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Stability and Stabilization of Differential Nonlinear Repetitive Processes with Applications ⋆

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Abstract: Repetitive processes are a class of two-dimensional systems that arise in the modeling of physical examples and also the control systems theory developed for them has, in the case of linear dynamics, been applied to design iterative learning control laws with experimental verification. This paper gives new results on the stability of nonlinear differential repetitive processes for applications where a linearized model is either very limited or not applicable. The stability results are then applied to the design of iterative learning control laws in the presence of uncertain parameters and to the same problem when random failures occur that are modeled by a homogeneous Markov chain with a finite set of states. In both cases the computations required are expressed as a finite set of linear matrix inequalities.

Keywords: Nonlinear repetitive processes, random failures, stability, iterative learning control, uncertain parameters

1. INTRODUCTION

Many industrial processes make a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length (Rogers et al., 2007). Once each pass is complete the process resets to the starting location ready for the start of the next pass. The output on each pass is termed the pass profile and acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile.

An industrial example described in (Rogers et al., 2007), with references to the original modeling work, is long-wall coal cutting where the pass profile is the height of the stone/coal interface above some datum line and the objective is to extract the maximum amount of coal without penetrating the stone/coal boundary. The cutting machine rests on the most recently produced pass profile during the production of the next pass profile and therefore this is an industrial repetitive process. The unique control problem for these processes is oscillations in the pass profiles generated that increase in amplitude from pass-to-pass.

If these oscillations occur in a particular mining operation then productive work must halt in order to enable their manual removal. The alternative is to use control action to prevent their occurrence but the stabilization problem for these processes cannot be solved using standard (or 1D) systems theory/algorithms. In particular, to apply standard control laws it is necessary to ignore their inherent 2D systems structure, i.e., information propagation occurs from pass-to-pass and along a given pass respectively and the initial conditions are reset before the start of each new pass.

To remove these deficiencies, a rigorous stability theory has been developed (Rogers et al., 2007) based on an abstract model of the dynamics in a Banach space setting that includes a very large number of processes with linear dynamics and a constant pass length as special cases. The existence of this theory has also led to the emergence of problem areas where using a repetitive process setting for analysis has advantages. An example is classes of Iterative Learning Control (ILC) laws where experimental verification has been reported (Hładowski et al., 2010). Another area is the analysis of OL-Nash games with a gas pipeline application (Azevedo-Perdicoulis and Jank, 2012).

The literature on the control of repetitive processes and other classes of 2D systems, is very largely based on a linear model of the dynamics. Comparatively much less work has been reported on the stability of nonlinear multidi-

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mensional systems, see, e.g., Pakshin et al. (2011b); Kurek (2012); Yeganefar et al. (2013) and references therein. Further support for the development of a stability and stabilization theory for nonlinear repetitive processes is supplied by examples such as ILC applied to bead morphology in laser metal deposition processes (Sammons et al., 2013).

This paper begins by developing new results on the stability of differential nonlinear repetitive processes using vector Lyapunov functions. These are then used to develop new stability results for processes where failures in operation can occur, which are modeled as random switching. In particular, the failures are modeled by a state-space model with jumps in the parameter values and/or structure governed by a Markov chain with a finite set of states, often termed Markovian jump systems or systems with random structure (Mariton, 1990; Kats and Martynyk, 2002). Finally, application to ILC design under parameter uncertainty and information, or sensor, failures is considered. All of the new results in this paper are Linear Matrix Inequality (LMI) based.

2. EXPONENTIAL STABILITY OF NONLINEAR DIFFERENTIAL REPETITIVE PROCESSES

This paper considers differential nonlinear repetitive processes with pass length $T < \infty$ described over $0 \le t \le T$ by the state-space model

$$\begin{aligned} \dot{x}_{k+1}(t) &= f_1(x_{k+1}(t), y_k(t), t), \\ y_{k+1}(t) &= f_2(x_{k+1}(t), y_k(t), t), \end{aligned} \tag{1}$$

where on pass k $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ pass profile vector and f_1 and f_2 are nonlinear functions such that $f_1(0,0,t) = 0$ and $f_2(0,0,t) = 0$. The boundary conditions, i.e, the pass state initial vector sequence and the initial pass profile are assumed to be known and of the form

$$\begin{array}{lcl} x_{k+1}(0) &=& d_{k+1},\ k \ge 0, \\ y_0(t) &=& f(t),\ 0 \le t \le T\ , y_0(t) = 0,\ t > T, \end{array} \tag{2}$$

where the entries in the $n \times 1$ vector d_{k+1} are known constants, $f(t)$ is an $m \times 1$ vector whose entries are known functions of t, $0 \le t \le T$. Moreover, if $|q|$ denotes the norm of a vector q, it is assumed that $f(t)$ and d_{k+1} satisfy

$$|f(t)|^2 \le M_f,\ |d_{k+1}|^2 \le \kappa_d z_d^k,\ k = 0, 1, ... \tag{3}$$

where $M_f > 0$ is a finite scalar and $0 < z_d < 1$ determines the rate of convergence of the pass state initial vector sequence.

Note 1. All references to the boundary conditions from this point onwards will assume that they satisfy (3).

Note 2. The stability theory for repetitive processes is defined in terms of the pass profile and the model and the results of this and the next section extend directly to the case when a current pass input is present.

Note 3. For ease of notation, the pass subscript k is omitted where the meaning is obvious.

Define norm of the pass profile vector as

$$||y_k|| = \sqrt{\int_0^T |y_k(t)|^2 dt}. \tag{4}$$

Definition 1. A differential nonlinear repetitive process described by (1) and (2) is said to be pass profile exponentially stable if

$$||y_k|| \le \kappa z^k,\ \ 0 < z < 1, \tag{5}$$

where κ depends on the pass length T and z, in general, depends on z_d.

The links between this definition of stability for nonlinear differential repetitive processes and that for their linear counterparts (Rogers et al., 2007) is given in Section 4.

To obtain conditions for pass profile exponential stability of a process described by (1) and (2), a vector Lyapunov function approach is used with candidate function

$$V(x, y) = \begin{bmatrix} V_1(x_{k+1}(t)) \\ V_2(y_k(t)) \end{bmatrix}, \tag{6}$$

where $V_1(x) > 0,\ x \ne 0,\ \ V_2(y) > 0,\ y \ne 0,\ \ V_1(0) = 0,\ V_2(0) = 0$ and the divergence operator of this function along the trajectories of (1) is defined as

$$\mathrm{div}V(x_{k+1}(t), y_k(t)) = \frac{dV_1(x_{k+1}(t))}{dt} + \Delta_k V_2(y_k(t)), \tag{7}$$

where $\Delta_k V_2(y_k(t)) = V_2(y_{k+1}(t)) - V_2(y_k(t))$.

Theorem 2. Consider a differential nonlinear repetitive process described by (1) and (2). Then pass profile exponential stability holds if there exist positive constants $c_1,\ c_2,\ c_3$ with $c_2 > c_3$ such that the function V of (6) and its divergence along the trajectories of (1) satisfy

$$c_1|x|^2 \le V_1(x) \le c_2|x|^2, \tag{8}$$

$$c_1|y|^2 \le V_2(y) \le c_2|y|^2, \tag{9}$$

$$\mathrm{div}V(x, y) \le -c_3(|x|^2 + |y|^2). \tag{10}$$

Proof. It follows from (8), (9) and (10) that

$$\begin{aligned} &\frac{dV_1(x_{k+1}(t))}{dt} + \lambda V_1(x_{k+1}(t)) \\ &+ V_2(y_{k+1}(t)) - \zeta V_2(y_k(t)) \le 0, \end{aligned} \tag{11}$$

where $\lambda = \frac{c_3}{c_2}$, $\zeta = 1 - \frac{c_3}{c_2} \in (0, 1)$. Solving inequality (11) with respect to $V_1(x_{k+1}(t))$ gives

$$\begin{aligned} V_1(x_{k+1}(t)) &\le V_1(x_{k+1}(0))\mathrm{e}^{-\lambda t} \\ &- \int_0^t \mathrm{e}^{-\lambda(t-s)}[V_2(y_{k+1}(s)) - \zeta V_2(y_k(s))]ds. \end{aligned} \tag{12}$$

Introducing

$$\begin{aligned} W_{k+1}(t) &= V_1(x_{k+1}(0))\mathrm{e}^{-\lambda t} - V_1(x_{k+1}(t)), \\ H_k(t) &= \int_0^t \mathrm{e}^{-\lambda(t-s)} V_2(y_k(s))ds. \end{aligned}$$

enables (12) to be rewritten as

$$H_{k+1}(t) \le \zeta H_k(t) + W_{k+1}(t). \tag{13}$$

Solving the inequality (13) gives

$$H_n(t) \le \zeta^n H_0(t) + \sum_{k=1}^{N} W_k(t)\zeta^{n-k} \tag{14}$$

or

$$\sum_{k=1}^{n} V_1(x_k(t))\zeta^{n-k} + \int_0^t \mathrm{e}^{-\lambda(t-s)} V_2(y_n(s))ds$$
$$\le \mathrm{e}^{-\lambda t} \sum_{k=1}^{n} V_1(x_k(0))\zeta^{n-k}$$
$$+\zeta^n \int_0^t \mathrm{e}^{-\lambda(t-s)} V_2(y_0(s))ds. \quad (15)$$

Also for the given boundary conditions

$$\mathrm{e}^{-\lambda t} \sum_{k=1}^{n} V_1(x_k(0))\zeta^{n-k} \le \mathrm{e}^{-\lambda t} \frac{c_2 \kappa_d}{1-\bar{\zeta}} \bar{\zeta}^n,$$

$$\zeta^n \int_0^t \mathrm{e}^{-\lambda(t-s)} V_2(y_0(s)) \le \zeta^n c_2 (1 - \mathrm{e}^{-\lambda t}) M_f,$$

where $\bar{\zeta} = \sqrt{\zeta_*}$, $\zeta_* = \max\{\zeta, z_d\}$, and from (15)

$$||y_n||^2 \le \tilde{\zeta}^n \frac{c_2}{c_1} \left(\frac{\kappa_d}{1-\bar{\zeta}} + (\mathrm{e}^{\lambda T} - 1) M_f \right),$$

where $\tilde{\zeta} = \max\{\zeta, \bar{\zeta}\}$. It follows immediately from the last inequality that (5) holds and the proof is complete.

3. STABILITY OF NONLINEAR DIFFERENTIAL REPETITIVE PROCESS WITH FAILURES

This section extends the results of the previous section to differential repetitive processes in the presence of failures. The failures are modeled by a state-space model with jumps in the parameter values and/or structure governed by a Markov chain with a finite set of states, often termed Markovian jump systems or systems with random structure (Mariton, 1990; Kats and Martynyk, 2002).

Results on the development of control theory for Markovian jump systems, which address issues such as stability, optimal and robust control problems, in the 1D case can be found in, e.g., (Costa et al., 2004) and the references therein. In (Gao et al., 2004; Wu et al., 2008) the results obtained for 1D Markovian jump systems are extended to investigate the problems of state feedback stabilization and H_∞ control of 2D discrete-time Markovian jump systems described by a Roesser model.

In (Pakshin et al., 2011a) a class of discrete linear repetitive processes with uncertain parameters and Markovian jumps was considered. A parametric description of pass profile based stabilizing control laws was developed using linear quadratic regular theory, which led to the development of LMI-based algorithms for the computation of the control law matrices. In (Pakshin et al., 2012) the stability theory for discrete linear repetitive processes developed in (Pakshin et al., 2011a) was used as the basis for the design of ILC laws for discrete linear systems with possible failures.

All of these previous results assume a discrete state-space model description of the dynamics, where for nonlinear dynamics discretization is somewhat more involved than in the linear case. Hence there is the need to consider differential nonlinear dynamics, which is the subject of this section, starting from the following state-space model

$$\begin{aligned} \dot{x}_{k+1}(t) &= \varphi_1(x_{k+1}(t), y_k(t), r(t)), \\ y_{k+1}(t) &= \varphi_2(x_{k+1}(t), y_k(t), r(t)), \end{aligned} \quad (16)$$

where $r(t)$ $(t \ge 0)$ is a Markov chain with discrete state-space $\mathbb{N} = \{1, \dots, \nu\}$ and transition probabilities are given by

$$\mathrm{P}(r(t+\tau) = j \mid r(t) = i)$$
$$= \begin{cases} \pi_{ij}\tau + o(\tau), & \text{if } j \ne i, \\ 1 + \pi_{ii}\tau + o(\tau), & \text{if } j = i, \end{cases} \quad (17)$$

$i, j = 1, \dots, \nu$, $\pi_{ij} > 0$, $\pi_{ii} = -\sum_{i \ne j}^{\nu} \pi_{ij}$ and φ_1 and φ_2 are nonlinear functions such that for all $r \in \mathbb{N}$ $\varphi_1(0,0,r) = 0$, $\varphi_2(0,0,r) = 0$. The rest of the notation is the same as in (1) and the boundary conditions are given by (2).

Take the norm of pass profile vector as

$$||y_k||_{\mathrm{E}} = \sqrt{\mathrm{E}[\int_0^T |y_k(t)|^2] dt}. \quad (18)$$

Then pass profile exponentially mean square (PPEM) stability of a differential repetitive process described by (16) and (2) is defined as follows.

Definition 3. A differential nonlinear repetitive process described by (16), (17) and (2) is said to be PPEM stable if there exist scalars $\kappa > 0$ and $0 < z < 1$ such that

$$||y_k||_{\mathrm{E}} \le \kappa z^k. \quad (19)$$

To obtain conditions for PPEM stability, consider the candidate Lyapunov vector function

$$V(x_{k+1}(t), y_k(t), r(t)) = \begin{bmatrix} V_1(x_{k+1}(t), r(t)) \\ V_2(y_k(t), r(t)) \end{bmatrix}, \quad (20)$$

where $V_1(x, r) > 0,\ x \ne 0,\ V_2(y, r) > 0,\ y \ne 0,\ V_1(0, r) = 0,\ V_2(0, r) = 0$

Introduce the operators $\mathcal{D}_1$ and $\mathcal{D}_2$ defined along the trajectories of system (16):

$$\mathcal{D}_1 V(\xi, \eta, i) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathrm{E}[V_1(x_{k+1}(t + \Delta t), r(t + \Delta t)$$
$$-V_1(x_{k+1}(t), r(t)) \mid x_{k+1}(t) = \xi, y_k(t) = \eta, r(t) = i],$$
$$\mathcal{D}_2 V(\xi, \eta, i) = \mathrm{E}[V_2(y_{k+1}(t), r(t)) - V_2(\eta_k, i) \mid x_{k+1}(t)$$
$$= \xi, y_k(t) = \eta, r(t) = i].$$

Also let $V_1(\xi, i)$ be differentiable in ξ for each $i \in \mathbb{N}$ and hence, using (16) and (17), it follows immediately that

$$\mathcal{D}_1 V(\xi, \eta, i) = \varphi_1^T(\xi, \eta, i) \frac{\partial V_1(\xi, i)}{\partial \xi} + \sum_{j=1}^{\nu} \pi_{i,j} V_1(\xi, j). \quad (21)$$

Define the operator $\mathcal{D}$ as stochastic counterpart of divergence operator of the previous section:

$$\mathcal{D}V(\xi, \eta, i) = \mathcal{D}_1 V(\xi, \eta, i) + \mathcal{D}_2 V(\xi, \eta, i)) \quad (22)$$

and the following theorem can be established.

Theorem 4. Consider a differential nonlinear repetitive process described by (16), (17) and (2) and suppose that there exist positive constants c_1, c_2, c_3 with $c_2 > c_3$ such that the function V along the trajectories of (16) and (17) satisfies the inequalities

$$c_1|\xi|^2 \le V_1(\xi, i) \le c_2|\xi|^2, \quad (23)$$
$$c_1|\eta|^2 \le V_2(\eta, i) \le c_2|\xi|^2, \quad (24)$$
$$\mathcal{D}V(\xi, \eta, i) \le -c_3(|\xi|^2 + |\eta|^2), \quad (25)$$

$i \in \mathbb{N}$. Then PPEM stability holds.

Proof. By the definition of the operator $\mathcal{D}_1$

$$\begin{aligned}&\mathrm{E}[V_1(x_{k+1}(t), r(t)) - V_1(x_{k+1}(0), r(0))]\\&= \int_0^t \mathrm{E}[\mathcal{D}_1 V(x_{k+1}(\tau), y_k(\tau), r(\tau))]d\tau.\end{aligned} \tag{26}$$

Also rewrite (26) in differential form as

$$\frac{d}{dt}\mathrm{E}[V_1(x_{k+1}(t), r(t))] = \mathrm{E}[\mathcal{D}_1 V(x_{k+1}(t), y_k(t), r(t))]$$

and it follows from (25), given (23) and (24), that

$$\begin{aligned}&\frac{d}{dt}\mathrm{E}[V_1(x_{k+1}(t), r(t))] + \lambda \mathrm{E}[V_1(x_{k+1}(t), r(t))]\\&+\mathrm{E}[V_2(y_{k+1}(t), r(t))] - \zeta \mathrm{E}[V_2(y_k(t), r(t))] \le 0,\end{aligned} \tag{27}$$

where $\lambda = \frac{c_3}{c_2}$, $\zeta = 1 - \frac{c_3}{c_2} \in (0, 1)$. Solving the inequality (27) with respect to $\mathrm{E}[V_1(x_{k+1}(t), r(t))]$ gives

$$\begin{aligned}\mathrm{E}[V_1(x_{k+1}(t), r(t))] &\le \mathrm{E}[V_1(x_{k+1}(0), r(0))\mathrm{e}^{-\lambda t}]\\&- \int_0^t \mathrm{e}^{-\lambda(t-s)}\mathrm{E}[V_2(y_{k+1}(s), r(s))]\\&-\zeta \mathrm{E} V_2(y_k(s), r(s))]ds.\end{aligned} \tag{28}$$

Introducing the notation

$$\begin{aligned}W_{k+1}(t) &= \mathrm{E}[V_1(x_{k+1}(0), r(0))\mathrm{e}^{-\lambda t}] - \mathrm{E}[V_1(x_{k+1}(t), r(t))],\\ H_k(t) &= \int_0^t \mathrm{e}^{-\lambda(t-s)}\mathrm{E}[V_2(y_k(s), r(s))]ds.\end{aligned}$$

enables (28) to be rewritten in the form (13) and the rest of the proof is the same as for the theorem of the previous section with obvious changes of notation.

4. ITERATIVE LEARNING CONTROL OF UNCERTAIN LINEAR CONTINUOUS-TIME SYSTEMS

In this section the stability results of the previous two sections are applied to ILC law design for linear systems described by the state-space model

$$\begin{aligned}\dot{x}(t) &= A(\delta)x(t) + B(\delta)u(t),\\ y(t) &= Cx(t),\end{aligned} \tag{29}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector and $\delta \in \mathbb{R}^N$ is the vector of uncertain time-invariant parameters. The uncertainty associated with the dynamics is assumed to be of the affine parallelotopic type with variations around a central nominal model defined by the matrices (A, B) along the axes (A_j, B_j) of the form

$$A(\delta) = A + \sum_{j=1}^N \delta_j A_j, \quad B(\delta) = B + \sum_{j=1}^N \delta_j B_j. \tag{30}$$

where N is the dimension of the uncertainty vector. Each δ_j in (29) and (30) is assumed to be bounded in an interval

$$\underline{\delta}_j \le \delta_j \le \overline{\delta}_j. \tag{31}$$

The set of uncertainties is denoted by $\boldsymbol{\Delta}$ and the finite set of extremal values, or vertices, is

$$\boldsymbol{\Delta}_v = \left\{ \ \delta = (\ \delta_1 \ \dots \ \delta_N \ ) : \delta_j \in \{\underline{\delta}_j, \overline{\delta}_j\} \ \right\}. \tag{32}$$

To formulate the ILC problem, let the integer k denote the pass, termed trial in most of the ILC literature, number and $u_k(t), x_k(t)$ and $y_k(t)$ the input, state and output vectors, respectively, at instant $0 \le t \le T < \infty$, where T denotes the pass length. Then the dynamics of the uncontrolled system are described by

$$\begin{aligned}\dot{x}_k(t) &= A(\delta)x_k(t) + B(\delta)u_k(t),\\ y_k(t) &= Cx_k(t).\end{aligned} \tag{33}$$

with assumed boundary conditions

$$y_0(t) = 0, \ 0 \le t \le T, \ x_k(0) = x_0, \ k = 0, 1, \dots \tag{34}$$

Also let $y_{ref}(t)$ denote the supplied reference vector over $0 \le t \le T$, where each entry in $y_{ref}(t)$ is assumed to be differentiable. Then $e_k(t) = y_{ref}(t) - y_k(t)$ is the error on pass k and the objective of constructing a sequence of input functions such that the performance achieved is gradually improving with each successive pass can be expressed as a convergence condition on the input and error, i.e.,

$$\lim_{k\to\infty} |e_k(t)| = 0, \ \lim_{k\to\infty} |u_k(t) - u_\infty(t)| = 0. \tag{35}$$

A commonly used ILC law is to select the input on the current pass as that used on the previous pass plus a correction. In this work the ILC law on pass $k+1$ is of the form

$$u_{k+1}(t) = u_k(t) + \Delta u_{k+1}(t), \tag{36}$$

where $\Delta u_{k+1}(t)$ is the correction term to be designed. The novel feature of ILC is all information generated on a completed pass is available for use in the computation of $\Delta u_{k+1}(t)$. This allows the use of temporal information that is non-causal in the standard sense provided it is generated and stored from a previous pass.

To write the ILC dynamics as a differential linear repetitive process, introduce, for analysis purposes only, the vector

$$\dot{\upsilon}_{k+1}(t) = x_{k+1}(t) - x_k(t), \tag{37}$$

and also

$$\begin{aligned}e_{k+1}(t) - e_k(t) &= -CA(\delta)\int_0^t (x_{k+1}(\tau) - x_k(\tau))d\tau\\&-CB(\delta)\int_0^t (u_{k+1}(\tau) - u_k(\tau))d\tau.\end{aligned} \tag{38}$$

Then the ILC dynamics can be written as a differential linear repetitive process with uncertainty of the form

$$\begin{aligned}\dot{\upsilon}_{k+1}(t) &= A(\delta)\upsilon_{k+1}(t) + B(\delta)\int_0^t \Delta u_{k+1}(\tau)d\tau,\\ e_{k+1}(t) &= -CA(\delta)\upsilon_{k+1}(t) + e_k(t)\\&- CB(\delta)\int_0^t \Delta u_{k+1}(\tau)d\tau.\end{aligned} \tag{39}$$

Consider the case when

$$\Delta u_{k+1}(t) = F_1\dot{\upsilon}_{k+1}(t) + F_2\dot{e}_k(t). \tag{40}$$

Then if (40) guarantees pass profile exponential stability of (39), it follows from Theorem 2 that the ILC law is convergent in the sense of (35). Also the result of applying (40) to (39) can be written as

$$\begin{aligned}\dot{\upsilon}_{k+1} &= [A(\delta) + B(\delta)F_1]\upsilon_{k+1}(t) \\ &\quad + B(\delta)F_2 e_k(t), \\ e_{k+1}(t) &= [-CA(\delta) - CB(\delta)F_1]\upsilon_{k+1}(t) \\ &\quad + [I - CB(\delta)F_2]e_k.\end{aligned} \tag{41}$$

and to construct stabilizing control law matrices F_1 and F_2 Theorem 2 is used.

Chose the candidate vector Lyapunov function as (6) with $V_1(\upsilon_{k+1}(t)) = \upsilon_{k+1}^T(t)P_1\upsilon_{k+1}(t)$, $V_2(e_k(t)) = e_k^T(t)P_2e_k(t)$, where P_1 and P_2 are compatibly dimensioned symmetric positive definite matrices, denoted by > 0 from this point onwards. Moreover, the divergence of the function (6) must satisfy (10). Calculating the divergence of this function along the trajectories of (41) gives the following sufficient conditions for pass profile exponential stability

$$\begin{aligned}&P_1 > 0,\ P_2 > 0,\ A_c^T(\delta)P^{1,0} + P^{1,0}A_c(\delta) \\ &+A_c^T(\delta)P^{0,1}A_c(\delta) - P^{0,1} < 0,\ \delta \in \mathbf{\Delta},\end{aligned} \tag{42}$$

where $P^{1,0} = \mathrm{diag}\{P_1\ 0,\ P^{0,1}\} = \mathrm{diag}\{0\ P_2\},\ A_c(\delta)$

$$= \begin{bmatrix} A(\delta) + B(\delta)F_1 & B(\delta)F_2 \\ -CA(\delta) - CB(\delta)F_1 & I - CB(\delta)F_2 \end{bmatrix}.$$

If the value of δ is fixed, these conditions are the same those for the stability along the pass property of differential linear repetitive processes (Rogers et al., 2007). Also (42) can be reduced to finite set of LMI's by first using the Schur's complement formula to rewrite (42) in the form

$$\begin{bmatrix} A_{c1}^T(\delta)P + PA_{c1}(\delta) - P^{0,1} & A_{c2}^T(\delta)P \\ PA_{c2}(\delta) & -P \end{bmatrix} < 0,\ P > 0,\ \delta \in \mathbf{\Delta},$$

where $P = \mathrm{diag}\{P_1\ P_2\}$,

$$A_{c2}(\delta) = \begin{bmatrix} 0 & 0 \\ -CA(\delta) - CB(\delta)F_1 & I - CB(\delta)F_2 \end{bmatrix},$$

$$A_{c1}(\delta) = \begin{bmatrix} A(\delta) + B(\delta)F_1 & B(\delta)F_2 \\ 0 & 0 \end{bmatrix}.$$

Define $X_1 = P_1^{-1},\ X_2 = P_2^{-1},\ Y_1 = F_1X_1$ and $Y_2 = F_2X_2$. Then routine calculations give the following LMI with respect to these new variables

$$\begin{bmatrix} D_{11}(\delta) & D_{12}(\delta) & 0 & D_{14}(\delta) \\ D_{12}(\delta)^T & -X_2 & 0 & D_{24}(\delta) \\ 0 & 0 & -X_1 & 0 \\ D_{14}(\delta)^T & D_{24}(\delta)^T & 0 & -X_2 \end{bmatrix} < 0. \tag{43}$$

$$X_1 > 0,\ X_2 > 0,\ \delta \in \mathbf{\Delta},$$

where $D_{11}(\delta) = A(\delta)X_1 + B(\delta)Y_1 + (A(\delta)X_1 + B(\delta)Y_1)^T$, $D_{12}(\delta) = B(\delta)Y_2,\ D_{14}(\delta) = [-CA(\delta)X_1 - CB(\delta)Y_1]^T$, $D_{24}(\delta) = (X_2 - CB(\delta)Y_2)^T$.

Theorem 5. Suppose that an ILC law of the form (36) and (40) is applied to a system described by (33) and (34). Suppose also that the LMI's of (43) with $\delta \in \mathbf{\Delta}$ are feasible. Then the resulting system with $F_1 = Y_1X_1^{-1}$ and $F_2 = Y_2X_2^{-1}$ satisfies the ILC convergence conditions of (35).

Introduce

$$\vartheta_{k+1}(t) = x_{k+1}(t) - x_k(t) \tag{44}$$

and suppose that $\Delta u_{k+1}(t)$ in the ILC law (36) is replaced by

$$\Delta u_{k+1}(t) = K_1\vartheta_{k+1}(t) + K_2e_k(t), \tag{45}$$

Then such a control law avoids the need to work with derivative information and hence the possibility of noise corruption is reduced. Also using (36) and (44) the resulting controlled dynamics can be written as

$$\begin{aligned}\dot{\vartheta}_{k+1} &= [A(\delta) + B(\delta)K_1]\vartheta_{k+1}(t) + B(\delta)K_2e_k(t), \\ e_{k+1}(t) &= -C\vartheta_{k+1}(t) + e_k,\end{aligned} \tag{46}$$

where

$$A_c(\delta) = \begin{bmatrix} A(\delta) + B(\delta)K_1 & B(\delta)K_2 \\ -C & I \end{bmatrix}.$$

This system is not stable along the pass (Rogers et al., 2007) and also the weaker property of asymptotic stability does not hold (this property requires that all eigenvalues of the bottom right sub-matrix have modulus strictly less than unity) Hence ILC error convergence cannot be achieved, but if the update law is formed using (40) Theorem 5 guarantees ILC convergence.

5. ITERATIVE LEARNING CONTROL OF UNCERTAIN LINEAR SYSTEMS WITH SENSOR FAILURES

Consider the system (29) under possible information or sensor failures. In this case the output equation in this state-space model becomes

$$y(t) = C(r(t))x(t), \tag{47}$$

where $r(t)$ is a Markov chain with a finite set of states $\mathbb{N} = \{1, \ldots, \nu\}$ corresponding to the number of possible failures with transition probabilities given by

$$\mathrm{P}[r(t+1) = j | r(t) = i] = \pi_{ij}. \tag{48}$$

The dynamics of a system obtained by applying an ILC law of the form (36) to a system with state dynamics described by (29) and output equation (47), under the conditions and notation previously defined, are described by

$$\begin{aligned}\dot{x}_k(t) &= A(\delta)x_k(t) + B(\delta)u_k(t), \\ y_k(t) &= C(r(t))x_k(t).\end{aligned} \tag{49}$$

Moreover, the stochastic nature of $r(t)$ requires the following modified definition of ILC convergence.

Definition 6. A system described by (49) is said to be convergent if for all $0 \le t \le T$

$$\mathrm{E}[|e_k(t)|^2] = \mathrm{E}[|y_{ref}(t) - y_k(t)|^2] \to 0,\ k \to \infty \tag{50}$$

and

$$\mathrm{E}[|u_k(t) - u_\infty(t)|^2] \to 0,\ k \to \infty. \tag{51}$$

Using (37) and (38) the dynamics with the ILC law applied can be written as

$$\dot{\upsilon}_{k+1}(t) = A(\delta)\upsilon_{k+1}(t) + B(\delta)\Delta u_{k+1}(t), \tag{52}$$

$$\begin{aligned}e_{k+1}(t) &= -C(r(t))A(\delta)\upsilon_{k+1}(t) + e_k(t) \\ &\quad - C(r(t))B(\delta)\Delta u_{k+1}(t).\end{aligned} \tag{53}$$

Consider also the case when

$$\Delta u_{k+1}(t) = F_1(i)\dot{\upsilon}_{k+1}(t) + F_2(i)\dot{e}_k(t),\ \text{if } r(t) = i. \tag{54}$$

Then if (54) guarantees PPEM stability of (52) it follows from Theorem 4 that this ILC law is convergent.

To construct stabilizing control law matrices $F_1(i)$ and $F_2(i)$, $i \in \mathbb{N}$, the stability conditions of Theorem 4 are employed. Chose the candidate vector Lyapunov function as (20) with $V_1(\upsilon_{k+1}(t), r(t)) = \upsilon_{k+1}^T(t) P_1(r(t)) \upsilon_{k+1}(t)$, $V_2(e_k(t), r(t)) = e_k^T(t) P_2(r(t)) e_k(t)$, with $P_1 > 0,\ P_2 > 0$. Also the stochastic divergence operator $\mathcal{D}$ of the function (20) in this case must satisfy (25). Calculating this operator along the trajectories of a system described by (52) and (54) gives the following sufficient conditions for PPEM stability

$$
\begin{aligned}
&P(i) = \mathrm{diag}\{P_1(i)\ P_2(i)\} > 0,\ A_{c1}^T(\delta, i)P(i) + \\
&P(i)A_{c1}(\delta, i) + \sum_{j=1}^{\nu} \pi_{ij} I^{1,0} P(j) I^{1,0} - I^{0,1} P(i) I^{0,1} \\
&\qquad + A_{c2}^T(\delta, i) P(i) A_{c2}(\delta, i) < 0,\ i \in \mathbb{N},\ \delta \in \mathbf{\Delta}, \qquad (55)
\end{aligned}
$$

where

$$
I^{1,0} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\quad I^{0,1} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.
$$

On setting $X(i) = P^{-1}(i)$, $Y(i) = F_1(i) X_1(i)$, $Y_2(i) = F_2(i) X_2(i)$, routine calculations and convexity properties give the following coupled set of LMI with respect to these variables

$$
\begin{bmatrix} S_{11}(\delta, i) & S_{12}(\delta, i) & S_{13}(i) \\ S_{12}^T(\delta, i) & -X(i) & 0 \\ S_{13}^T(i) & 0 & S_{33}(i) \end{bmatrix} < 0,
$$

$$
X(i) > 0,\ \delta \in \mathbf{\Delta}_{\upsilon},\ i \in \mathbb{N}, \qquad (56)
$$

where $S_{11}(\delta, i) = \begin{bmatrix} A_{c11}(\delta, i) & B(\delta)Y_1(i) \\ (B(\delta)Y_1(i))^T & -X_2(i) \end{bmatrix}$, $S_{12}(\delta, i) = \begin{bmatrix} 0 & 0 \\ A_{c12}(\delta, i) & A_{c22}(\delta, i) \end{bmatrix}^T$, $A_{c11}(\delta, i) = A(\delta)X(i) + B(\delta)Y_1(i) + (A(\delta)X(i) + B(\delta)Y_1(i))^T + \pi_{ii} X_1(i)$, $A_{c12}(\delta, i) = -C(i)A(\delta)X_1(i) - C(i)B(\delta)Y_1(i)$, $A_{c22}(\delta, i) = X_2(i) - C(i)B(\delta)Y_2(i)$, $S_{13}(i) = [\pi_{i1}^{\frac{1}{2}} X(i) I^{1,0} \ldots \pi_{i\,i-1}^{\frac{1}{2}} X(i) I^{1,0}\ \pi_{i\,i+1}^{\frac{1}{2}} X(i) I^{1,0} \ldots \pi_{i\nu}^{\frac{1}{2}} X(i) I^{1,0}]$, $S_{33}(i) = \mathrm{diag}[-X(1) \ldots - X(i-1)\ - X(i+1) \ldots - X(\nu)]$ and the following result has been established.

Theorem 7. Consider the ILC dynamics described by (49) and (54) and suppose that the LMI's (56) with $\delta \in \mathbf{\Delta}_{\upsilon}$, $i \in \mathbb{N}$, are feasible and set $F_1(i) = Y_1(i) X_1^{-1}(i)$ and $F_2(i) = Y_2(i) X_2^{-1}(i)$, $i \in \mathbb{N}$. Then ILC convergence occurs.

6. CONCLUSIONS

The vast majority of the existing control and systems theory for repetitive processes, a distinct class of 2D linear systems with applications areas and control problems that cannot be solved by either 1D systems theory or that for other classes of 2D systems, assumes linear dynamics. This paper has produced new results on the stability of differential nonlinear repetitive processes with potential applications areas. To demonstrate their role for the latter, they have been applied to ILC design, including the case when failures may arise. These results provide a basis for further research to fully exploit their potential.

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