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Discrete Kalman filter based on quasi steady state modelling in the delta-domain

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Abstract: This paper develops the Kalman filter for systems with reduced order models based on the implicit quasi steady state method of Eitelberg and Boje (2007). This approach is shown to give improved state estimates when compared to the standard singular perturbation method.

Keywords: Quasi-steady state, Kalman filter, model order reduction, singular perturbations

1. INTRODUCTION

The problem of model order reduction for general non-linear systems was considered by Eitelberg and Boje (2007) using a so-called implicit quasi steady state (QSS) technique. This technique follows a slightly different approach to that of the well-established singular perturbation method and (for linear systems) has been shown to have better frequency domain accuracy than the singular perturbation method. For an overview of the singular perturbation method, please see Kokotović, Khalil and O'Reilly (1986); and Kokotović, O'Malley and Sannuti (1976).

Kailasa Rao and Naidu (1984) developed the Kalman filter for discrete time systems using the singular perturbation method and Shim and Sawan (1999) developed the Kalman filter for singularly perturbed discrete time systems represented using Middleton and Goodwin's (1986) delta operator. We will follow this approach but note that other approaches are possible: Kando, (1997) examines the problem starting from the continuous time representation, and it is also possible to make use of the Tustin (bilinear) transform representation of the discrete time system which retains the $j\omega$ axis stability bound of the continuous time system and has been shown to be very useful for discrete time control system design (Eitelberg, 1988; and Eitelberg and Boje, 1991).

The dynamics of the state estimates are not the same as the dynamics of the physical system states because the former includes the feedback of the output estimate via the Kalman gain (which very roughly depends on the ratio of state to output noise covariances). Because of this, it is possible that slow states in the physical system become fast states in the estimator and vice versa, leading to problems where deriving the full order Kalman filter model and then implementing only the reduced order version of it makes sense. This theme is developed in the continuous time by Qaddour (1998) but will not be pursued in this paper. Our interest is in understanding how to approach the state estimation problem if the physical model is already reduced using the quasi-steady state approach.

The remainder of the paper is organised as follows: Section 2 develops the quasi steady method for discrete time linear systems described using the delta transform. Section 3 develops the Kalman filter equations for the resulting reduced order quasi steady state model. Section 4 presents an example of a simple second order system to illustrate the method. The example also makes a comparison between Kalman filter results based on the quasi steady state and singular perturbation approaches.

2. QUASI STEADY STATE REPRESENTATION OF DISCRETE TIME LINEAR SYSTEMS

Using conventional notation, we consider the linear system in the delta-domain, $\delta\{\bullet\} = \frac{z-1}{T}\{\bullet\}$,

$$\begin{pmatrix} \delta \boldsymbol{X}_1(z) \\ \varepsilon\delta \boldsymbol{X}_2(z) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{X}_1(z) \\ \boldsymbol{X}_2(z) \end{pmatrix} + \begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{pmatrix}\boldsymbol{U}(z) + \begin{pmatrix} \boldsymbol{N}_1(z) \\ \boldsymbol{N}_2(z) \end{pmatrix}$$
$$\boldsymbol{Y}(z) = \begin{pmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{pmatrix}\begin{pmatrix} \boldsymbol{X}_1(z) \\ \boldsymbol{X}_2(z) \end{pmatrix} + \boldsymbol{DU}(z) + \boldsymbol{V}(z) \quad (1)$$

In (1), the state vector is divided into two components, $\boldsymbol{X}^{\mathrm{T}} = \left(\boldsymbol{X}_1^{\mathrm{T}}, \boldsymbol{X}_2^{\mathrm{T}}\right)$, representing slow and fast behavior respectively. $\mathbf{A}_{jk}$, $\mathbf{B}_j$, $\mathbf{C}_k$, and **D** are (partitions of) the state, input, output and throughput matrices respectively. Zero mean, white state noise, $\mathbf{N}_j$ and output noise, **V** are included. ε is a variable representing the time scale separation and formally, $\varepsilon \to 0$ results in a singular perturbation. For the model order reduction to make sense, A_{22} must be stable and full rank (the latter condition is not onerous as A_{22} represents the dynamics of the *fast* subsystem). Representing (1) in the time-domain with time index, superscript i,

$$\begin{pmatrix} \left(\mathbf{x}_1^{i+1} - \mathbf{x}_1^i\right)/T \\ \varepsilon\left(\mathbf{x}_2^{i+1} - \mathbf{x}_2^i\right)/T \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}\begin{pmatrix} \mathbf{x}_1^i \\ \mathbf{x}_2^i \end{pmatrix} + \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}\mathbf{u}^i + \begin{pmatrix} \mathbf{n}_1^i \\ \mathbf{n}_2^i \end{pmatrix}$$
$$\mathbf{y}^i = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{pmatrix}\begin{pmatrix} \mathbf{x}_1^i \\ \mathbf{x}_2^i \end{pmatrix} + \mathbf{Du}^i + \mathbf{v}^i \quad (2)$$

The relationship between the system, **A**, and input, **B**, matrices in the continuous (superscript c) and delta (superscript δ) domains is,

$$\mathbf{A}^{\delta}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{pmatrix}=\frac{1}{T}\left(e^{\mathbf{A}^{c}T}-\mathbf{I}\right)\approx\mathbf{A}^{c},\qquad \mathbf{B}^{\delta}=\begin{pmatrix}\mathbf{B}_{1}\\ \mathbf{B}_{2}\end{pmatrix}=\mathbf{B}^{c}\frac{1}{T}\int_{0}^{T}e^{\mathbf{A}^{c}\tau}d\tau\approx\mathbf{B}^{c} \tag{3}$$

where the approximations are good if the sampling time is small relative to the eigenvalues of the system matrix ($T/|\lambda|_{max} \ll 1$). Note in (3) that jω-axis eigenvalues of A^c do not map into the jω-axis in the discrete time as they would if a Tustin transform were used. (They map onto a circle in the left hand plane with radius $1/T$, centered on $-1/T$.)

2.1 Implicit Quasi Steady State Recursion

As has been argued in Eitelberg and Boje (2007) (for the general continuous-time, nonlinear case), the singular perturbation solution, $\mathbf{X}_2=-\mathbf{A}_{22}^{-1}(\mathbf{A}_{21}\mathbf{X}_1+\mathbf{B}_2\mathbf{U}+\mathbf{N}_2)$ for the fast subsystem obtained via $\lim \varepsilon\to 0$ in (1), may be adequate with respect to modelling of $\mathbf{X}_2$ but may not be adequate for the dynamic model of $\mathbf{X}_1$. The exact solution for $\mathbf{X}_2$ in (1) is,

$$\boldsymbol{X}_2=-\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\boldsymbol{X}_1-\boldsymbol{A}_{22}^{-1}\boldsymbol{B}_2\boldsymbol{U}-\boldsymbol{A}_{22}^{-1}\boldsymbol{N}_2+\boldsymbol{A}_{22}^{-1}\varepsilon\delta\boldsymbol{X}_2 \tag{4}$$

After a *first substitution*, of (4) into (1), the exact $\mathbf{X}_1$ in (1) is,

$$\delta\boldsymbol{X}_1=\left(\boldsymbol{A}_{11}-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\right)\boldsymbol{X}_1+\left(\boldsymbol{B}_1-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{B}_2\right)\boldsymbol{U}+\left(\boldsymbol{N}_1-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{N}_2\right)+\varepsilon\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\delta\boldsymbol{X}_2 \tag{5}$$

The *second substitution* (eq.(4) into eq.(5)) yields

$$\begin{aligned}\left(\boldsymbol{I}+\varepsilon\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-2}\boldsymbol{A}_{21}\right)\delta\boldsymbol{X}_1&=\left(\boldsymbol{A}_{11}-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\right)\boldsymbol{X}_1\\&+\left(\boldsymbol{B}_1-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{B}_2\right)\boldsymbol{U}-\varepsilon\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-2}\boldsymbol{B}_2\delta\boldsymbol{U}\\&+\left(\boldsymbol{N}_1-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{N}_2\right)-\varepsilon\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-2}\delta\boldsymbol{N}_2\\&+\varepsilon^2\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-2}\delta^2\boldsymbol{X}_2\end{aligned} \tag{6}$$

An important difference between the quasi-steady state approximation (6) and the corresponding result from the singular perturbation method is that (6) includes the first differential of the input (or derivative in the continuous time case). This improves the frequency domain accuracy of the approximation (*ibid.*). The above substitutions could be continued indefinitely in principle but this would defeat the objective of simplifying the model! After the k^{th} substitution, the retained exact model can be written in the form,

$$\begin{aligned}&\left(\delta\boldsymbol{I}-\boldsymbol{A}_{11}+\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\sum_{r=0}^{k-1}\left(\varepsilon\delta\boldsymbol{A}_{22}^{-1}\right)^{r}\boldsymbol{A}_{21}\right)\boldsymbol{X}_1=\\&\left(\boldsymbol{B}_1-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\sum_{r=0}^{k-1}\left(\varepsilon\delta\boldsymbol{A}_{22}^{-1}\right)^{r}\boldsymbol{B}_2\right)\boldsymbol{U}+\left(\boldsymbol{A}_{12}(\varepsilon\delta)^{k}\boldsymbol{A}_{22}^{-k}\right)\boldsymbol{X}_2\end{aligned} \tag{7}$$

The eventual outcome is the implicit solution obtained by eliminating $\mathbf{X}_2$ in the first row of (1), for example using the binomial expansion of,

$$\begin{aligned}\delta\boldsymbol{X}_1&=\boldsymbol{A}_{11}\boldsymbol{X}_1+\boldsymbol{B}_1\boldsymbol{U}+\boldsymbol{N}_1\\&+\boldsymbol{A}_{12}\left(\boldsymbol{I}\varepsilon\delta-\boldsymbol{A}_{22}\right)^{-1}\left(\boldsymbol{A}_{21}\boldsymbol{X}_1+\boldsymbol{B}_2\boldsymbol{U}+\boldsymbol{N}_2\right)\end{aligned} \tag{8}$$

3. KALMAN FILTER DERIVATION BASED ON QUASI-STEADY STATE APPROXIMATION

If the second order and higher terms in $\left(\varepsilon\delta\boldsymbol{A}_{22}^{-1}\right)$ are small, the last term in (6) can be ignored and the result written in the (discrete) time domain to yield a reduced order model that can be used to set up a Kalman filter to estimate the slow state vector, $\mathbf{x}_1^i$,

$$\begin{aligned}\left(\boldsymbol{I}+\varepsilon\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-2}\boldsymbol{A}_{21}\right)\delta\boldsymbol{X}_1+\varepsilon\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-2}\left(\boldsymbol{B}_2\delta\boldsymbol{U}+\delta\boldsymbol{N}_2\right)&=\\\left(\boldsymbol{A}_{11}-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\right)\boldsymbol{X}_1+\left(\boldsymbol{B}_1-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{B}_2\right)\boldsymbol{U}&\\+\left(\boldsymbol{N}_1-\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{N}_2\right)&\end{aligned} \tag{9}$$

$$\begin{aligned}\boldsymbol{Y}&=\left(\boldsymbol{C}_1\;\vdots\;\boldsymbol{C}_2\right)\begin{pmatrix}\boldsymbol{X}_1\\ \boldsymbol{X}_2\end{pmatrix}+\boldsymbol{D}\boldsymbol{U}+\boldsymbol{V}\\&\approx\left(\boldsymbol{C}_1-\boldsymbol{C}_2\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\right)\boldsymbol{X}_1+\left(\boldsymbol{D}-\boldsymbol{C}_2\boldsymbol{A}_{22}^{-1}\boldsymbol{B}_2\right)\boldsymbol{U}\\&+\left(0\;\vdots\;-\boldsymbol{C}_2\boldsymbol{A}_{22}^{-1}\right)\boldsymbol{N}+\boldsymbol{V}\end{aligned} \tag{10}$$

or, simplifying notation,

$$\begin{aligned}\boldsymbol{a}\delta\boldsymbol{X}_1+\boldsymbol{b}\delta\boldsymbol{U}+\boldsymbol{c}\delta\boldsymbol{N}&=\boldsymbol{d}\boldsymbol{X}_1+\boldsymbol{e}\boldsymbol{U}+\boldsymbol{f}\boldsymbol{N}\\\boldsymbol{Y}&=\boldsymbol{g}\boldsymbol{X}_1+\boldsymbol{h}\boldsymbol{U}+\boldsymbol{k}\boldsymbol{N}+\boldsymbol{V}\end{aligned} \tag{11}$$

with,

$$\begin{aligned}&\mathbf{a}=\mathbf{I}+\varepsilon\mathbf{A}_{12}\mathbf{A}_{22}^{-2}\mathbf{A}_{21} &&\mathbf{f}=\left(\mathbf{I}\;\vdots\;-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\right)\\&\mathbf{b}=\varepsilon\mathbf{A}_{12}\mathbf{A}_{22}^{-2}\mathbf{B}_2 &&\mathbf{g}=\mathbf{C}_1-\mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\\&\mathbf{c}=\varepsilon\mathbf{A}_{12}\mathbf{A}_{22}^{-2}\left(\mathbf{0}\;\vdots\;\mathbf{I}\right) &&\mathbf{h}=\mathbf{D}-\mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{B}_2\\&\mathbf{d}=\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} &&\mathbf{k}=\left(0\;\vdots\;-\mathbf{C}_2\mathbf{A}_{22}^{-1}\right)\\&\mathbf{e}=\mathbf{B}_1-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{B}_2\end{aligned}$$

In order to deal with differentials of the input and state noise in (9) or (11), define a modified state variable,

$$\mathbf{z}_1^i=\mathbf{x}_1^i+\mathbf{a}^{-1}\mathbf{b}\mathbf{u}^i+\mathbf{a}^{-1}\mathbf{c}\mathbf{n}^i \tag{12}$$

Note that for small $\varepsilon\mathbf{A}_{12}\mathbf{A}_{22}^{-2}$, $\mathbf{z}_1^i\approx\mathbf{x}_1^i$. Application of (12) leads to the model,

$$\begin{aligned}\mathbf{z}^{i+1}&=\mathbf{A}_d\mathbf{z}^i+\mathbf{B}_d\mathbf{u}^i+\mathbf{E}_d\mathbf{n}^i\\\mathbf{y}^i&=\mathbf{C}_d\mathbf{z}^i+\mathbf{D}_d\mathbf{u}^i+\mathbf{F}_d\mathbf{n}^i+\mathbf{v}^i\end{aligned} \tag{13}$$

with,

$$\begin{aligned}&\mathbf{A}_d=I+T\mathbf{a}^{-1}\mathbf{d} &&\mathbf{C}_d=\mathbf{g}\\&\mathbf{B}_d=T\mathbf{a}^{-1}\left(\mathbf{e}-\mathbf{d}\mathbf{a}^{-1}\mathbf{b}\right) &&\mathbf{D}_d=\mathbf{h}-\mathbf{g}\mathbf{a}^{-1}\mathbf{b}\\&\mathbf{E}_d=T\mathbf{a}^{-1}\left(\mathbf{f}-\mathbf{d}\mathbf{a}^{-1}\mathbf{c}\right) &&\mathbf{F}_d=\mathbf{k}-\mathbf{g}\mathbf{a}^{-1}\mathbf{c}\end{aligned}$$

Obtaining an unbiased, minimum variance estimate for the state, $\mathbf{z}^i$, in (13) is a straight-forward Kalman filtering problem (for example see Eitelberg (1991); Brown and

Hwang (1992); or Kailath, Sayed and Hassibi (2000)). The only "variation on the theme" is that there is state noise in the output equation and this means that the filter equations need to take the special structure of cross-covariance between state and measurement noise into account. The filter equations are:

Gain

$$\mathbf{K}^i = \left(\mathbf{A}_d \mathbf{P}^i \mathbf{C}_d^{\mathrm{T}} + \mathbf{S}^i\right)\left(\mathbf{C}_d \mathbf{P}^i \mathbf{C}_d^{\mathrm{T}} + \mathbf{R}^i\right)^{-1} \tag{14}$$

Estimate:

$$\hat{\mathbf{z}}^{i+1} = \mathbf{A}_d \hat{\mathbf{z}}^i + \mathbf{B}_d \mathbf{u}^i + \mathbf{K}^i\left(\mathbf{y}^i - \mathbf{C}_d \hat{\mathbf{z}}^i - \mathbf{D}_d \mathbf{u}^i\right) \tag{15}$$

Error covariance:

$$\mathbf{P}^{i+1} = \mathbf{Q}^i + \mathbf{A}_d \mathbf{P}^i \mathbf{A}_d^{\mathrm{T}} - \mathbf{K}^i\left(\mathbf{C}_d \mathbf{P}^i \mathbf{A}_d^{\mathrm{T}} + \left(\mathbf{S}^i\right)^{\mathrm{T}}\right) \tag{16}$$

with noise covariances:

$$\begin{aligned} \boldsymbol{Q}^i &= \boldsymbol{E}_d \operatorname{cov}\left\{\boldsymbol{n}^i\right\}\boldsymbol{E}_d^{\mathrm{T}} \\ \boldsymbol{S}^i &= \boldsymbol{E}_d \operatorname{cov}\left\{\boldsymbol{n}^i\right\}\boldsymbol{F}_d^{\mathrm{T}} \\ \boldsymbol{R}^i &= \boldsymbol{F}_d \operatorname{cov}\left\{\boldsymbol{n}^i\right\}\boldsymbol{F}_d^{\mathrm{T}} + \operatorname{cov}\left\{\boldsymbol{v}^i\right\} \end{aligned} \tag{17}$$

From the estimate of $\mathbf{z}^i$, an unbiased (but possibly not the minimum variance) estimate of $\mathbf{x}^i$, is obtained:

$$\begin{pmatrix}\mathbf{x}_1^i \\ \mathbf{x}_2^i\end{pmatrix} \approx \begin{pmatrix}\mathbf{I} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\end{pmatrix}\mathbf{z}_1^i + \begin{pmatrix}-\mathbf{a}^{-1}\mathbf{b} \\ \mathbf{A}_{22}^{-1}\left(\mathbf{A}_{21}\mathbf{a}^{-1}\mathbf{b} - \mathbf{B}_2\right)\end{pmatrix}\mathbf{u}^i + \begin{pmatrix}\mathbf{a}^{-1}\mathbf{c} \\ \mathbf{A}_{22}^{-1}\left(\mathbf{A}_{21}\mathbf{a}^{-1}\mathbf{c} - (\mathbf{0} \;\vdots\; \mathbf{I})\right)\end{pmatrix}\mathbf{n}^i \tag{18}$$

$$\mathscr{E}\left\{\begin{pmatrix}\mathbf{x}_1^i \\ \mathbf{x}_2^i\end{pmatrix}\right\} \approx \begin{pmatrix}\mathbf{I} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\end{pmatrix}\hat{\mathbf{z}}_1^i - \begin{pmatrix}-\mathbf{a}^{-1}\mathbf{b} \\ \mathbf{A}_{22}^{-1}\left(\mathbf{A}_{21}\mathbf{a}^{-1}\mathbf{b} - \mathbf{B}_2\right)\end{pmatrix}\mathbf{u}^i \tag{19}$$

$$\operatorname{cov}\left\{\boldsymbol{x}^i\right\} \approx \begin{pmatrix}\boldsymbol{I} \\ -\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\end{pmatrix}\boldsymbol{P}^i\begin{pmatrix}\boldsymbol{I} \\ -\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\end{pmatrix}^{\mathrm{T}} + \begin{pmatrix}-\boldsymbol{a}^{-1}\boldsymbol{c} \\ \boldsymbol{A}_{22}^{-1}\left(\boldsymbol{A}_{21}\boldsymbol{a}^{-1}\boldsymbol{c} - (\boldsymbol{0} \;\vdots\; \boldsymbol{I})\right)\end{pmatrix}\operatorname{cov}\left\{\boldsymbol{n}^i\right\}\begin{pmatrix}-\boldsymbol{a}^{-1}\boldsymbol{c} \\ \boldsymbol{A}_{22}^{-1}\left(\boldsymbol{A}_{21}\boldsymbol{a}^{-1}\boldsymbol{c} - (\boldsymbol{0} \;\vdots\; \boldsymbol{I})\right)\end{pmatrix}^{\mathrm{T}} \tag{20}$$

This section has developed the QSS equations for a system in the delta-domain into a form that allows the application of the discrete time Kalman filter.

In some applications one may account for the deleted $\varepsilon^2 \mathbf{A}_{12}\mathbf{A}_{22}^{-2}\delta^2\mathbf{X}_2$ in (6) by considering this as an additional "noise" signal in both the state difference equation and the output equation. Of course it is not a random signal and is correlated with the signal $\mathbf{X}_2$ but in many applications, the whitening effect of finding the second order difference may make such an approximation useful.

Using higher order approximations as in (7) will improve the approximation accuracy with respect to ε for both singular perturbation and QSS methods. In the case of the QSS method, the renaming of the state vector in (12) to avoid numerical differentiation of the input signal will only work (and give the corresponding frequency domain accuracy improvement over the singular perturbation method) up to first order. Thereafter, the benefit can only be realised if the numerical differentials of the input are calculated (or if the input is known to be sufficiently smooth that the higher differentials are small when scaled according to (7)).

4. EXAMPLE

The following example compares Kalman filter solutions of a second order system to those of first order reduced order representations based on the singular perturbation method and the QSS method. It is derived from the example in Kokotovic, *et al* (1986, Ch. 2, Example 7.1), and discussed in Eitelberg and Boje, (2007). Consider the system,

$$\begin{aligned} \delta x_1 &= -x_1 + x_2 + n_1 \\ \varepsilon\delta x_2 &= -x_2 + u + n_2 \\ y &= x_1 + v \end{aligned} \tag{21}$$

with white noises, $\boldsymbol{n}$, and other parameters,

$$\operatorname{cov}(\mathbf{n}) = \begin{pmatrix}20 & 1 \\ 1 & 0.3\end{pmatrix}\delta_{ij}, \quad \operatorname{cov}(v) = 0.4\delta_{ij}$$

$$\varepsilon = 0.1, \quad T = 0.05.$$

Eq. (21) has the state difference equation,

$$\mathbf{x}_{full}^{i+1} = \begin{pmatrix}1-T & T \\ 0 & (1-T)/\varepsilon\end{pmatrix}\mathbf{x}_{full}^i + \begin{pmatrix}0 \\ T/\varepsilon\end{pmatrix}u^i + \begin{pmatrix}T & 0 \\ 0 & T/\varepsilon\end{pmatrix}\mathbf{n}\,. \tag{22}$$

The singular perturbation model is,

$$\begin{aligned} x_{1\,sp}^{i+1} &= (1-T)x_{1sp}^i + Tu^i + T\left(n_1^i + n_2^i\right) \\ y_{sp}^i &= x_{1\,sp}^i + v^i \end{aligned} \tag{23}$$

with the original state vector and covariance at time, i, recovered via,

$$\begin{aligned} \hat{\mathbf{x}}_{sp}^i &= \begin{pmatrix}1 \\ 0\end{pmatrix}\hat{x}_{1\,sp}^i + \begin{pmatrix}0 \\ 1\end{pmatrix}u^i \\ \operatorname{cov}(\mathbf{x}_{sp}) &= \begin{pmatrix}1 \\ 0\end{pmatrix}\operatorname{cov}(x_{1\,sp})\begin{pmatrix}1 \\ 0\end{pmatrix}^{\mathrm{T}} + \begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\operatorname{cov}(\mathbf{n})\begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}^{\mathrm{T}} \end{aligned} \tag{24}$$

The modified QSS model from (13) is,

$$\begin{aligned} z_{qss}^{i+1} &= (1-T)z_{qss}^i + T(1+\varepsilon)u^i + T\left(n_1^i + (1-\varepsilon)n_2^i\right) \\ y_{qss}^i &= z_{qss}^i - \varepsilon u^i + (0 \quad \varepsilon)\mathbf{n}^i + v^i \end{aligned} \tag{25}$$

with the original state vector and covariance at time, i, recovered using (20) and (21) respectively to give,

$$\hat{\mathbf{x}}^i_{qss} = \begin{pmatrix}1\\0\end{pmatrix}\hat{z}^i_{qss} + \begin{pmatrix}-\varepsilon\\1\end{pmatrix}u^i$$
$$\mathrm{cov}(\mathbf{x}) = \begin{pmatrix}1\\0\end{pmatrix}\mathrm{cov}(z_{qss})\begin{pmatrix}1\\0\end{pmatrix}^{\mathrm{T}} + \begin{pmatrix}0 & -\varepsilon\\0 & 1\end{pmatrix}\mathrm{cov}(\mathbf{n})\begin{pmatrix}0 & -\varepsilon\\0 & 1\end{pmatrix}^{\mathrm{T}} \tag{26}$$

The system (21) is simulated over 100 samples with $u^i = \begin{cases}0 & i<50\\1 & i>50\end{cases}$. Figs. 1 and 2 show compare the Kalman filter outputs of the full-order, singular perturbation, and QSS models. Initial conditions are $\hat{\mathbf{x}}^0 = (1 \quad 0)^{\mathrm{T}}$ and initial state covariance is $\mathbf{P}^0 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

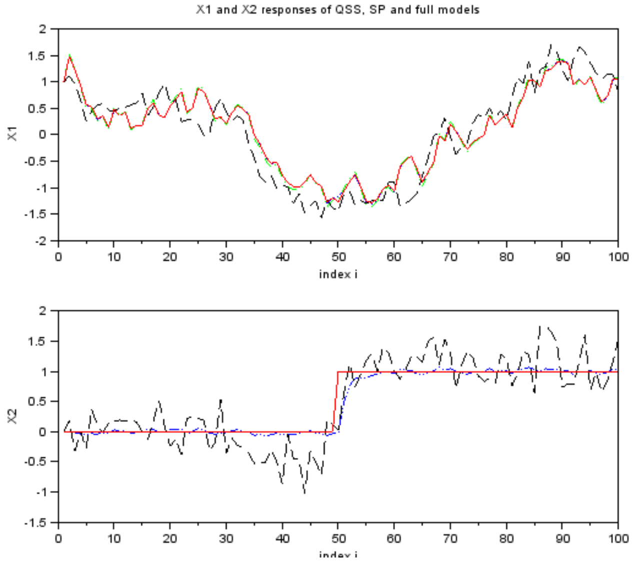

Fig. 1. Kalman filter for QSS (solid, red), singular perturbation (dashed, green), full model (dash dot, blue) and simulation (dotted, black).

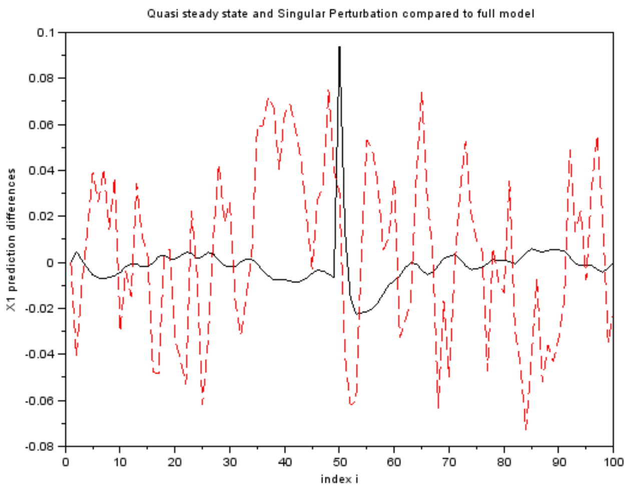

Fig. 2. QSS $(\hat{x}^i_{1\,full} - \hat{x}^i_{1\,qss})$ (solid, black) and singular perturbation $(\hat{x}^i_{1\,full} - \hat{x}^i_{1\,sp})$ (broken, red) compared with respect to full-order state estimates.

Note that the Fig. 1 does not show the actual state vector as this is in principle not recoverable. Instead, the figure compares the reduced order estimates to the best linear unbiased estimate generated by the Kalman filter solution of the full-order model. From Fig. 1 it is observed that the estimates of x_1 using either model are almost indistinguishable and that the reduced order estimates for x_2 do not track the state as well as the second order method. In Fig. 2, the differences between the full order model estimate of x_1 (N.B. again, the full order Kalman filter solution and not the actual state) and the estimates of x_1 using the two reduced order methods are compared. This finer detail highlights the small but clearly improved performance of the QSS method over the singular perturbation method in this example.

The state covariance matrices at the end of the simulation are as follows.

$$\boldsymbol{P}_{full} = \begin{pmatrix}0.153 & 0.042\\0.042 & 0.099\end{pmatrix}, \tag{27a}$$
$$\boldsymbol{P}_{sp} = \begin{pmatrix}0.143 & 0\\0 & 0.30\end{pmatrix}, \tag{27b}$$
$$\boldsymbol{P}_{qss} = \begin{pmatrix}0.169 & -0.03\\-0.03 & 0.30\end{pmatrix} \tag{27c}$$

One would expect that the covariance (diagonal elements of P) of the full order model should be lower than that of the reduced order model which is the case for the quasi steady state method. The singular perturbation result is too optimistic.

5. DISCUSSION AND CONCLUSIONS

This paper has developed a reduced order, discrete time Kalman filter based on the continuous time QSS model order reduction technique of Eitelberg and Boje (2007). Because the QSS method has better *frequency domain* asymptotic accuracy than the singular perturbation approach, in typical (low-pass system) Kalman filtering problems, it should out-perform the singular perturbation method. To some extent, this is achieved by sleight-of-hand as there is a renaming of the state vector in (12) (to bring input derivatives that are found in the continuous time case into the discrete time framework). On the other hand, the underlying state vector can be recovered easily and the method works well. We have used the delta-transform representation of the discrete-time system for ease of application of the underlying continuous-time ideas but the *w*-domain representation would result in equivalent analysis and results. This initial work suggests the following avenues for further work:

1) Application to higher order linear system problems and practical examples.
2) Approximating for the state error caused by unmodelled dynamics (the obvious effect of the model order reduction) via increased state noise covariance in the reduced order system.

3) Application of the QSS based model order reduction to reduced order extended Kalman filter and sigma-point type of Kalman filter designs

4) Application to time scale separation problems.

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