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Conservatism of Analysis and Controller Synthesis of Decomposable Systems

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Abstract: Recently, distributed controller synthesis approaches for decomposable systems, a subclass of distributed systems with identical subsystems, where the interconnection can be described as LFT interconnection, have been proposed. In order to make these approaches tractable for systems containing a very large number of subsystems, constraints on the Lyapunov matrix and the multiplier matrices are introduced that render the complexity of analysis and controller synthesis smaller and in best case independent on the number of subsystems. Those assumptions have to be paid for with conservatism, which is investigated in this work. It is proven that the conservatism is not reduced if either only the Lyapunov or only the multiplier matrices are constrained, when compared with having constraints on both simultaneously.

Keywords: Distributed control, conservatism, large-scale systems, complexity reduction, linear parametrically varying (LPV) methodologies

1. INTRODUCTION

Distributed systems have received considerable interest over the last years. A distributed system consists of a number of subsystems that interact with each other to reach a common goal. Here the interaction can be either physical as in spatially distributed systems, see D'Andrea and Dullerud (2003), or realized by information exchange as in wireless sensor networks or multi-agent systems, (Murray, 2007). The very general framework of *decomposable systems*, introduced by Massioni and Verhaegen (2009), is well suited to capture both cases even at once. It is assumed that the subsystems are identical or have at least identical LPV-structure, which is the case in many practical examples, e.g. formation control of a swarm of agents, or structural control using PDE's. Based on that framework, approaches for distributed control have been presented there. Distributed control approaches provide identical controllers for each subsystem, that interact along the same interconnection topology as the plant. One approach, presented in Massioni and Verhaegen (2010), is based on the full block S-procedure, a tool well-known for LPV control, (Scherer, 2001). Starting with static state feedback controllers, it has been extended to convex synthesis of distributed dynamic output feedback controllers for time-varying symmetric interconnection topologies in Hoffmann et al. (2013) and dynamic output feedback controllers for general time-varying topologies in Eichler et al. (2013b).

One issue with decomposable systems is that, although the individual subsystems may be rather simple, the whole system becomes rather complex with increasing number of subsystems. Therefore distributed control approaches have in common that assumptions and constraints are introduced, such that the synthesis complexity does not scale with the number of subsystems and is at best only of the complexity of one single subsystem. This is crucial for very large distributed systems, which otherwise would be intractable. It is clear that the reduction of complexity usually has to be paid for with conservatism. This will be further investigated in this work. We point out which assumptions are made, why conservatism is introduced at this point and how to keep it small. One source of conservatism for every LPV system is that solutions are found on the convex hull of an uncertain parameter set. In case of decomposable systems, the interconnection structure is the uncertainty, thus the question arises and is discussed in the following, for which interconnections in addition to those of interest the found solution is valid.

1.1 Outline

In Section 2 the concept of decomposable systems is reviewed and the considered analysis and synthesis approaches are presented. Section 3 examines the assumptions and simplifications in the analysis and synthesis conditions in further detail and analyses the introduced conservatism. This is illustrated with a numerical example in Section 4. Section 5 concludes the work.

2. PRELIMINARIES

2.1 Decomposable Systems

A decomposable system consists of N identical subsystems, which are interconnected by the pattern matrix P. Consider the decomposable closed-loop system

$$\breve{\mathcal{T}}: \begin{bmatrix} \dot{\xi} \\ z \end{bmatrix} = \begin{bmatrix} \breve{\mathcal{A}} & \breve{\mathcal{B}}_{\mathrm{p}} \\ \breve{\mathcal{C}}_{\mathrm{p}} & \breve{\mathcal{D}}_{\mathrm{pp}} \end{bmatrix} \begin{bmatrix} \xi \\ d \end{bmatrix}. \quad (1)$$

with state vector $\xi \in \mathbb{R}^{Nn_\xi}$, disturbance $d \in \mathbb{R}^{Nn_d}$ and performance output $z \in \mathbb{R}^{Nn_z}$. The state vector is composed of the subsystems' state vectors ξ_k, $k = 1, \ldots, N$ as

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$\xi = [\xi_1^T, \ldots, \xi_N^T]^T$. The remaining signals are partitioned accordingly. The time dependency of the signals is omitted here for the sake of brevity. Such a system is called decomposable, indicated by the notation $\breve{\bullet}$, if there exists a common, possibly time-varying pattern matrix P, with respect to which $\breve{\mathcal{A}}$, $\breve{\mathcal{B}}_\mathrm{p}$, $\breve{\mathcal{C}}_\mathrm{p}$ and $\breve{\mathcal{D}}_\mathrm{pp}$ are decomposable, i.e. are in the set $\breve{\boldsymbol{\mathcal{D}}}_{p\times q}(P)$ defined below with respective dimensions p and q.

Definition 1. (Massioni and Verhaegen (2009)). A matrix $\breve{M}$ is called decomposable, if there exists a matrix $P(t) : \mathbb{R} \to \mathbb{R}^{N\times N}$, such that

$$\breve{M} \in \breve{\boldsymbol{\mathcal{D}}}_{p\times q}(P) := \big\{\breve{M} \in \mathbb{R}^{Np\times Nq} \,\big|\, \exists M^\mathsf{d}, M^\mathsf{i} \in \mathbb{R}^{p\times q} : \breve{M} = I_N \otimes M^\mathsf{d} + P \otimes M^\mathsf{i}\big\}.$$

The superscript d labels the *decentralized* and the superscript i the *interconnected* part. The decentralized system matrices describe how the local signals are handled, while the interconnected ones describe the handling of the signals of the interaction.

Since (1) is decomposable, an LFT representation

$$\breve{\mathcal{T}} : \begin{cases} \begin{bmatrix} \dot{\xi} \\ w \\ z \end{bmatrix} = \begin{bmatrix} I_N \otimes \mathcal{A}^\mathsf{d} & I_N \otimes \mathcal{B}_\mathrm{i} & I_N \otimes \mathcal{B}_\mathrm{p}^\mathsf{d} \\ I_N \otimes \mathcal{C}_\mathrm{i} & 0 & I_N \otimes \mathcal{D}_\mathrm{ip} \\ I_N \otimes \mathcal{C}_\mathrm{p}^\mathsf{d} & I_N \otimes \mathcal{D}_\mathrm{pi} & I_N \otimes \mathcal{D}_\mathrm{pp}^\mathsf{d} \end{bmatrix} \begin{bmatrix} \xi \\ v \\ d \end{bmatrix}, \\ v = (P \otimes I_{n_\Lambda})\, w, \quad \forall P \in \boldsymbol{\mathcal{P}} \end{cases} \tag{2}$$

can be found with $\mathcal{C}_\mathrm{i} = [\mathcal{A}^\mathsf{i} \;\; \mathcal{C}_\mathrm{p}^\mathsf{i}]^T$, $\mathcal{D}_\mathrm{ip} = [\mathcal{B}_\Theta^\mathsf{i} \;\; \mathcal{D}_{\mathrm{p}\Theta}^\mathsf{i}]^T$, $\mathcal{B}_\mathrm{i} = [I_{n_\xi} \;\; 0]$, $\mathcal{D}_\mathrm{pi} = [0 \;\; I_{n_z}]$ and $\boldsymbol{\mathcal{P}}$ being the set of all admissible topologies. Here the interconnection part is pulled out of the system as an uncertainty channel with $v, w \in \mathbb{R}^{n_\Lambda}$ and $n_\Lambda = N(n_\xi + n_z)$ here. More advanced LFT representations can be found in (Eichler et al., 2014), which are however not considered in this work.

2.2 Analysis of Decomposable Systems

Multiplier-based analysis conditions that offer ways to trade complexity versus conservatism for systems of the form (2) can be derived by the full block S-procedure (Scherer, 2001) as in (Eichler et al., 2013a). Note that $[\star]^T Y Z$ abbreviates $Z^T Y Z$.

Theorem 2. Given a set of topologies $\boldsymbol{\mathcal{P}}$ and a decomposable system $\breve{\mathcal{T}}$ with an uncertain time-varying topology $P(t) \in \boldsymbol{\mathcal{P}}$, $\forall t \geq 0$. System $\breve{\mathcal{T}}$ is stable and has an induced $\mathcal{L}_2$ gain less than γ, if there exist $X^\mathsf{d} = X^{\mathsf{d}^T} > 0$, $Q = Q^{\mathsf{d}^T}$, $R^\mathsf{d} = R^{\mathsf{d}^T}$ and S^d such that

$$[\star]^T \begin{bmatrix} I_N \otimes Q^\mathsf{d} & I_N \otimes S^\mathsf{d} \\ I_N \otimes S^{\mathsf{d}T} & I_N \otimes R^\mathsf{d} \end{bmatrix} \begin{bmatrix} P \otimes I_{n_\Lambda} \\ I_{Nn_\Lambda} \end{bmatrix} > 0, \; \forall P \in \boldsymbol{\mathcal{P}}, \tag{3}$$

$$[\star]^T \left[\begin{array}{cc|cc|cc} 0 & X^\mathsf{d} & & & & \\ X^\mathsf{d} & 0 & & & & \\ \hline & & Q^\mathsf{d} & S^\mathsf{d} & & \\ & & S^{\mathsf{d}T} & R^\mathsf{d} & & \\ \hline & & & & -\gamma^2 I & 0 \\ & & & & 0 & I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \mathcal{A}^\mathsf{d} & \mathcal{B}_\mathrm{i} & \mathcal{B}_\mathrm{p}^\mathsf{d} \\ \hline 0 & I & 0 \\ \mathcal{C}_\mathrm{i} & 0 & \mathcal{D}_\mathrm{ip} \\ \hline 0 & 0 & I \\ \mathcal{C}_\mathrm{p}^\mathsf{d} & \mathcal{D}_\mathrm{pi} & \mathcal{D}_\mathrm{pp}^\mathsf{d} \end{array}\right] < 0. \tag{4}$$

Proof. A proof is given in (Eichler et al., 2013a), but some aspects are worth pointing out for the benefit of the following discussion on sources of conservatism. Note that if the bounded real lemma is applied with the full block S-procedure as in (Scherer, 2001), then stability and an induced $\mathcal{L}_2$ gain less than γ of $\breve{\mathcal{T}}$ is guaranteed, if there exist $X = X^T > 0$ and $Q = Q^T$, $R = R^T$ and S such that

$$[\star]^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} P \otimes I_{n_\Lambda} \\ I_{Nn_\Lambda} \end{bmatrix} > 0, \; \forall P \in \boldsymbol{\mathcal{P}} \tag{5}$$

$$[\star]^T \left[\begin{array}{cc|cc|cc} 0 & X & & & & \\ X & 0 & & & & \\ \hline & & Q & S & & \\ & & S^T & R & & \\ \hline & & & & -\gamma^2 I & 0 \\ & & & & 0 & I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ I_N \otimes \mathcal{A}^\mathsf{d} & I_N \otimes \mathcal{B}_\mathrm{i} & I_N \otimes \mathcal{B}_\mathrm{p}^\mathsf{d} \\ \hline 0 & I & 0 \\ I_N \otimes \mathcal{C}_\mathrm{i} & 0 & I_N \otimes \mathcal{D}_\mathrm{ip} \\ \hline 0 & 0 & I \\ I_N \otimes \mathcal{C}_\mathrm{p}^\mathsf{d} & I_N \otimes \mathcal{D}_\mathrm{pi} & I_N \otimes \mathcal{D}_\mathrm{pp}^\mathsf{d} \end{array}\right] < 0 \tag{6}$$

With the structural assumptions $X = I \otimes X^\mathsf{d}$, $Q = I \otimes Q^\mathsf{d}$ and S and R structured respectively, (5) and (6) are equivalent to (3) and (4).

Extension to LPV systems is, as shown in (Hoffmann et al., 2013; Eichler et al., 2013b), easily possible. Here analysis is reduced to decomposable LTI systems, to focus on the sources of conservatism by the interconnection. Note that while (4) does not scale with the number of subsystems N due to the structural constraints on the Lyapunov matrix X and the multiplier matrices Q,S and R, (3) does. In (Hoffmann et al., 2013; Eichler et al., 2013b) methods for the reduction of complexity in (3) are presented for different cases. The approaches for time-varying but always symmetric or possibly non-symmetric interconnections are discussed in Section 3.2 and 3.3 with regard to the conservatism they introduce.

2.3 Controller Synthesis for Decomposable Systems

Controller synthesis for decomposable systems is proposed in (Massioni and Verhaegen, 2010; Hoffmann et al., 2013; Eichler et al., 2013b) for state feedback and constant interconnection topologies by convex, for output feedback and symmetric time-varying topologies by convex and more general time-varying topologies by non-convex conditions. Here the convex case of state feedback is considered, in order to focus on the conservatism and to avoid the discussion of suboptimal solutions for non-convex conditions.

Assume a decomposable open-loop system and a decomposable controller are given as

$$\breve{\mathcal{G}} : \begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \left[\begin{array}{c|cc} \breve{A} & \breve{B}_\mathrm{p} & \breve{B}_\mathrm{u} \\ \hline \breve{C}_\mathrm{p} & \breve{D}_\mathrm{pp} & \breve{D}_\mathrm{pu} \end{array}\right] \begin{bmatrix} x \\ d \\ u \end{bmatrix}, \quad \breve{\mathcal{K}} : u = \breve{K} x \tag{7}$$

with the same interconnection topology P, i.e. all subsystem are controlled by identical controllers, interconnected by the same topology as the plant. The resulting closed loop is decomposable if $B_\mathrm{u}^\mathsf{i}{=}0$ and $D_\mathrm{pu}^\mathsf{i}{=}0$. Stability and performance bounds can be proven by Theorem 2, which is not linear in the controller, Lyapunov and multiplier matrices. Using the duals of (3) and (4) and the variable transformation $\breve{M} = I_N \otimes Y^\mathsf{d} \breve{K} = I_N \otimes (Y^\mathsf{d} K^\mathsf{d}) + P \otimes (Y^\mathsf{d} K^\mathsf{i})$ with $Y^\mathsf{d} = (X^\mathsf{d})^{-1}$ leads to convex synthesis conditions, see (Massioni and Verhaegen, 2010).

3. SOURCES OF CONSERVATISM

3.1 Conservatism due to Constraints on Lyapunov and Multiplier Matrices

In the proof of Theorem 2 the Lyapunov matrix as well as the multiplier are structurally constrained. Here the introduced conservatism is considered. Given is a decomposable system $\breve{\mathcal{T}}$, and $\gamma_{\mathrm{c,c}}$ is the upper performance bound calculated by applying Theorem 2.

Lemma 3. Assume $\gamma_{\mathrm{c,f}}$ is the upper performance bound of $\breve{\mathcal{T}}$, if in conditions (5) and (6) only the Lyapunov matrix is constrained to $X = I \otimes X^{\mathsf{d}}$, but $Q, T, S \in \mathbb{R}^{Nn_\Lambda \times Nn_\Lambda}$ are full. Then $\gamma_{\mathrm{c,c}} = \gamma_{\mathrm{c,f}}$.

Proof. Condition (6) with the constraints on Lyapunov and multiplier matrices can be expanded and permuted to

$$I\otimes\begin{bmatrix} X^{\mathsf{d}}\mathcal{A}^{\mathsf{d}}+\mathcal{A}^{\mathsf{d}T}X & X^{\mathsf{d}}\mathcal{B}^{\mathsf{d}}_{\mathrm{i}} & X^{\mathsf{d}}\mathcal{B}^{\mathsf{d}}_{\mathrm{p}} \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix} + I\otimes\begin{bmatrix} C_{\mathrm{i}}^T R^{\mathsf{d}} C_{\mathrm{i}} & C_{\mathrm{i}}^T S^{\mathsf{d}T} & C_{\mathrm{i}}^T R^{\mathsf{d}} D_{\mathrm{ip}} \\ \star & Q^{\mathsf{d}} & S^{\mathsf{d}} D_{\mathrm{ip}} \\ \star & \star & D_{\mathrm{ip}}^T R^{\mathsf{d}} D_{\mathrm{ip}} \end{bmatrix}$$
$$+I\otimes\begin{bmatrix} C_{\mathrm{p}}^{\mathsf{d}T} C^{\mathsf{d}}_{\mathrm{p}} & C_{\mathrm{p}}^{\mathsf{d}T} D^{\mathsf{d}}_{\mathrm{pi}} & C_{\mathrm{p}}^{\mathsf{d}T} D^{\mathsf{d}}_{\mathrm{pp}} \\ \star & D_{\mathrm{pi}}^{\mathsf{d}T} D^{\mathsf{d}}_{\mathrm{pi}} & D_{\mathrm{pi}}^{\mathsf{d}T} D^{\mathsf{d}}_{\mathrm{pp}} \\ \star & \star & D_{\mathrm{pp}}^{\mathsf{d}T} D^{\mathsf{d}}_{\mathrm{pp}} - \gamma_{\mathrm{c,c}}^2 I \end{bmatrix} = I\otimes V < 0 \tag{8}$$

which is equivalent to (4). To achieve optimal performance, the optimization task to solve is to minimize $\gamma_{\mathrm{c,c}}$ subject to (8) and (3). For $\gamma = \gamma_{\mathrm{c,c}}$ we attain the boundary, where the maximal eigenvalue of V is zero. If the multiplier is not constrained but a full block with $R_{ij} \in \mathbb{R}^{n_\Lambda \times n_\Lambda}$, the second term of (8) has the form

$$\left[\begin{array}{ccc|c|ccc} C_{\mathrm{i}}^T R_{11} C_{\mathrm{i}} & C_{\mathrm{i}}^T S_{11}^T & C_{\mathrm{i}}^T R_{11} D_{\mathrm{ip}} & & C_{\mathrm{i}}^T R_{1N} C_{\mathrm{i}} & C_{\mathrm{i}}^T S_{1N}^T & C_{\mathrm{i}}^T R_{1N} D_{\mathrm{ip}} \\ \star & Q_{11} & S_{11} D_{\mathrm{ip}} & \cdots & \star & Q_{1N} & S_{1N} D_{\mathrm{ip}} \\ \star & \star & D_{\mathrm{ip}}^T R_{11} D_{\mathrm{ip}} & & \star & \star & D_{\mathrm{ip}}^T R_{1N} D_{\mathrm{ip}} \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline C_{\mathrm{i}}^T R_{N1} C_{\mathrm{i}} & C_{\mathrm{i}}^T S_{N1}^T & C_{\mathrm{i}}^T R_{N1} D_{\mathrm{ip}} & & C_{\mathrm{i}}^T R_{NN} C_{\mathrm{i}} & C_{\mathrm{i}}^T S_{NN}^T & C_{\mathrm{i}}^T R_{NN} D_{\mathrm{ip}} \\ \star & Q_{N1} & S_{N1} D_{\mathrm{ip}} & \cdots & \star & Q_{NN} & S_{NN} D_{\mathrm{ip}} \\ \star & \star & D_{\mathrm{ip}}^T R_{N1} D_{\mathrm{ip}} & & \star & \star & D_{\mathrm{ip}}^T R_{NN} D_{\mathrm{ip}} \end{array}\right]. \tag{9}$$

It is clear that for (8) with the second term as in (9), at least $\gamma_{\mathrm{c,f}} = \gamma_{\mathrm{c,c}}$ can be reached, by setting $R_{ij} = 0$ for $i \neq j$, $R_{ii} = R_{jj}$ for $i, j = 1, \ldots, N$ and Q and S respectively. The question is whether it is possible to achieve $\gamma_{\mathrm{c,f}} < \gamma_{\mathrm{c,c}}$: in order for (8) with (9) to be negative definite, all leading principal minors have to be negative definite. Consider only the first $n_\xi + n_\Lambda + n_z$ diagonal block, which is equal to (4). Thus the optimal solution considering this first block only is $\gamma_{\mathrm{c,c}}$, and the maximum eigenvalue of that first block is zero. In (Horn and Johnson, 2012) it is shown that given a symmetric matrix F and

$$\hat{F} = \begin{bmatrix} F & y \\ y^T & f \end{bmatrix} \quad \text{then} \quad \lambda_1 \leq \hat{\lambda}_1$$

with λ_1 being the largest eigenvalue of F and $\hat{\lambda}_1$ that of $\hat{F}$. Thus if we consider the first $n_\xi + n_\Lambda + n_z + 1$ block, its maximum eigenvalue would be larger than zero taking the same solution as for the first $n_\xi + n_\Lambda + n_z$ block and setting the off diagonal elements unequal to zero. To recover negative definiteness, $\gamma_{\mathrm{c,f}}$ would have to increase. This can be done iteratively for each missing line and column. Thus $\gamma_{\mathrm{c,f}} \geq \gamma_{\mathrm{c,c}}$, which concludes the proof.

Lemma 4. Assume $\gamma_{\mathrm{f,c}}$ is the upper performance bound of $\breve{\mathcal{T}}$, if in conditions (5) and (6) only the multiplier matrices are constrained to $Q = I \otimes Q^{\mathsf{d}}$, $R = I \otimes R^{\mathsf{d}}$ and $S = I \otimes S^{\mathsf{d}}$, but $X \in \mathbb{R}^{Nn_\xi \times Nn_\xi}$ are full. Then $\gamma_{\mathrm{c,c}} = \gamma_{\mathrm{f,c}}$.

Proof. The proof follows the line of Lemma 3.

Lemma 5. Assume $\gamma_{\mathrm{f,f}}$ is the upper performance bound of $\breve{\mathcal{T}}$, if in conditions (5) and (6) neither Lyapunov matrix nor multiplier matrices are constrained, but $Q, T, S \in \mathbb{R}^{Nn_\Lambda \times Nn_\Lambda}$ and $X \in \mathbb{R}^{Nn_\xi \times Nn_\xi}$ are full. Then $\gamma_{\mathrm{c,c}} \geq \gamma_{\mathrm{f,f}}$.

Proof. If neither Lyapunov nor multiplier matrices are constrained, the non diagonal structure of the uncertainty due to the interconnection in (5) can be captured by the full multiplier and compensated by the full Lyapunov matrix in (6). Thus $\gamma_{\mathrm{c,c}} \geq \gamma_{\mathrm{f,f}}$.

3.2 Conservatism for Symmetric Interconnections

If (4) has been solved such that (3) is fulfilled for all $P \in \boldsymbol{\mathcal{P}}$, the corresponding solution is also valid for all $P \in \boldsymbol{\mathcal{P}}_{\mathrm{c}}$, where $\boldsymbol{\mathcal{P}}_{\mathrm{c}}$ is the convex hull of $\boldsymbol{\mathcal{P}}$. If all $P \in \boldsymbol{\mathcal{P}}$ are symmetric, the solution is also valid for a larger region; this case will be discussed in the following. In (Massioni and Verhaegen, 2010; Hoffmann et al., 2013), the following has been shown.

Lemma 6. Given is a set of interconnection pattern $\boldsymbol{\mathcal{P}}$ with $P = P^T$ for all $P \in \boldsymbol{\mathcal{P}}$, then (3) is equivalent to

$$[\star]^T \begin{bmatrix} Q^{\mathsf{d}} & S^{\mathsf{d}} \\ S^{\mathsf{d}T} & R^{\mathsf{d}} \end{bmatrix} \begin{bmatrix} \lambda_{\min} I_{n_\Lambda} \\ I_{n_\Lambda} \end{bmatrix} > 0 \text{ and } [\star]^T \begin{bmatrix} Q^{\mathsf{d}} & S^{\mathsf{d}} \\ S^{\mathsf{d}T} & R^{\mathsf{d}} \end{bmatrix} \begin{bmatrix} \lambda_{\max} I_{n_\Lambda} \\ I_{n_\Lambda} \end{bmatrix} > 0 \tag{10}$$

where $\lambda_{\min}$ is the minimal eigenvalue of all $P \in \boldsymbol{\mathcal{P}}$ and $\lambda_{\max}$ the maximal one.

Thus the respective solution is also valid for all symmetric interaction topologies with its eigenvalues in $[\lambda_{\min}, \lambda_{\max}]$. In the following it will be examined for what kind of interconnection topologies this is the case. Here we will have a closer look at the eigenvalues of the adjacency and normalized adjacency matrix, which are well-established matrix representations of graphs. An interconnection topology is represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with the node set $\mathcal{N} = \{v_1, \ldots, v_N\}$, which represents the subsystems, and the edge set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ describing the interconnection topology. If there is an edge $\{ij\} \in \mathcal{E}$ then subsystem i receives information from subsystem j and, since undirected graphs are considered that lead to symmetric interconnection topologies, vice versa. The adjacency matrix A and the normalized $\bar{A}$ are defined as

$$A = [a_{ij}] = \begin{cases} 1 \text{ if } i \neq j \text{ and } \{ij\} \in \mathcal{E} \\ 0 \text{ otherwise} \end{cases} \tag{11}$$

$$\bar{A} = \operatorname{diag}\left(\frac{1}{\sqrt{d_1}}, \ldots, \frac{1}{\sqrt{d_N}}\right) A \operatorname{diag}\left(\frac{1}{\sqrt{d_1}}, \ldots, \frac{1}{\sqrt{d_N}}\right) \tag{12}$$

with d_i being the cardinality of $\{v_j \mid \{ij\} \in \mathcal{E}\}$. If the corresponding graph is connected, then the non-negative A is an irreducible matrix with eigenvalues $\lambda_1 > \ldots \geq \lambda_N$, where λ_1 is positive. Different approximations exist: due to the Perron-Frobenius theorem $\min_i(d_i) \leq \lambda_1$ and $\lambda_1 \leq \max_{\{ij\}\in\mathcal{E}} \sqrt{d_i d_j}$ with equality for regular or bipartite graphs, see (Berman and Zhang, 2001). For bipartite graphs $\lambda_N = -\lambda_1$ otherwise $|\lambda_N| < \lambda_1$. If any entry in A increases/decreases (e.g. a link is added or deleted) then λ_1 increases/decreases as well. Nothing can be said about λ_N. For the normalized adjacency with eigenvalues $\bar{\lambda}_1 > \ldots \geq \bar{\lambda}_N$ we have $\bar{\lambda}_1 = 1$ with the corresponding eigenvector $x = [\sqrt{d_1} \ldots \sqrt{d_N}]^T$. For bipartite graphs $\hat{\lambda}_N = -1$ otherwise $|\hat{\lambda}_N| < \hat{\lambda}_1$.

3.3 Conservatism using a Second-Stage Multiplier

If not all $P \in \boldsymbol{\mathcal{P}}$ are symmetric, (10) can not be applied. To reduce complexity in this case, in (Eichler et al., 2013a) the application of the full block S-procedure on the multiplier condition has been proposed. Assume the interconnection topology can be represented as

$$\begin{aligned} P(\upsilon(t)) &= P_0 + \upsilon_1(t)P_1 + \ldots + \upsilon_m(t)P_m \\ &= P_0 + P_{21}\Upsilon(t)P_{21} = \Upsilon(t)\star\left[\begin{array}{c|c} 0 & P_{12} \\ \hline P_{21} & P_0 \end{array}\right] \end{aligned} \tag{13}$$

with $\Upsilon(t) = \mathrm{diag}(\upsilon_1 I_{r_1}, \ldots, \upsilon_m I_{r_m}) \in \mathbb{R}^{n_\upsilon \times n_\upsilon}$, $\upsilon_i \in [-1, 1]$ and $\upsilon(t) = [\upsilon_1, \ldots, \upsilon_m]^T$ with m being the number of connections that my change independently in time. Then (3) is equivalent to

$$[\star]^T \left[\begin{array}{cc|cc} Q_p & S_p & & \\ S_p^T & R_p & & \\ \hline & & I_N \otimes Q^{\mathsf{d}} & I_N \otimes S^{\mathsf{d}} \\ & & I_N \otimes {S^{\mathsf{d}}}^T & I_N \otimes R^{\mathsf{d}} \end{array}\right] \left[\begin{array}{cc} 0 & P_{12} \otimes I_{n_\Lambda} \\ I_{n_\upsilon n_\Lambda} & 0 \\ \hline P_{21} \otimes I_{n_\Lambda} & P_0 \otimes I_{n_\Lambda} \\ 0 & I_{N n_\Lambda} \end{array}\right] > 0, \tag{14}$$

$$[\star]^T \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix} \begin{bmatrix} \Upsilon(\upsilon) \otimes I \\ I \end{bmatrix} < 0, \ \forall \upsilon \in \boldsymbol{\upsilon} \tag{15}$$

where $\boldsymbol{\upsilon}$ denotes the compact set of all admissible $\upsilon(t)$. Note that using (14) and (15) instead of (3) does not introduce any further conservatism, but hardly reduces complexity or in some case even increases it as discussed in (Eichler et al., 2013a). By the constraints given in the following, the complexity is reduced, but conservatism is introduced. Assume that the new multiplier matrices in (15) are restricted to $Q_p = I_{n_\upsilon} \otimes Q_p^{\mathsf{d}}$ with $Q_p^{\mathsf{d}} \in \mathbb{R}^{n_\Lambda \times n_\Lambda}$ and S_p, R_p accordingly, then (15) is equivalent to

$$[\star]^T \begin{bmatrix} Q_p^{\mathsf{d}} & S_p^{\mathsf{d}} \\ {S_p^{\mathsf{d}}}^T & R_p^{\mathsf{d}} \end{bmatrix} \begin{bmatrix} \upsilon_i I \\ I \end{bmatrix} < 0 \ \ \forall \upsilon_i \in [-1, 1]. \tag{16}$$

In (Meinsma et al., 1997) it has been proven that for one diagonal uncertainty block with one real scalar uncertainty, as we have in (16), no conservatism is introduced if D-G scalings are used instead of full block multipliers. Therefore using (14) and (16) with full R_p^{d}, S_p^{d} and Q_p^{d} leads to the same solution as using (14) with D-G scalings $R_p^{\mathsf{d}} = -Q_p^{\mathsf{d}}$ and $S_p^{\mathsf{d}} = -{S_p^{\mathsf{d}}}^T$. Condition (16) is due to the D-G scalings always being fulfilled and need not be considered. This is referred to as constrained 2 stage full block S-procedure (FPSP2 (constr.)) to distinguish it from the single stage full block S-procedure (FBSP1) using (3). The FBSP2 (constr.) leads to an enormous reduction in the number of variables and the size of the LMI to solve, determined from the diagonal concatenation of all single LMIs, as can be seen from Table 1 (Eichler et al., 2013b). For the reduction of complexity we pay by conservatism, which is introduced by the structural restriction on the new multiplier matrices compared to FBSP1.

Lemma 7. Given are the interconnections

$$\mathbf{P}^{1,2}(\upsilon) = P_0 + \upsilon_1 P_1 + \upsilon_2 P_2 + \textstyle\sum_{k=3}^{m} \upsilon_k P_k,$$
$$\mathbf{P}^{1+2}(\upsilon) = P_0 + \upsilon_l (P_1 + P_2) + \textstyle\sum_{k=3}^{m} \upsilon_k P_k$$

with $\upsilon_{1,2,l,k} \in [-1, 1]$. Assume that $\gamma_{1,2}$ is the performance achieved for $\mathbf{P}^{1,2}$ by the FBSP2 (constr.), then it possible to achieve γ_{1+2}, the performance of $\mathbf{P}^{1+2}$ by the FBSP2 (constr.), such that $\gamma_{1+2} \leq \gamma_{1,2}$ holds.

Proof. The LFT representation of $\mathbf{P}^{i,j}(\upsilon)$ is

$$\mathbf{P}^{i,j}(\upsilon) = \mathrm{diag}(\upsilon_1 I_{r_1}, \upsilon_2 I_{r_2}, \upsilon_3 I_{r_3}, \ldots, \upsilon_m I_{r_m}) \star \left[\begin{array}{c|c} 0 & P_{12}^{1,2} \\ \hline P_{21}^{1,2} & P_0 \end{array}\right]$$

with $P_{12}^{1,2} = [\, P_{12_1}^T \ P_{12_2}^T \ P_{12_3}^T \ \cdots \ P_{12_m}^T \,]^T$,
$P_{21}^{1,2} = [\, P_{21_1} \ P_{21_2} \ P_{21_3} \ \cdots \ P_{21_m} \,]$,

and $P_{21_1}P_{12_1} = P_1$ and $P_{21_2}P_{12_2} = P_2$, resulting in the performance $\gamma_{1,2}$. The LFT representation of $\mathbf{P}^{i+j}(\upsilon)$ is

$$\mathbf{P}^{i,j}(\upsilon) = \mathrm{diag}(\upsilon_1 I_{r_l}, \upsilon_3 I_{r_3}, \ldots, \upsilon_m I_{r_m}) \star \left[\begin{array}{c|c} 0 & P_{12}^{1+2} \\ \hline P_{21}^{1+2} & P_0 \end{array}\right]$$

with $P_{12}^{1+2} = [P_{12_l}^T \ P_{12_3}^T \ \cdots \ P_{12_m}^T]^T$, $P_{21}^{1+2} = [P_{21_l} \ P_{21_3} \ \cdots \ P_{21_m}]$, and $P_{21l}P_{12l} = P_i + P_j$. A possible choice of the LFT representation, is $P_{12l} = [P_{12i}^T \, P_{12j}^T]^T$ and $P_{21l} = [P_{121i} \, P_{21j}]$ and thus leads to $\gamma_{1+2} = \gamma_{1,2}$. Other LFT representations may yield to better performance.

Note that if $\mathrm{rank}(P_i + P_j) < \mathrm{rank}(P_i) + \mathrm{rank}(P_j)$ the LFT representation with $P_{12l} = [P_{12i}^T \, P_{12j}^T]^T$ and $P_{21l} = [P_{121i} \, P_{21j}]$ has not the smallest possible size. Due to Lemma 7 it can be expected, that FBSP2 (constr.) introduces less conservatism if n_υ/m is small.

An alternative approach is to use D-G scalings already for Q_p, S_p and Q_p. This is labeled FBSP2 (DG). With the FBSP2 (DG) the size of LMIs to be solved is the same as for the FBSP2 (constr.), since then (15) is always fulfilled and does not need to be checked. On the other hand, the number of variables increases, as shown in Table 1. It is bounded below by $2n_\Lambda^2 + n_\Lambda + n_\upsilon n_\Lambda^2$, which is achieved if all $r_i = 1$ and $m = n_\upsilon$, and thus increases at best linearly with n_υ. The upper bound is given by $2n_\Lambda^2 + n_\Lambda + (n_\Lambda n_\upsilon)^2$, which increases quadratically with n_υ. This worst case occurs if $m = 1$ and $r_1 = n_\upsilon$. It can be expected that with FBSP2 (DG) the conservatism compared to FBSP2 (constr.) is reduced, but problems may occur for large n_υ.

Table 1. Complexity of multiplier conditions

	Size of LMIs	No. of decision Var.
FBSP1	$n_\Lambda N 2^m$	$2n_\Lambda^2 + n_\Lambda$
FBSP2 (constr.)	$n_\Lambda(N + n_\upsilon)$	$3n_\Lambda^2 + n_\Lambda$
FBSP2 (DG)	$n_\Lambda(N + n_\upsilon)$	$2n_\Lambda^2 + n_\Lambda + \sum_{i=1}^{m}(n_\Lambda r_i)^2$ with $\sum_{i=1}^{m} r_i = n_\upsilon$

Table 2. Uncertain link sets

No.	link set	No.	link set	No.	link set
1	$\{P_1+P_2+P_3+P_4\}$	6	$\{P_1+P_2+P_4,\ P_3\}$	11	$\{P_4,\ P_1+P_3,\ P_2\}$
2	$\{P_1+P_3+P_4,\ P_2\}$	7	$\{P_1+P_4,\ P_2+P_3\}$	12	$\{P_2+P_4,\ P_3,\ P_1\}$
3	$\{P_2+P_3+P_4,\ P_1\}$	8	$\{P_1+P_4,\ P_3,\ P_2\}$	13	$\{P_4,\ P_2+P_3,\ P_1\}$
4	$\{P_3+P_4,\ P_1+P_2\}$	9	$\{P_2+P_4,\ P_1+P_3\}$	14	$\{P_4,\ P_3,\ P_1+P_2\}$
5	$\{P_3+P_4,\ P_1,\ P_2\}$	10	$\{P_4,\ P_1+P_2+P_3\}$	15	$\{P_4,\ P_3,\ P_1,\ P_2\}$

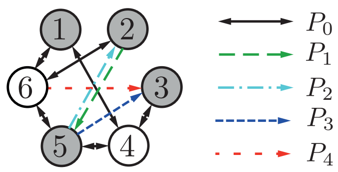

Fig. 1. Graph topologies

4. NUMERICAL RESULTS

To investigate the conservatism introduced by the different constraints in order to simplify dealing with time-varying topologies, state feedback controller design for the system

$$\begin{bmatrix} \dot{x} \\ \hline z_z \end{bmatrix} = \left[\begin{array}{c:c:c} I_N \otimes \begin{bmatrix} 0.1 & -0.2 & 0.7 \\ -0.9 & -0.6 & -0.4 \\ -0.9 & 0.6 & -0.5 \end{bmatrix} + P \otimes \begin{bmatrix} 0.1 & 0.1 & -0.1 \\ -0.3 & -0.1 & -0.1 \\ -0.2 & -0.1 & 0 \end{bmatrix} & I_N \otimes \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} & I_N \otimes \begin{bmatrix} 0.7 \\ 0 \\ 0.5 \end{bmatrix} \\ \hdashline I_N \otimes \begin{bmatrix} 0.9 & 0.2 & 0.3 \\ 0 & 0 & 0 \end{bmatrix} + P \otimes \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} & 0 & I_N \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}\right] \begin{bmatrix} x \\ \hdashline w_z \\ \hdashline u \end{bmatrix} \quad (17)$$

used in (Massioni and Verhaegen, 2010; Eichler et al., 2013a) is performed for 15 different topology set, within which the interconnection can switch arbitrarily. The different topology sets all consist of an identical nominal topology P_0, shown in Fig. 1 (used in e.g. (Massioni and Verhaegen, 2010; Eichler et al., 2013a,b)) but different link sets may be switched on additionally. The components of the link sets, which are summarized in Table 4, consist of one up to four single links P_1, P_2, P_3 and P_4, shown in Fig. 1. Links which are added in the link sets can only switch on and off together. Thus the uncertain topologies no.1 and no.2 can e.g. be described as

$$\mathbf{P}^1(\upsilon) = P_0 + \upsilon_1(P_1+P_2+P_3+P_4) \text{ and}$$
$$\mathbf{P}^2(\upsilon) = P_0 + \upsilon_1(P_1+P_3+P_4) + \upsilon_2 P_2 \text{ with } \upsilon_1, \upsilon_2 \in \{0,1\}$$

respectively. Here $P_0, \ldots, P_4$ are described by the corresponding adjacency matrices.

4.1 Conservatism Results due to Constraints on Lyapunov and Multiplier Matrices

In Section 3.1 the conservatism is discussed, that is introduced by the structural constraints on the Lyapunov and multiplier matrices in order to prevent the nominal condition (4) from scaling with the number of subsystems. In Fig. 2 the calculation times and the performance bounds for the 15 different interconnection topology sets are shown. The solid green curves show the results for the case where neither the Lyapunov matrix nor the multiplier matrices are constrained. Here the nominal condition (6) of size $N(n_\xi + n_\Lambda + n_d) = 54$ has to be solved with the multiplier condition (5). The red dashed dotted lines show the results for the constrained Lyapunov and multiplier matrices as given in Theorem 2, where the nominal condition (4) is of size $n_\xi + n_\Lambda + n_d = 9$ and the multiplier condition (3) has to be solved, labeled as FBSP1. As stated in Lemma 5 the performance bounds for the constrained case are larger; in average they increase by a factor of 1.5 compared to the unconstrained case. The results for the constrained multiplier but unconstrained Lyapunov matrix and vice versa are not shown, since as stated in Lemma 3 and 4 they achieve exactly (up to a numerical accuracy of 10^{-8}) the same performance bounds as when both are constrained. Looking at the calculation time, the calculation time for the unconstrained synthesis is on average 29.6 times longer than with constraints. Note that here a rather small distributed system with 6 subsystems each with 3 states has been considered. For larger systems the unconstrained solution will at some point become intractable. With only the multiplier unconstrained the time is in average 19.6 times longer than with both matrices constrained and with only the Lyapunov matrix unconstrained it is 3 times longer.

4.2 Symmetric Interconnection Topologies with Identical Eigenvalue Set

In Section 3.2 it was pointed out that according to Lemma 6, if (4) has been solved with the multiplier condition (10) for $\lambda_{\min}$ and $\lambda_{\max}$ leading to a performance bound γ, the system is stable with that performance bound for arbitrary switchings between symmetric interconnection topologies within that eigenvalue region. It is clear that if the eigenvalue region and thus the uncertainty region gets smaller, at least as small performance bounds are achieved. Thus the question arises, whether it would not be best to have $P = 0$ and thus no interconnection. But if the interconnection appears in the fictitious performance channels, a change in the interconnection topology changes the defined performance goal. If for example something like consensus is demanded, at least an interconnection topology representing a connected graph is required.

Consider the symmetric interconnection graph P_0 in Fig. 1. The corresponding adjacency matrix has an eigenvalue region of $\lambda \in [-2.236, 2.236]$ and achieves a performance of $\gamma = 0.2779$. According to Section 3.2, for any symmetric graphs, if links are added, those results are not valid, since the maximum eigenvalue increases. Since the graph is bipartite ($\lambda_{\min} = -\lambda_{\max}$, the two node sets are shown in different colors), for any graph where edges are deleted (as long as the graph stays connected) the same performance bounds are valid. The normalized adjacency matrix has eigenvalues $\lambda \in [-1, 1]$ and thus every graph presented by the normalized adjacency matrix would lead to at least the same performance of $\gamma = 0.1878$.

4.3 Conservatism Incurred by FBSP2

To analyze the conservatism introduced by using FBSP2 as discussed in Section 3.3, system (17) with the graph sets in table 4 is considered. In Fig. 2(a) and (b) the resulting calculation times and the performance bounds of FBSP1 are compared to FBSP2 (constr.) in black dashed and to FBSP2 (DG) in blue dotted. In Fig. 2(a) the rank of the components of the corresponding link sets is shown. Each component in a link set is represented by a bar of different color where its length is determined by its rank. Thus the number of different colors in one bar is m and the total length n_υ. The calculation time for the FBSP2 (constr.) is –as expected– reasonably constant for all topologies, since the number of unknowns is constant at 80 and the size of the LMIs changes linearly with n_υ, which is either 3 or 4, resulting in 45 or 50 LMIs to be solved. Using the FBSP2 (DG) the same number of LMIs have to be solved, while the number of unknowns is between 145 and 325. This seems to have almost no effect for those small systems. (For more subsystems, e.g. $N \geq 10$, the calculation time already doubles.) Using the FBSP1 the number of LMIs to solve depends exponentially on the cardinality of the link set m. For $m = 1, 2$ the number of LMIs for FBSP1 (with only 55 unknowns) is with 30 and 60, respectively, smaller than or in the region of FBSP2 (constr.) and thus the calculation time is similar to FBSP (constr.). But for the graphs 5,8,11,12,13 and 14 with $m = 3$ and graph 15 with $m = 4$ the calculation time rises exponentially with 2^m. As expected the performance bound γ for FBSP2 (constr.) is, due to the constraints larger than for FBSP1,

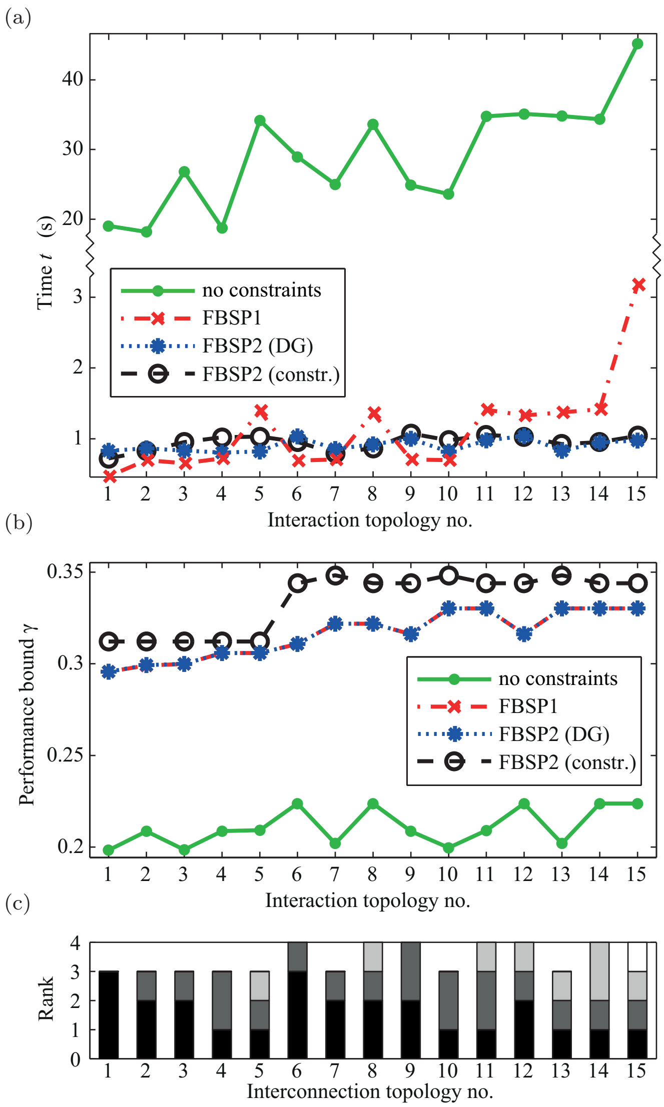

Fig. 2. (a) Calculation time, (b) performance bounds and (c) ranks of the different graph sets in Table 4

but increases in average only by a factor of 1.06. For graphs 5 and 15, where $n_v/m = 1$, as expected a very small deviation is achieved. With Lemma 7 it is clear that for the interconnection topology sets 10, 13 and thus 7 not the best possible performance is achieved, since it should be at least as good as set 15, when applying its LFT representations. Since here the LFT representation of minimal size is used, which is 3 for 7,10 an 13 and 4 for 15, it is obvious that this is not always the best choice.

It seems that with FBSP2 (DG) for the considered topologies no conservatism is introduced compared to FBSP1. In general this has only been shown (see Meinsma et al. (1997)) for one scalar uncertainty blocks as we have in graph 1, but not for more, as in the other graphs. It can be expected, that due to the structure imposed on the first multiplier and the (in general sparse) LFT matrices P_0, P_{12} and P_{21} in (14), some parts of the second multiplier have no influence.

5. CONCLUSION

This work points out and analyzes different sources of conservatism in analysis and controller synthesis for decomposable systems as proposed in Massioni and Verhaegen (2010); Hoffmann et al. (2013); Eichler et al. (2013a). There restrictions are applied to the Lyapunov matrix and the multiplier matrices to solve a nominal condition of the size of a single subsystem. This work proves that the conservatism can not be reduced, if only one of both is constrained, while no constraints on both leads to very long calculation times already for small networks. For symmetric topologies an interesting conclusion is drawn, that the same results are not only valid for the convex hull of all considered switching symmetric topologies, but for all symmetric topologies with their eigenvalues in the convex hull of the eigenvalues of all considered switching symmetric topologies. For non-symmetric switching topologies, with the 2nd stage FBSP an approach to reduce the complexity of the multiplier condition is analyzed with regard to conservatism. Explicit cases, where more and where less conservatism is introduced compared to the 1st stage FBSP, are discussed.

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