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Global Output Feedback Sliding Mode Control of Nonlinear Systems with Multiple Time Delays ⋆

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Abstract: This paper proposes a sliding-mode control scheme for a class of nonlinear systems with multiple time delays, in the state variables and in the output signal. The unmeasured state of the system is estimated by an asymptotic observer for the zero dynamics and high-gain observers connected in cascade for a chain of integrators with a nonlinear input disturbance which compose the complete state. The proposed control strategy guarantees global asymptotic stability of the closed-loop system using only output feedback. The use of observers prevents undesirable chattering phenomena. Simulation results illustrate the effectiveness of this scheme.

Keywords: Variable-structure control, Sliding-mode control, Output feedback, Time delay, Nonlinear systems, Global stability, Observers.

1. INTRODUCTION

Sliding-mode control (SMC) is an attractive methodology for nonlinear systems, being robust to parameter uncertainties and disturbances [Utkin et al., 1999]. On the other hand, actuator and sensor delays are among the most common dynamic phenomena that arises in control engineering practice [Krstic, 2009]. In SMC, time delays deteriorate the control performance, since they cause chattering and may even destabilize the system. Despite of this, sliding-mode controllers for systems with state delays were proposed by Li and DeCarlo [2003], Orlov et al. [2003] and Gouaisbaut et al. [2004] assuming full-state feedback. The use of state observers is an alternative for output-feedback stabilization of systems with state delay, as developed in [Niu et al., 2004, Yan et al., 2010, 2013]. However, such observers may not be applied to a wide class of systems. Adaptive stabilizers based on compensators (e.g., [Bobtsov et al., 2013]) have well known noise sensitivity.

Observers can be applied in SMC to avoid chattering caused by small time lags due to unmodelled dynamics in the measurement system [Utkin et al., 1999, Sec. 8.3]. However, delayed output signals may impair the convergence of the estimated state to the true state and, consequently, the control may become unstable.

Surprisingly, few results are available for SMC of systems with input or output delayed signals. Basin et al. [2003] and Feng et al. [2006] proposed state-feedback SMC for systems with delayed input signals as well as input signals free of delay, which facilitate the control. Liu et al. [2009] applied Padé approximations to transform the SMC of systems with delayed output signal into the problem of controlling non-minimum phase systems. However, it is known that such approximations may be unrealistic for long delays. Furthermore, only local stability could be guaranteed for known parameters and known time delay.

This paper proposes an output-feedback SMC for nonlinear systems with multiple time delays, in the system state and output. The state delay is assumed time-varying and there is no limitation on the rate of its change. The output delay can be arbitrary provided that it is constant. The proposed controller is based on a cascade of high-gain observers [Ahmed-Ali et al., 2012] and an asymptotic observer for the zero dynamics to estimate the system state. The conjunction of these estimators of two kinds allows the inclusion of state-delays in the estimation problem not considered by Ahmed-Ali et al. [2012]. Global asymptotic stability of the closed-loop system is also guaranteed. It is important to stress that the scheme allows ideal sliding mode, even in the presence of delays. Indeed, the estimated sliding variable provided by the cascade observers becomes null after some finite time, therefore, avoiding undesirable chattering phenomena.

1.1 Preliminaries

The following notation and basic concepts are considered: **(a)** OSS means output-to-state-stable as in [Sontag and Wang, 1997]. **(b)** Classes $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions are defined as in [Khalil, 2002, p. 144]. **(c)** The maximum and minimum eigenvalues of a symmetric matrix P are denoted by $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$, respectively. **(d)** The Euclidean norm of a vector x and the corresponding induced norm of a matrix A are denoted by $\|x\|$ and $\|A\|$, respectively. **(e)** The definition of Filippov [1964] for the solution of discontinuous differential equations is adopted. **(f)** As is usual in the time-delay literature [Gu et al., 2003, Sec. 1.2], the initial conditions of a system with state $z \in \mathbb{R}^n$ are given by $z(t) = z_0(t)$, $t \in [-d_{\max}, 0]$, where $z_0(t)$ is a

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vector function continuous in $t \in [-d_{\max}, 0]$, and $d_{\max}$ is the maximum time delay. **(g)** A vector signal $\pi(t)$ is an exponentially decaying term dependent on the initial conditions $z_0(t)$, if $\exists k, \lambda > 0$ such that $\|\pi(t)\| \le k e^{-\lambda t} z_0^*$, $\forall t \ge 0$, where $z_0^* = \max\{\|z_0(t)\| : t \in [-d_{\max}, 0]\}$.

2. PROBLEM STATEMENT

Consider nonlinear systems with multiple time delays

$$\dot{\eta}(t) = A_0 \eta + f_0(\eta, t) + f_1(\eta(t - d_\eta(t)), t) + f_2(y, t)\,, \quad (1)$$
$$\dot{x}(t) = Ax(t) + B\left[k_p(t) u + w(\eta, x, t)\right]\,, \quad (2)$$
$$y(t) = Cx(t - d_x)\,, \quad (3)$$

where $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the measured output, $x \in \mathbb{R}^l$ and $\eta \in \mathbb{R}^{n-l}$ are unmeasured state vectors. For convenience, the subscript d is introduced to denote the time-delayed signals [Yan et al., 2010]

$$\eta_d(t) := \eta(t - d_\eta(t))\,, \qquad x_d(t) := x(t - d_x)\,. \quad (4)$$

Multiple time delays are allowed, in the state (d_η) and in the output signal (d_x). The initial conditions are given by

$$\eta(t) = \eta_0(t)\,, \quad x(t) = x_0(t)\,, \quad t \in [-d_{\max}, 0]\,, \quad (5)$$

where $\eta_0(t)$ and $x_0(t)$ are vector functions continuous in $t \in [-d_{\max}, 0]$, and $d_{\max} := \max\{d_x, \sup_{t \in \mathbb{R}_+}(d_\eta(t))\}$ is the maximum allowed time delay. The triple

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; C = [\,1 \;\, 0 \; \cdots \; 0\,] \quad (6)$$

is in the Brunovsky's controller form, i.e., it represents a chain of l integrators [Khalil, 2002, p. 514].

The following assumptions should be satisfied:

(A1) The time delay $d_\eta(t)$ is a piecewise-continuous function bounded by $0 < \underline{d} \le d_\eta(t) \le \bar{d} < +\infty$, $\forall t$. The time delay $d_x > 0$ in the output signal is constant.

(A2) The matrix A_0 is Hurwitz with *stability margin* satisfying

$$\gamma_0 < -\max_i\{\mathrm{Re}(\gamma_i)\}\,, \quad (7)$$

where $\{\gamma_i\}$ are the eigenvalues of A_0, and a constant $c_1 \ge 1$ is known such that [Cunha et al., 2008]

$$\|e^{A_0 t}\| \le c_1 e^{\gamma_0 t}\,, \quad \forall t \ge 0\,. \quad (8)$$

(A3) $\exists \varphi_f(\cdot) \in \mathcal{K}_\infty$ such that $\|f_2(y,t)\| \le \varphi_f(|y|)$, $\forall y \in \mathbb{R}$, $\forall t \in \mathbb{R}_+$. The output dependent nonlinear function $f_2(y,t)$ is piecewise continuous in t, locally Lipschitz and continuous with respect to y.

(A4) The nonlinear functions $f_0(\eta, t)$ and $f_1(\eta_d, t)$ are piecewise continuous in t, $f_0(0,t) = f_1(0,t) = 0$, $\forall t \in \mathbb{R}_+$, globally Lipschitz and continuous in the other arguments, satisfying

$$\|f_0(\eta, t) - f_0(\eta', t)\| \le \mu_0 \|\eta - \eta'\|\,,$$
$$\|f_1(\eta_d, t) - f_1(\eta'_d, t)\| \le \mu_1 \|\eta_d - \eta'_d\|\,, \quad (9)$$

$\forall(\eta, \eta') \in \mathbb{R}^{n-l} \times \mathbb{R}^{n-l}$, $\forall(\eta_d, \eta'_d) \in \mathbb{R}^{n-l} \times \mathbb{R}^{n-l}$, $\forall t \in \mathbb{R}_+$, where $\mu_0, \mu_1 \ge 0$ are known constants.

(A5) The function $k_p(t)$ is piecewise continuous in t, its sign is known and constant, and $|k_p(t)| \ge \underline{k}_p > 0$, $\forall t$, where $\underline{k}_p$ is a known constant lower bound for $|k_p(t)|$.

(A6) The function $w(\eta, x, t)$ is globally Lipschitz, uniformly in t with respect to η and x, i.e., there exists a constant $\beta > 0$ such that the inequality

$$|w(\eta, x, t) - w(\eta', x', t)| \le \beta\left[\|\eta - \eta'\| + \|x - x'\|\right]\,, \quad (10)$$

is verified $\forall(\eta, \eta') \in \mathbb{R}^{n-l} \times \mathbb{R}^{n-l}$, $\forall(x, x') \in \mathbb{R}^l \times \mathbb{R}^l$, $\forall t \in \mathbb{R}$. In addition, constants $k_\eta, k_x \ge 0$ and a piecewise-continuous uniformly-bounded scalar function $\varphi_w(t) \ge 0$ are known such that the upper bound

$$|w(\eta, x, t)| \le k_\eta \|\eta(t)\| + k_x \|x(t)\| + \varphi_w(t)\,, \quad (11)$$

is valid $\forall \eta \in \mathbb{R}^{n-l}$, $\forall x \in \mathbb{R}^l$, $\forall t \in \mathbb{R}_+$.

The time delay $d_\eta(t)$ is allowed to be time-varying and there is no limitation on the rate of change of the time delay ($\dot{d}_\eta(t)$). From this point of view, assumption (A1) is less restrictive than the condition $\dot{d}_\eta(t) < 1$ assumed in [Han et al., 2010, Theorem 5.2].

The control objective is to guarantee global asymptotic stability of the closed-loop system based on output-feedback.

3. OBSERVER FOR THE ZERO DYNAMICS STATE

To design the control law, here is proposed the following observer for the state η of the subsystem (1):

$$\dot{\hat{\eta}}(t) = A_0 \hat{\eta}(t) + f_0(\hat{\eta}(t), t) + f_1(\hat{\eta}(t - d_\eta(t)), t) + f_2(y, t)\,, \quad (12)$$

where $\hat{\eta} \in \mathbb{R}^{n-l}$ is the estimated state vector. To analyze this observer, equation (12) is subtracted from (1), such that, the dynamic equation of the estimation error ($\tilde{\eta} := \eta - \hat{\eta}$) can be obtained:

$$\dot{\tilde{\eta}}(t) = A_0 \tilde{\eta}(t) + f_0(\eta(t), t) - f_0(\hat{\eta}(t), t)$$
$$+ f_1(\eta(t - d_\eta(t)), t) - f_1(\hat{\eta}(t - d_\eta(t)), t)\,, \quad (13)$$

with initial condition $\tilde{\eta}(t) = \tilde{\eta}_0(t)$, $t \in [-d_{\max}, 0]$. The solution of this differential equation is given by, $t \ge 0$:

$$\tilde{\eta}(t) = e^{A_0 t} \tilde{\eta}_0(0) + e^{A_0 t} * \left[f_0(\eta(t), t) - f_0(\hat{\eta}(t), t)\right.$$
$$\left. + f_1(\eta(t - d_\eta(t)), t) - f_1(\hat{\eta}(t - d_\eta(t)), t)\right]. \quad (14)$$

From assumptions (A2) and (A4), the norm of the estimation error can be bounded by

$$\|\tilde{\eta}(t)\| \le c_1 \tilde{\eta}_0^* e^{-\gamma_0 t} + c_1 e^{-\gamma_0 t} * \left[\|f_0(\eta(t), t) - f_0(\hat{\eta}(t), t)\|\right.$$
$$\left. + \|f_1(\hat{\eta}(t - d_\eta(t)), t) - f_1(\hat{\eta}(t - d_\eta(t)), t)\|\right]$$
$$\le c_1 \tilde{\eta}_0^* e^{-\gamma_0 t} + c_1 e^{-\gamma_0 t} * \left[\mu_0 \|\tilde{\eta}(t)\| + \mu_1 \|\tilde{\eta}(t - d_\eta(t))\|\right]$$
$$\le r(t)\,, \quad (15)$$

where

$$r(t) := c_1 e^{-\gamma_0 t} * \left[\mu_0 \|\tilde{\eta}(t)\| + \mu_1 \tilde{\eta}_{\sup}(t)\right] + c_1 \tilde{\eta}_0^* e^{-\gamma_0 t}\,, \quad (16)$$
$$\tilde{\eta}_0^* := \sup_{t \in [-d_{\max}, 0]} \|\tilde{\eta}_0(t)\|\,, \quad \tilde{\eta}_{\sup}(t) := \sup_{\tau \in [\underline{d}, \bar{d}]} \|\tilde{\eta}(t - \tau)\|\,.$$

The constants $c_1 \ge 1$ and $\gamma_0 > 0$ must satisfy assumption (A2). For design purposes, c_1 and γ_0 can be computed for the matrix A_0 using the method in [Cunha et al., 2008].

The function $r(t)$ is the solution of the differential equation

$$\dot{r}(t) = -\gamma_0 r(t) + c_1\left[\mu_0 \|\tilde{\eta}(t)\| + \mu_1 \tilde{\eta}_{\sup}(t)\right]\,, \quad (17)$$

$r(t) = c_1 \tilde{\eta}_0^*$, $\forall t \in [-d_{\max}, 0]$. Since $r(t) \ge \|\tilde{\eta}(t)\|$, $\forall t \ge -\bar{d}$, then

$$\tilde{\eta}_{\sup}(t) \le \sup_{\tau \in [\underline{d}, \bar{d}]} r(t - \tau)\,, \quad \forall t \ge 0\,. \quad (18)$$

In addition, $\dot{r} \geq -\gamma_0 r$ and by using [Filippov, 1964, Comparison Theorem 7], one has from (18) that, $\forall t \geq 0$,

$$\tilde{\eta}_{\text{sup}}(t) \leq \sup_{\tau \in [\underline{d}, \bar{d}]} \mathrm{e}^{-\gamma_0[(t-\tau)-t]} r(t) \leq \mathrm{e}^{\gamma_0 \bar{d}} r(t) \,. \tag{19}$$

Upon substituting $\|\tilde{\eta}\|$ by r and $\tilde{\eta}_{\text{sup}}$ by $\mathrm{e}^{\gamma_0 \bar{d}} r$ in (17), the differential equation

$$\dot{\bar{r}}(t) = \left(c_1\mu_0 + c_1\mu_1 \mathrm{e}^{\gamma_0 \bar{d}} - \gamma_0\right) \bar{r}(t), \;\; \bar{r}(0) = r(0) = c_1 \tilde{\eta}_0^*, \tag{20}$$

can be obtained. The solution of this equation satisfies $\bar{r}(t) \geq r(t) \geq \|\tilde{\eta}(t)\|$, $\forall t \geq 0$. So, it can be concluded that, if

$$\lambda_0 := \gamma_0 - c_1\mu_0 - c_1\mu_1 \mathrm{e}^{\gamma_0 \bar{d}} > 0 \,, \tag{21}$$

then (20) and, consequently, (13) are exponentially stable,

$$\|\tilde{\eta}(t)\| \leq c_1 \tilde{\eta}_0^* \mathrm{e}^{-\lambda_0 t} \,, \quad t \geq 0 \,, \tag{22}$$

and $\tilde{\eta}(t) \to 0$ as $t \to +\infty$. Note that the condition (21) on λ_0 implies the transcendental inequality

$$\gamma_0 > c_1\mu_0 + c_1\mu_1 \mathrm{e}^{\gamma_0 \bar{d}} \,. \tag{23}$$

Such inequality is useful to determine the maximum allowed delay. Rewriting (23) as $\mathrm{e}^{\gamma_0 \bar{d}} \leq (\gamma_0 - c_1\mu_0)/(c_1\mu_1)$ and then applying $\ln(\cdot)$ in both sides yields

$$\bar{d} \leq \frac{1}{\gamma_0} \ln\left(\frac{\gamma_0 - c_1\mu_0}{c_1\mu_1}\right) \,. \tag{24}$$

Remark 1. (Maximum state delay). As in Nam [2009], we can realize that the maximum time-delay is limited by the stability margin γ_0 of A_0, and the limitation imposed by the argument of $\ln(\cdot)$ in (24), which must be positive.

Remark 2. (Output-to-state stability). Assumption (A3) and the condition (21) imply that the zero dynamics of (1) with $y \equiv 0$ is exponentially stable. Moreover, this subsystem is OSS, as can be concluded from the analysis presented in [Coutinho et al., 2013].

4. HIGH-GAIN CASCADE OBSERVERS

In this section, the cascaded high-gain observers proposed by Ahmed-Ali et al. [2012] are briefly reviewed. These observers will be used to estimate the state x of the nonlinear subsystem (2) with delayed output signal (3).

According to Ahmed-Ali et al. [2012], the function

$$\phi(\eta, x, u, t) := B\left[k_p(t)u + w(\eta, x, t)\right] \tag{25}$$

should be globally Lipschitz with respect to η and x and, uniformly in u and t, which is ensured by assumptions (A5) and (A6). The analysis presented in [Ahmed-Ali et al., 2012] has not considered any kind of uncertainties or disturbances in the nonlinear function $\phi(\eta, x, u, t)$. However, it is possible to show that such cascade observers are robust to exponentially decaying terms since, according to (10), (22) and (25),

$$|\phi(\eta, x, u, t) - \phi(\hat{\eta}, x, u, t)| \leq \beta \|\eta - \hat{\eta}\| \leq \beta\, c_1 \tilde{\eta}_0^* \mathrm{e}^{-\lambda_0 t}, \tag{26}$$

$\forall t \geq 0$, and thus these signals can be absorbed in the observer analysis as additive terms without affecting the observer exponential convergence property.

4.1 Design of High-Gain Observers Connected in Cascade

The cascade connection of observers is an approach to deal with larger delays [Germani et al., 2002, Ahmed-Ali et al., 2012] using the idea of delay distribution [Michiels and Niculescu, 2007, Remark 7.8]. The number of observers (m) is proportional to the time delay d_x and should satisfy

$$m \geq \frac{d_x}{d_1} \,, \tag{27}$$

where $d_1 > 0$ is the maximum delay admitted by a single stage high-gain observer [Ahmed-Ali et al., 2012].

Each observer ($j = 1, \ldots, m$) estimates a delayed state vector,

$$\hat{x}_j(t) = \hat{x}\left(t - d_x + j\frac{d_x}{m}\right) \,, \tag{28}$$

where the delay is equally distributed among all observers (d_x/m). The delayed control signal u and the delayed state $\hat{\eta}$ estimated by the observer (12) are represented by:

$$u_j(t) := u\left(t - d_x + j\frac{d_x}{m}\right), \;\; \hat{\eta}_j(t) := \hat{\eta}\left(t - d_x + j\frac{d_x}{m}\right). \tag{29}$$

Thus, the observers connected in cascade are given by:

$$\begin{aligned}
\dot{\hat{x}}_1 &= A\hat{x}_1 + \phi\left(\hat{\eta}_1, \hat{x}_1, u_1, t - d_x + \frac{d_x}{m}\right) \\
&\quad - \theta\Delta^{-1}K\left[C\hat{x}_1\left(t - \frac{d_x}{m}\right) - y\right] \,, \\
\hat{y}_1 &= C\hat{x}_1 \,, \\
&\vdots \\
\dot{\hat{x}}_j &= A\hat{x}_j + \phi\left(\hat{\eta}_j, \hat{x}_j, u_j, t - d_x + j\frac{d_x}{m}\right) \\
&\quad - \theta\Delta^{-1}K\left[C\hat{x}_j\left(t - \frac{d_x}{m}\right) - \hat{y}_{j-1}\right] \,, \\
\hat{y}_j &= C\hat{x}_j \,, \\
&\vdots \\
\dot{\hat{x}}_m &= A\hat{x}_m + \phi\left(\hat{\eta}_m, \hat{x}_m, u, t\right) \\
&\quad - \theta\Delta^{-1}K\left[C\hat{x}_m\left(t - \frac{d_x}{m}\right) - \hat{y}_{m-1}\right] \,, \\
\hat{y}_m &= C\hat{x}_m \,,
\end{aligned} \tag{30}$$

where the vector $K \in \mathbb{R}^l$ is chosen such that the matrix $A - KC$ is Hurwitz, $\theta > 0$ is the high gain, and

$$\Delta = \operatorname{diag}\left\{1, \ldots, \frac{1}{\theta^{i-1}}, \ldots, \frac{1}{\theta^{l-1}}\right\} \,. \tag{31}$$

Remark 3. The vector $\hat{x}_j(t)$ is an estimate of the delayed state $x_j(t) := x\left(t - d_x + j\frac{d_x}{m}\right)$. The state vector $\hat{x}(t) := \hat{x}_m(t)$ is an estimate of the current state $x(t)$ of the time-delay subsystem (2)–(3).

4.2 Analysis of High-gain Observers Connected in Cascade

After showing the convergence of the state estimated by a single stage observer for a small delay d_1, Ahmed-Ali et al. [2012] had also proved that a sufficient number of cascade high-gain observers estimate the state of the system (2)–(3), for arbitrary long constant time delay. This is stated in [Ahmed-Ali et al., 2012, Theorem 1], rewritten below.

Lemma 4. Consider the system described in (2)–(3). Then, for any constant and known delay d_x, there exist a sufficiently large positive constant θ and an integer m such that the state estimated by the last observer $\hat{x}_m$ in (30) converges exponentially to the state x of the system (2)–(3). Moreover, the state estimation errors $\tilde{x}_j(t) := x_j(t) - \hat{x}_j(t)$, $\forall j \in \{1, \ldots, m\}$, converge exponentially to zero.

5. OUTPUT-FEEDBACK SLIDING MODE CONTROL

Here, we propose the sliding mode controller which applies the observer (12) for the state delayed subsystem (1) and high-gain cascade observers (30) for the output delayed subsystem (2)–(3) discussed above.

To design the output-feedback SMC, consider the system (2)–(3). Then, substituting u by $u + K_m x/k_p(t) - K_m x/k_p(t)$, one has

$$\dot{x} = A_m x + B\left[k_p(t)u + w(\eta, x, t) - K_m x\right], \tag{32}$$
$$y = Cx(t - d_x), \tag{33}$$

where $x(t-d_x) = [y, \dot{y}, \ldots, y^{(l-1)}]^T$ and $A_m := A + BK_m$. The matrix $K_m := -[a_0, \ldots, a_{l-1}]$ is chosen such that the characteristic polynomial

$$p(s) = s^l + a_{l-1}s^{l-1} + \ldots + a_0, \tag{34}$$

of the matrix A_m is Hurwitz. The state equation (32) can be rewritten as

$$\dot{x} = A_m x + Bk_p(t)\left[u + w_u(\eta, x, t)\right], \tag{35}$$

where the equivalent input disturbance is given by

$$w_u(\eta, x, t) = \left[w(\eta, x, t) - K_m x\right]/k_p(t). \tag{36}$$

If the full-state vector x were available for feedback, one could choose $\sigma = Sx = 0$ as the ideal sliding surface. Here, since only the output signal y is available for feedback, the sliding surface can be chosen as [Cunha et al., 2009]

$$\hat{\sigma} := S\hat{x} = 0, \qquad S := [b_0 \ \cdots \ b_{l-2} \ 1], \tag{37}$$

where $b_0, \ldots, b_{l-2}$ are coefficients of the polynomial

$$p_S(s) = s^{l-1} + b_{l-2}s^{l-2} + \ldots + b_0, \tag{38}$$

which is chosen such that the transfer function $p_S(s)/p(s) = S(sI - A_m)^{-1}B$ is Strictly Positive Real (SPR) [Khalil, 2002, Section 6.3]. The signal $\hat{x} := \hat{x}_m$ is the estimate of the system state x via observer (30).

The proposed control law u is

$$u = -\text{sgn}(k_p)\varrho(\hat{\eta}, \hat{x}, t)\text{sgn}(\hat{\sigma}(t)), \tag{39}$$

where the modulation function $\varrho(\hat{\eta}, \hat{x}, t)$ is a non-negative scalar function absolutely continuous in $\hat{\eta}$ and $\hat{x}$, piecewise continuous and bounded in t for each fixed $\hat{\eta}, \hat{x}$. It will be shown that, if the inequality

$$\varrho(\hat{\eta}, \hat{x}, t) \geq |w_u(\eta, x, t)| + \delta \tag{40}$$

is satisfied *modulo* exponentially decaying terms, then global stabilization can be guaranteed. The parameter $\delta \geq 0$ is an arbitrary constant. For instance, recalling assumptions (A5) and (A6) (eq. (11)), a function which satisfies (40) is

$$\varrho(\hat{\eta}, \hat{x}, t) = \left[|K_m\hat{x}| + k_\eta\|\hat{\eta}\| + k_x\|\hat{x}\| + \varphi_w(t)\right]/\underline{k}_p + \delta. \tag{41}$$

6. CLOSED-LOOP STABILITY ANALYSIS

The following lemma shows that the system (35)–(36) with state vector x and output signal $\tilde{\sigma} := S\tilde{x}$ is OSS, where the state estimation error is defined as $\tilde{x} := x - \hat{x}$. Therefore, according to next lemma, it will be demonstrated that if $\|\tilde{x}(t)\| \to 0$, then $\|x(t)\| \to 0$.

Lemma 5. ***(OSS property from $S\tilde{x}$ to x)*** Consider the dynamic system (35)–(36) with state vector x, output signal $\tilde{\sigma} = S\tilde{x} = Sx - S\hat{x}$, and the control law u (39) with modulation function ϱ (41). Then, (35)–(36) is OSS with respect to $\tilde{\sigma}$ and $\exists k_e > 0$ such that the following inequality holds

$$\|x(t)\| \leq k_e|S\tilde{x}(t)| + \pi_e(t), \quad \forall t \geq 0, \tag{42}$$

where $\pi_e(t)$ is an exponentially decaying term dependent on the initial conditions.

Proof. In the following, k_i's denote appropriate positive constants and π_i's denote exponentially decaying terms which depend on the initial conditions of the closed-loop system. Introduce the transformation

$$\bar{x} = Tx, \quad T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ b_0 & b_1 & \cdots & b_{l-2} & 1 \end{bmatrix}, \tag{43}$$

and the auxiliary signal $\sigma := Sx$ as in (37), such that the system (35) can be represented in the normal form [Khalil, 2002, Section 13.2] with state $\bar{x}(t - d_x) = [y(t), \dot{y}(t), \ldots, y^{(n-2)}(t), \sigma(x(t-d_x))]^T$. Since the triple $\{A_m, B, S\}$ (see (37) and (38)) is SPR, one can conclude that (35) is OSS from the output σ to the state $\bar{x}$. Therefore, $\bar{x}$ satisfies $\|\bar{x}\| \leq k_1|\sigma| + \pi_1(t)$. Since from (43), $\|x\| \leq \|T^{-1}\|\,\|\bar{x}\|$, one has

$$\|x\| \leq k_2|Sx| + \pi_2(t). \tag{44}$$

In what follows, two cases will be considered: $|Sx| > |S\tilde{x}|$ or $|Sx| \leq |S\tilde{x}|$. In the first case, $|Sx| > |S\tilde{x}|$ implies $\text{sgn}(\hat{\sigma}) = \text{sgn}(Sx)$. From the Kalman-Yakubovich Lemma [Khalil, 2002, Sec. 6.3], there exist $P = P^T > 0$ and $Q = Q^T > 0$ matrices which satisfy $A_m^T P + PA_m = -Q$ and $B^T P = S$. Using a Lyapunov function candidate $V = x^T Px$, one concludes that the time derivative of V along the solutions in (35) satisfies $\dot{V} \leq -\lambda_{\min}(Q)\|x\|^2 - 2k_p(t)|B^T Px|[\varrho - |w_u|]$, or equivalently,

$$\dot{V} \leq -\lambda_{\min}(Q)\|x\|^2 - 2|Sx|\left[\underline{k}_p\varrho - |k_p(t)w_u|\right], \tag{45}$$

where $\underline{k}_p \leq |k_p(t)|$ (see (A5)). Thus, considering that the modulation function ϱ in (41) verifies the inequality (40), one has $\dot{V} \leq -\lambda_{\min}(Q)\|x\|^2$. Then, one can conclude that $x \to 0$ and $Sx \to 0$ exponentially, for any initial condition, which in conjunction with the second case $|Sx| \leq |S\tilde{x}|$, leads to the conclusion that $|Sx| \leq |S\tilde{x}| + \pi_3$. Then applying the last inequality in (44), one concludes that $\|x\| \leq k_2|S\tilde{x}| + \pi_4(t)$. Thus, the dynamics which governs x is OSS with respect to $S\tilde{x}$, according to (42). □

In order to present the main result for global stability, the state of the closed-loop system (including plant and observers) is defined as

$$z(t) := [\eta^T(t)\ x^T(t)\ \hat{\eta}^T(t)\ \hat{x}_1^T(t) \cdots \hat{x}_m^T(t)]^T. \tag{46}$$

Theorem 6. ***(Global Asymptotic Stability)*** Consider the nonlinear system with multiple time delays given in (1)–(3), the control law (39) with modulation function (41), and the observers (12) and (30). If assumptions (A1)–(A6) and inequality (23) hold, then the equilibrium point $z = 0$ of the closed-loop system is globally asymptotically stable and all signals are uniformly bounded.

Proof. From Lemma 4, the estimated states $\hat{x}_j(t)$ converge exponentially to the states $x_j(t)$ and, consequently, the estimation errors $\tilde{x}_j(t)$, $\forall j \in \{1,\ldots,m\}$, converge exponentially to zero, i.e.,

$$\|\tilde{x}(t)\| \to 0\,, \quad \text{as} \quad t \to +\infty\,. \tag{47}$$

According to inequality (42) in Lemma 5, the norm of the state x is bounded by the estimation error norm plus an exponentially decaying term $\pi_e(t)$. Therefore, using (47) in (42) and reminding that $\hat{x} = x - \tilde{x}$, one can conclude that $x(t)$ and $\hat{x}(t)$ tend exponentially to zero. This implies that all the delayed estimated states $\hat{x}_j(t)$ and the output signal y tend exponentially to zero. From inequality (22), $\tilde{\eta}(t)$ tends exponentially to zero. From Remark 2, $\eta(t)$ and $\hat{\eta}(t)$ tend to zero asymptotically. Consequently, $\exists \beta_z \in \mathcal{KL}$ such that the state can be bounded by

$$\|z(t)\| \leq \beta_z(z_0^*, t)\,, \quad \forall t \geq 0\,, \tag{48}$$

where $z_0^* = \sup_{t\in[-d_{\max},0]} \|z_0(t)\|$ and $z(t) = z_0(t)$, $t \in [-d_{\max}, 0]$. Since the estimated states $\hat{\eta}$ and $\hat{x}$ tend to zero and the function $\varphi_w(t)$ is uniformly bounded in view of assumption (A6), the modulation function given by (41) is uniformly bounded. This implies the boundedness of the control signal u and all closed-loop system signals. □

Corollary 7. ***(Global Exponential Stability)*** In addition to the assumptions in Theorem 6, if $\exists \mu_2 \geq 0$ such that $\|f_2(y,t)\| \leq \mu_2 \|y\|$, $\forall y, t$, then the equilibrium point $z = 0$ of the closed-loop system is globally exponentially stable.

Proof. The proof follows the same steps in Theorem 6. Since y decays exponentially to zero, the signal $f_2(y,t)$ is also exponentially decaying. Therefore, from the OSS property of the η-dynamics, the state vectors $\eta(t)$ and $\hat{\eta}(t)$ tend to zero exponentially, completing the proof. □

Corollary 8. ***(Ideal Sliding Mode)*** In addition to the assumptions in Theorem 6, if $\delta > 0$ in (41), then the sliding surface $\hat{\sigma}(t) \equiv 0$ is reached in finite time.

Proof. The last observer in (30) can be rewritten as

$$\dot{\hat{x}} = Bk_p(t)\,[u + w(\hat{\eta}, \hat{x}, t)] + \zeta\,, \tag{49}$$

with

$$\zeta = A\hat{x} - \theta\Delta^{-1}K\left[C\hat{x}\left(t - \frac{d}{m}\right) - C\hat{x}_{m-1}\right]. \tag{50}$$

Now, consider the quadratic function $V_\sigma = \hat{\sigma}^2/2$. Then, calculating $\dot{V}_\sigma$ along the solution $\hat{x}(t)$ of (49),

$$\dot{V}_\sigma = \hat{\sigma}\dot{\hat{\sigma}} = \hat{\sigma}S\dot{\hat{x}} = \hat{\sigma}S\left[Bk_p\left(u + w(\hat{\eta}, \hat{x}, t)\right) + \zeta\right]\,. \tag{51}$$

Since $SB = 1$ and the control signal is given by (39), the function $\dot{V}_\sigma$ can be rewritten as

$$\begin{aligned}\dot{V}_\sigma &= \hat{\sigma}\left[k_p\left(-\text{sgn}(k_p)\varrho\,\text{sgn}(\hat{\sigma}) + w(\hat{\eta}, \hat{x}, t)\right) + S\zeta\right] \\ &\leq |\hat{\sigma}|\,|k_p|\left[-\varrho + |w(\hat{\eta}, \hat{x}, t)| + |k_p^{-1}S\zeta|\right]\,. \end{aligned} \tag{52}$$

Since the modulation function ϱ satisfies (40), the following inequality is valid $\dot{V}_\sigma \leq |\hat{\sigma}|\,|k_p|\left[-\delta + |k_p^{-1}S\zeta|\right]$. Note that, according to Theorem 6, x and $\hat{x}$ tend to zero. Then, $\exists T_1 > 0$ such that $|k_p^{-1}S\zeta| \leq \delta_1$, $\forall t \geq T_1$, with some constant $0 < \delta_1 < \delta$. Therefore, $\dot{V}_\sigma \leq |\hat{\sigma}|\,|k_p|\,[-\delta + \delta_1] < 0$, and the condition $\hat{\sigma}\dot{\hat{\sigma}} < 0$ for the existence of a sliding mode in some finite time is verified [Utkin et al., 1999, Sec. 2.5]. □

Remark 9. The asymptotic observer (30) can eliminate chattering, even in the presence of the time delay, since the ideal sliding mode occurs in an auxiliary observer loop rather than in the main control loop, as described by [Utkin et al., 1999, Sec. 8.3] and [Cunha et al., 2009].

7. SIMULATION RESULTS

Consider the following time-delay system [1]:

$$\dot{\eta} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\eta + \begin{bmatrix} -0.5\eta_1 \sin\eta_2 + 0.2\eta_{d1}\tanh(\eta_{d2}) + y^2 \\ 0.5\eta_2\cos\eta_1 + 0.2(\eta_{d2} - \eta_{d1}) - y^3 \end{bmatrix}, \tag{53}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_p(t)u(t) + w(\eta, x, t)]\,, \tag{54}$$

$$y = [1\ 0]\,x(t - d_x)\,, \tag{55}$$

where $k_p(t) \equiv 1$, $w(\eta, x, t) = 0.5x^1 + 0.5\tanh(x^2) + 2\|\eta\|$, η_d is defined in (4), the state vectors are $x = [x^1, x^2]^T$ and $\eta = [\eta_1, \eta_2]^T$. The nonlinear terms in η and in η_d presented in (53) satisfy (9) with $\mu_0 = 0.5$ and $\mu_1 = 0.2$. The time-varying state delay is given by $d_\eta(t) = 0.3\sin(5t) + 0.5$ (sec.). The upper bound $\bar{d} = 0.8$ s is required to design the observer for the zero dynamics state (see eq. (12)) in order to satisfy the condition (23). To satisfy (21), (23) and (24), $\gamma_0 = 1$ rad/s, $c_1 = 1$, and $\lambda_0 = 0.0549$ rad/s.

The parameters of the cascade observers (30) are set to $\theta = 4$ and $K = [0.85, 0.24]^T$. Since the considered output delay is $d_x = 0.4$ s, then one observer would not be sufficient to estimate the state of system (54). In this case, the number of observers connected in cascade is $m = 2$.

To design the control law (39), the matrix $S = [1, 1]$ is chosen to define the sliding surface as in (37). The modulation function ϱ is given by (41), where $K_m = [-2, -3]$. The function $\varphi_w(t) = 0.6$ and the constants $k_w = 0.6$, $k_\eta = 2.1$ were chosen to satisfy (11) in (A6), i.e., $|w(\eta, x, t)| \leq 2.1\,\|\eta\| + 0.6\,\|x\| + 0.6$. The constant $\underline{k}_p = 0.9$ was chosen to satisfy (A5). The parameter $\delta = 0.1$ guarantees the existence of the sliding mode in view of Corollary 8.

The simulation results in Figs. 1 and 2 show the convergence of the state (x, η) to the origin as expected. The estimation errors ($\tilde{x}$, $\tilde{\eta}$) converge to zero, as can be concluded from Fig. 1. The actual state η of the subsystem (53) and the delayed state η_d shown in Fig. 2 illustrate the distortion due to the time-varying delay $d_\eta(t)$.

8. CONCLUSIONS

An output-feedback sliding-mode controller was developed for a class of nonlinear systems with arbitrary constant time delay in the output signal. Time varying delay in part of the state variables is also allowed. Based on a combination of an asymptotic observer and high-gain cascade observers to estimate the state of the system, the proposed sliding mode control strategy guaranteed global asymptotic stability of the closed-loop system and the ideal sliding mode can be achieved in finite time. To the best of our knowledge, the proposed approach is a new sliding mode controller for nonlinear systems with multiple time delays and the results attained are relevant since output delayed systems are more difficult to control via output feedback than those with delayed states only.

[1] Following the notation in [Ahmed-Ali et al., 2012], x^1 and x^2 denote the elements of the state vector x (see Remark 3).

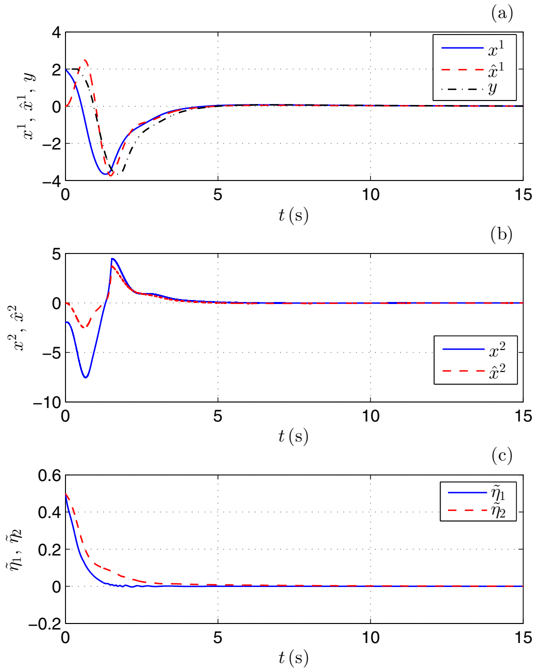

Fig. 1. Actual states (x^1 (a), x^2 (b)), estimated states ($\hat{x}^1$ (a), $\hat{x}^2$ (b)) and delayed output y (a). Estimation errors ($\tilde{\eta}_1$, $\tilde{\eta}_2$) for the zero dynamics state (c).

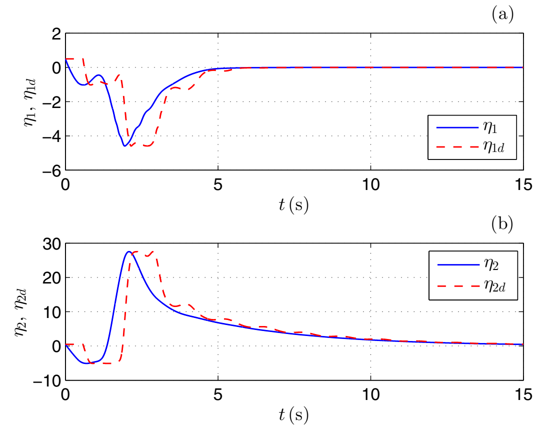

Fig. 2. Actual (η_1, η_2) and delayed (η_{d1}, η_{d2}) states.

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