if A is aperiodic), where $\alpha > 0$ and $\theta_1 \neq \theta_2$.

*Assumption 6.* $\text{rank}(\text{col}\{C_i, C_j\}) = 2$ and $(A, C_i)$ is observable for any $1 \leq i \neq j \leq d$.

*Remark 11.* If $\text{rank}(\text{col}\{C_i, C_j\}) = 1$ and $\text{rank}(C_i) \neq 0$ for any $i \neq j$, then $C_i$ and $C_j$ are dependent. This means that receiving the packet from sensor $i$ is equivalent to that of sensor $j$. We can combine sensors $i$ and $j$, and endow it a smaller packet loss probability $1 - (1 - p_i)(1 - p_j)$. This implies that there is no loss of generality to adopt Assumption 6.

For each sensor, a *Congruent Set* is introduced, which is key to the rank analysis of regression matrix $O_N$.

*Definition 1.* For each sensor $1 \leq i \leq d$, a congruent set $\mathcal{J}_i$ is defined as $\mathcal{J}_i = \{j | \exists k_{ij} \in \mathbb{N}, \text{s.t. span}\{C_i\} = \text{span}\{C_j A^{k_{ij}}\}$ and $0 \leq k_{ij} < \tau\}$.

Since $i \in \mathcal{J}_i$, the congruent set is not empty. The probability that all sensors in $\mathcal{J}_i$ lose their packets at the same time is computed by

$$p_i^* = \prod_{j \in \mathcal{J}_i} (1 - p_j). \tag{37}$$

Using Theorems 4 and 9, the stability condition for the case of second-order systems can be explicitly expressed as a simple inequality as below, whose proof is omitted.

*Theorem 12.* Consider the second-order system (1)-(2) under Assumptions 4-6, the MMSE estimator is stable if and only if

$$\frac{\prod_{i=1}^d (1 - p_i)^\tau}{\min_{j=1}^d p_j^*} \alpha^{2\tau} < 1. \tag{38}$$

*Remark 13.* Substituting the number of sensor $d = 1$ into Theorem 12, our result is the same as Theorem 7 in You et al. [2011] for i.i.d. packet loss model. Thus, it generalizes a result in You et al. [2011] for a single sensor case.

## 5. CONCLUSION

In this paper, we have studied the networked estimation problem of a stochastic discrete-time system with multiple sensors. The networked MMSE estimate was recursively computed using the technique of the Kalman filter. Then we study the stability of the MMSE estimator under a general Markovian packet loss process. We have derived the necessary and sufficient condition for the stability of the networked MMSE estimator for diagonalizable systems. A method is proposed to compute the stability condition. For second-order systems under the i.i.d. packet loss model, the stability condition can be given by a simple inequality.

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