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An Overview on the Modeling of Oilwell Drilling Vibrations

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Abstract: In drilling operations, the drillstring interaction with the borehole gives rise to a wide variety of undesired oscillations. The main types of drilling vibrations are torsional (stick-slip), axial (bit-bounce) and lateral (whirling). The analysis and modeling of rotary drilling vibrations is a topic whose economical interest has been renewed by recent oilfields discoveries leading to a growing literature. This paper summarizes the most popular modeling strategies allowing the oscillatory behavior analysis of the physical system.

Keywords: Mechanical vibrations, drilling system, stick-slip, bit-bounce, friction models.

1. INTRODUCTION

The presence of drillstring vibrations is the main cause of loss of performance in the perforation process for oil and gas. It provokes premature wear and tear of drilling equipment resulting in fatigue and induced failures such as pipe wash-out and twist-off [22]. It also cause significant wastage of drilling energy [21] and may induce wellbore instabilities reducing the directional control and its overall shape [10]. In the oil industry, the improvement of drilling performance is a matter of crucial economical interest.

Many studies have been conducted to identify and recognize the different types of vibrations during drilling operation. These have led to the identification and classification of vibrations into three separate and distinctive categories namely torsional (stick-slip oscillations), axial (bit-bouncing phenomenon) and lateral (whirl motion due to the out-of-balance of the drillstring). Below is a brief description of each one of them.

- Torsional vibration. Downhole measurements show that applying a constant rotary speed at the surface does not necessarily translate into a steady rotational motion of the bit. In fact, the downhole torsional speed typically exhibits large amplitude fluctuations during a significant fraction of the drilling time. This self-excited rotational motion, also known as stick-slip, is induced by the nonlinear relationship between the torque and the angular velocity at the bit [16]. The torsional flexibility of the drilling assembly exacerbates a non-uniform oscillatory behavior causing rotational speeds as high as ten times the nominal rotary table speed or a total standstill of the bit [33]. Torsional vibrations provoke fatigue to drill collar connections, damages the drill bit and slow down the drilling operation thereby prolonging the overall drilling process. They are detectable at the drillfloor by fluctuations in the power needed to maintain a constant rate of surface rotation.

- Axial vibration. This vibration mode consists of irregular movements of the drilling components along its longitudinal axis causing bit-bounce and rough drilling behavior that destroys the drill bit, damages the Bottom Hole Assembly (BHA) and increases total drilling time. Additionally, due to downhole coupling mechanisms, it also excites lateral displacements of the string [32]. The bit-bounce pattern may be detected at the surface, it is likely to develop when drilling with a bit of roller-cone type, also called tricone or rock bit, consisting of multiple lobes which leads an erratic interaction of the bit with the bottom of the well making the bit to loose contact with the rock formation.

- Lateral vibration. One of the most destructive drillstring oscillations is the whirling phenomenon, since it may be unleashed with no indication at the surface. Deep in the hole, the rotating BHA interacts with the borehole wall generating shocks from lateral vibrations. The collisions with the borehole wall will produce eccentric hole and the shocks can damage components of the BHA [23]. The lateral oscillations of the drillstring cause severe damage to the borehole wall and affect the overall drilling direction [15]. Drill collars whirling are simply the centrifugally induced bowing of the drill collar resulting from rotation. If the center of gravity of the drill collar is not initially located precisely on the centerline of the hole, then as the collar rotates, a centrifugal force acts at the center of gravity causing the collar to bend [37]. Forward and backward whirling behaviors can further intensify due to the combined effect of fluid damping, stabilizer clearance, and friction of the drilling assembly against the borehole wall [36].

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Currently, drilling of deepwater wells for oil and gas production have opened new horizon for petroleum engineers and experts to try to mitigate the influence of vibration during drilling operation. Even though new technology has been deployed, such phenomena still occurs, significantly affecting on drilling costs and daily operations. Before the 1960s, studies were focused on material strength of the drillstring components but the trends have since changed to emphasize on its dynamic behavior [15].

The great practical significance of oilwell drillstrings has attracted the attention of many researchers. In order to reduce the costs of failures, extensive research effort has been conducted in the last five decades to suppress the drillstring vibrations, several methodologies have been proposed, both from practical and theoretical viewpoints, see for example the references cited in the survey [19].

This paper compiles the main modeling techniques reproducing axial and torsional vibrations in a vertical oil well drilling system.

2. DRILLING SYSTEM DESCRIPTION

The main process during well drilling for oil is the creation of borehole by a rock-cutting tool called a bit. The drillstring consists of the BHA and drillpipes screwed end to end to each other to form a long pipe. The BHA comprises the cutting device (bit), stabilizers (at least two spaced apart) which prevent the drillstring from unbalancing, and a series of pipe sections which are relatively heavy known as drill collars. While the length of the BHA remains constant, the total length of the drill pipes increases as the borehole depth does. An important element of the process is the drilling mud or fluid which among others, has the function of cleaning, cooling and lubricating the bit. The drillstring is rotated from the surface by an electrical motor located at the rotary table.

Drilling system complexity brings out an important modelling problem. An appropriate model must accurately describe the most important phenomena arising in real wells and has to be simple enough for analysis and control purposes.

3. TORSIONAL DYNAMICS

In general, the models that have been used in literature to describe drillstring systems can be classified into two main categories: distributed parameter models and lumped parameter ones.

To keep the analysis simple, many contributions consider the drillstring as a torsional pendulum described by a lumped parameter model with one or multiple Degrees of Freedom (DOF). In [14] and [20] a one DOF model is considered, in [7], [16], [24] and [34], a two DOF model is proposed, in [25], a discontinuous model of four DOF is introduced.

Figure 1 shows the simplified torsional model of a conventional vertical drillstring proposed in [24]. J_r and J_b are the inertias of the top rotary system and the BHA, respectively. The inertias are connected one to each other by a linear spring with torsional stiffness k and torsional damping c. The equations of motion are:

$$J_r\ddot{\Phi}_r + c\left(\dot{\Phi}_r - \dot{\Phi}_b\right) + k\left(\Phi_r - \Phi_b\right) = T_m - T_d(\dot{\Phi}_r),$$
$$J_b\ddot{\Phi}_b - c\left(\dot{\Phi}_r - \dot{\Phi}_b\right) - k\left(\Phi_r - \Phi_b\right) = -T_b(\dot{\Phi}_b),$$

where Φ_r and Φ_b are the angular displacements of the rotary table and the BHA respectively, T_m is the drive torque coming from the rotary table transmission box which is driven by a DC electric motor, and it may be considered as $T_m = k_m u$ with k_m the motor parameter and u the input to the system, T_d and T_b are the friction torques associated with J_r and J_b, respectively. T_b represents the torque-on-bit (TOB) and the nonlinear frictional forces along the drill collars.

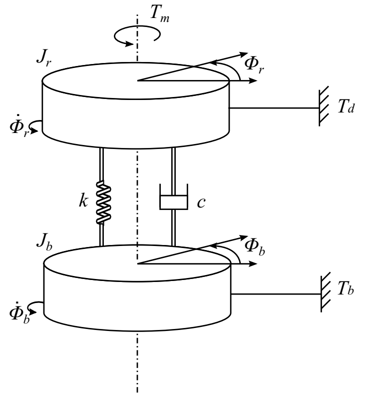

Figure 1. Lumped parameter model of a rotary drilling rig.

Lumped parameters models have been shown to be easy enough for the analysis, but in some cases, the distributed nature of the system cannot be neglected. It is natural to think that a distributed parameter model of the system is more suitable to properly describe the oscillation phenomena.

The wave equation is widely used to reproduce the oscillatory behavior of physical systems. In the 1960s, some analyses have been carried out using the classical wave equation to describe the torsional behavior of drillstring assemblies [2], [11]. This modeling strategy was also adopted in [8] and [35].

The purely torsional excitations of a drillstring of length L, described by the rotary angle $\Phi(\xi, t)$ can be modeled by the wave equation [4], [5], [8], [13], [35]:

$$GJ\frac{\partial^2\Phi}{\partial\xi^2}(\xi,t) - I\frac{\partial^2\Phi}{\partial t^2}(\xi,t) - \beta\frac{\partial\Phi}{\partial t}(\xi,t) = 0,\ \xi \in (0, L), \quad (1)$$

with the twist angle Φ depending on the length coordinate ξ and time t. The parameters I, G and J are the inertia, the shear modulus and the geometrical moment of inertia respectively. A damping $\beta \geq 0$ which includes the viscous and structural damping, is assumed along the structure.

The boundary conditions are chosen according to the dynamics taking place at the upper and lower ends of the drill string. In [8], the following boundary conditions are considered:

$$\Phi(0,t) = \Omega t, \quad (2)$$

$$GJ\frac{\partial\Phi}{\partial\xi}(L,t) + I_B\frac{\partial^2\Phi}{\partial t^2}(L,t) = -T\left(\frac{\partial\Phi}{\partial t}(L,t)\right). \quad (3)$$

A lumped inertia I_B is chosen to represent the assembly at the bottom hole. It is assumed that the speed at the surface ($\xi = 0$) is restricted to a constant value Ω, the other extremity ($\xi = L$), which symbolizes the bit, being subject to a torque T, which is a function of the bit speed.

The boundary condition (3) satisfactorily reproduces the behavior at the ground level, however (2) does not reflect an important dynamic occurring at the top extremity. The angular velocity coming from the rotor Ω does not match the rotational speed of the load $\frac{\partial \Phi}{\partial t}(0,t)$, this sliding speed results in the local torsion of the drillstring. In order to take into account this phenomenon, the following boundary condition is considered in [30] and [31]:

$$GJ\frac{\partial \Phi}{\partial \xi}(0,t) = c_a\left(\frac{\partial \Phi}{\partial t}(0,t) - \Omega(t)\right). \quad (4)$$

Alternative boundary conditions to represent the drilling behavior at the upper extremity are derived from the classical Newton equation such as the following one introduced in [9]:

$$J_T\frac{\partial^2 \Phi}{\partial t^2}(0,t) + \Theta(t) = -T_{top}(t) \quad (5)$$

where J_T is the effective moment of inertia of the top-drive, the forcing function $\Theta(t)$ denotes the external torque delivered by the top-drive, taken as control input with the following feedback structure:

$$\Theta(t) = \kappa_p\left(\frac{\partial \Phi}{\partial t}(0,t) - \Omega_0\right) + \kappa_i\left(\Phi(0,t) - \Omega_0 t\right)$$

where κ_p and κ_i are the rotary speed control parameters and Ω_0 is the target angular speed. The function T_{top}, describes the transmitted torque and damping due to viscous effects, it is given by:

$$T_{top} = k\frac{\partial \Phi}{\partial \xi}(0,t) + \beta_v\frac{\partial \Phi}{\partial t}(0,t),$$

where $k > 0$ is the spring constant of the drillstring regarded as a torsional spring and $\beta_v > 0$ is a viscous damping constant.

Normally, when a distributed parameter model is subject to nonlinearities and uncertainties (in our case study, those arising from the rock-bit interaction), analysis and simulations are not easy tasks. However, it is possible to derive a simpler model involving only the variables of main interest. By means of a direct transformation, an input-output model described by a neutral-type time-delay equation which clearly simplifies the analysis and simulations, is obtained. The procedure allowing to transform the PDE model to a delay system of neutral type was presented for the first time in [1], see also [4], [5], [12] and [30].

Integration along characteristics of the hyperbolic PDE allows the association of certain system of functional differential equations to the mixed problem, more precisely, a one-to-one correspondence may be established and proved between the solutions of the mixed problem for hyperbolic PDE and the initial value problem for the associated system of functional equations [26], [27].

By reducing the boundary value problem to a neutral-type time-delay equation, we are able to exploit techniques from dynamic systems theory to gain insight into the complexity involved in the analysis and simulation of infinite dimensional systems.

Since most of the energy dissipation in drilling systems is taking place at the bit-rock interface, we consider that the damping along the structure β is negligible. The distributed parameter model (1) then reduces to the unidimensional wave equation:

$$\frac{\partial^2 \Phi}{\partial \xi^2}(\xi,t) = p^2\frac{\partial^2 \Phi}{\partial t^2}(\xi,t), \quad \xi \in (0,L), \quad (6)$$

where $p = \sqrt{\frac{I}{GJ}}$. A direct transformation of the wave equation model (6) with boundary conditions (3)-(4) allows to obtain the following neutral-type time-delay model describing the drilling behavior [30]:

$$\begin{aligned}&\ddot{\Phi}_b(t) - \Upsilon\ddot{\Phi}_b(t-2\Gamma) = -\Psi\dot{\Phi}_b(t) - \Upsilon\Psi\dot{\Phi}_b(t-2\Gamma)\\ &-\frac{1}{I_B}T\left(\dot{\Phi}_b(t)\right) + \frac{1}{I_B}\Upsilon T\left(\dot{\Phi}_b(t-2\Gamma)\right) + \Pi\Omega(t-\Gamma),\end{aligned} \quad (7)$$

where $\dot{\Phi}_b$ is the angular velocity at the bottom extremity, and $\Pi = \frac{2\Psi c_a}{c_a+\sqrt{IGJ}}$, $\Upsilon = \frac{c_a-\sqrt{IGJ}}{c_a+\sqrt{IGJ}}$, $\Psi = \frac{\sqrt{IGJ}}{I_B}$, $\Gamma = \sqrt{\frac{I}{GJ}}L$.

In [30], the transformation techniques (D'Alembert and Laplace methods) allowing the simplified model derivation are explained in detail.

A similar reduction of analogous PDE boundary value problems to time-delay equations, and the techniques exploited here, have relevance to a broad range of other engineering, physical and biological problems. These include power transmission line networks [6], laser optical fibres, sonar/radar ranging technologies [3], cardiovascular system dynamics and many other applications.

3.1 Modeling of the rock-bit interaction

Torsional vibrations are characterized by stick phases, during which the rotation stops completely, and slip phases, during which the angular velocity of the tool increases up to more than twice the nominal angular velocity. This phenomenon occurs when a section of the rotating drillstring is momentarily caught by friction against the borehole, then released. The bit might eventually get stuck and then, after accumulating energy in terms of torsion, be suddenly released, the collar rotation speeds up dramatically and large centrifugal accelerations occur.

It is usually assumed that the growth of instabilities eventually leading to stick-slip oscillations arises from the friction model, which empirically captures the bit-rock interaction.

There are different modeling strategies to reproduce the bit-rock interaction, we present below some of the main ones:

- Velocity weakening law.

In [8], the model:

$$T(\dot{U},\dot{\Phi}_b) = \zeta\dot{U}e^{-\alpha\dot{\Phi}_b}$$

is chosen to represent the rock-bit interaction, $\dot{U}$, and $\dot{\Phi}_b$ stand for the axial and angular velocities at the bottom extremity and ζ denotes the ability of the rock to be cut.

- Stiction plus Coulomb friction.

In [16], [34], the torque on the bit is modeled with the expression:

$$T(\dot{\Phi}_b(t)) = c_b\dot{\Phi}_b(t) + T_{fb}(\dot{\Phi}_b(t))$$

where c_b is the damping viscous coefficient at the bit level and T_{fb} is the classical Coulomb plus static friction (dry friction) model, that is,

$$T_{fb}(\dot{\Phi}_b(t)) = \begin{cases} T_{cb}\text{sign}(\dot{\Phi}_b(t)) & \text{if } \dot{\Phi}_b(t) \neq 0 \\ |T_{fb}| \leq T_{sb} & \text{if } \dot{\Phi}_b(t) = 0 \end{cases} \quad (8)$$

with $T_{sb} = \mu_{sb}W_{ob}R_b$ and $T_{cb} = \mu_{cb}W_{ob}R_b$ the static and Coulomb friction torques, μ_{sb}, $\mu_{cb} \in (0, 1)$ the static and Coulomb friction coefficients, W_{ob} is the weight on the bit and R_b is the bit radius.

The use of this modeling strategy is explained in [34] as follows: the maximum torque T_{sb} clamping the bit to zero speed is substantially larger than the Coulomb friction T_{cb} experienced when the bit is rotating . If $\dot{\Phi}_b = 0$, the friction torque will adjust to the torque in the drillstring maintaining a static equilibrium of the bit (see Figure 2).

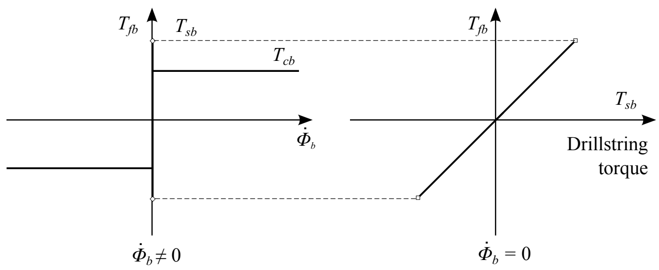

Figure 2. Coulomb and static friction.

- Dry friction plus Karnopp's model.

Another common model for $T_{f_b}(\dot{\Phi}_b(t))$ is defined below

$$T_{f_b}(\dot{\Phi}_b(t)) = \begin{cases} T_{eb} & \text{if } \left|\dot{\Phi}_b(t)\right| < D_v,\ |T_{eb}| \leq T_{sb} \\ & \text{(stick)} \\ T_{sb}\text{sign}(T_{eb}) & \text{if } \left|\dot{\Phi}_b(t)\right| < D_v,\ |T_{eb}| > T_{sb} \\ & \text{(stick-to-slip transition)} \\ T_{cb}\text{sign}(\dot{\Phi}_b(t)) & \text{if } \left|\dot{\Phi}_b(t)\right| \geq D_v, \\ & \text{(slip)} \end{cases}$$

where $D_v > 0$ specifies a small enough neighborhood of $\dot{\Phi}_b(t) = 0$ and T_{eb} is the applied external torque that must overcome the static friction torque T_{sb} to make the bit move, T_{eb} is modeled as follows:

$$T_{eb} = c\left(\dot{\Phi}_r - \dot{\Phi}_b\right) + k\left(\Phi_r - \Phi_b\right) - c_b\dot{\Phi}_b.$$

This model combines the dry friction model (8) with the Karnopp's model introduced in [17] and introduces a zero velocity band (see Figure 3).

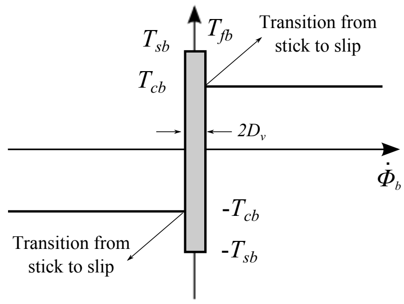

Figure 3. Dry friction plus Karnopp's model.

- Karnopp's model with a decaying friction term.

The function governing friction in the slip phase is chosen as a decaying function inspired by the experimental results given in [7]:

$$T_{f_b}(\dot{\Phi}_b(t)) = \begin{cases} \min\{|T_{eb}, T_{sb}|\}\text{sign}(T_{eb}) & \text{if } \left|\dot{\Phi}_b(t)\right| < D_v \\ f_b(\dot{\Phi}_b(t))\text{sign}(\dot{\Phi}_b(t)) & \text{if } \left|\dot{\Phi}_b(t)\right| \geq D_v \end{cases}$$

with

$$f_b(\dot{\Phi}_b(t)) = W_{ob}R_b\mu_b(\dot{\Phi}_b(t))$$
$$\mu_b(\dot{\Phi}_b(t)) = \frac{\mu_{sb} - \mu_{cb}}{1 + \gamma_b\left|\dot{\Phi}_b(t)\right|} + \mu_{cb}$$

where μ_b is the dry friction coefficient at the bit and γ_b is a positive constant defining the decaying velocity of T_{f_b} (see Figure 4).

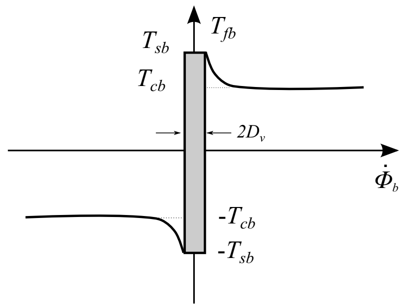

Figure 4. Karnopp's model with a decaying friction term.

- Karnopp's model with an exponential decaying friction term.

An alternative model for the torque on the bit defines an exponential decaying term in the slip phase [25]. It consider the expression for T_{f_b} defined above with:

$$f_b(\dot{\Phi}_b(t)) = W_{ob}R_b\mu_b(\dot{\Phi}_b(t))$$
$$\mu_b(\dot{\Phi}_b(t)) = \mu_{cb} + (\mu_{sb} - \mu_{cb})e^{-\gamma_b\left|\dot{\Phi}_b(t)\right|}.$$

Simulations results presented in [29] validate the proposed model.

- Simplified torque on bit model.

In [18] the following model is introduced:

$$T(\dot{\Phi}_b(t)) = \frac{2k\dot{\Phi}_b(t)}{\dot{\Phi}_b^2(t) + k^2}$$

where k is a positive parameter. This model is simpler than those presented above and effectively reproduces the behavior of the friction at the bit level (see Figure 5).

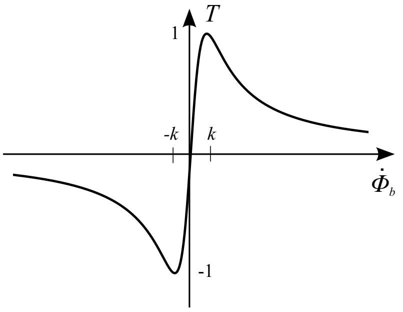

Figure 5. Simplified torque on bit model.

4. AXIAL-TORSIONAL COUPLED DYNAMICS

Axial and torsional vibrations are generally quite complex in nature. They are self-excited oscillations intimately coupled together and may occur simultaneously. For example, the high bit speed level caused by stick-slip torsional motion can excite severe axial vibrations in the BHA, which may cause bit-bounce, excessive bit wear and reduction in the penetration rate.

Axial and torsional excitations of a drillstring of length L, described by the longitudinal position $U(s,t)$ and the rotary angle $\Phi(s,t)$ can be modeled by a pair of coupled wave equations [4], [5], [18]:

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2}(s,t) &= c^2 \frac{\partial^2 U}{\partial s^2}(s,t), \quad c = \sqrt{\frac{E}{\rho}} \\ E\Gamma \frac{\partial U}{\partial s}(0,t) &= \alpha \frac{\partial U}{\partial t}(0,t) - H(t) \\ M\frac{\partial^2 U}{\partial t^2}(L,t) &= -E\Gamma \frac{\partial U}{\partial s}(L,t) - p\,F\left(\frac{\partial U}{\partial t}(L,t)\right) \end{aligned} \tag{9}$$

and

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial t^2}(s,t) &= \tilde{c}^2 \frac{\partial^2 \Phi}{\partial s^2}(s,t), \quad \tilde{c} = \sqrt{\frac{G}{\rho}} \\ G\Sigma \frac{\partial \Phi}{\partial s}(0,t) &= \beta \frac{\partial \Phi}{\partial t}(0,t) - \Omega(t) \\ J\frac{\partial^2 \Phi}{\partial t^2}(L,t) &= -G\Sigma \frac{\partial \Phi}{\partial s}(L,t) - \tilde{p}\,F\left(\frac{\partial U}{\partial t}(L,t)\right) \end{aligned} \tag{10}$$

where, in equation (9), $H(t)$ is the brake motor control and $\alpha\frac{\partial U}{\partial t}(L,t)$ represents a friction force of viscous type. For equation (10), the right hand side of the second equation designates the difference between the motor speed and rotational speed of the first pipe. The spatial variable s is chosen such that $s = 0$ denotes the top of the drillstring and $s = L$ its bottom. The physical parameters of the model (9)-(10) are: G is the shear modulus of the drillstring steel, E the elasticity Young's modulus, ρ is the density, M is the BHA mass, $J = Mr^2$ is the inertia, r is taken as the averaged radius of drillpipe, Γ is the drillstring's cross-section and Σ its second moment of area. The parameters p and $\tilde{p}$ together with the function F, appearing in the boundary conditions at the bottom, account for the friction resulting from the interaction between the drill bit and the rock.

An alternative modeling strategy is presented in [28], the following dymanical model allows to represent the coupled axial-torsional drilling vibrations:

$$\begin{aligned} I\frac{d^2\Phi}{dt^2} + C(\Phi - \Phi_0) &= T_0 - T\left({}^t_0\Phi, {}^t_0 U\right), \\ M\frac{d^2 U}{dt^2} &= W_0 - W\left({}^t_0\Phi, {}^t_0 U\right), \end{aligned} \tag{11}$$

where U and Φ stand for the vertical and angular positions of the bit, respectively. The mechanical elements representing the BHA are: M the point mass and I the moment of inertia; C is the spring stiffness representing the torsional stiffness of the drillpipe. In this modeling strategy a drag bit consisting of n identical radial blades regularly spaced by an angle equal to $2\pi/n$ is considered. When such a bit is drilling rock, the depth of cut per blade d_n (i.e., the thickness of the rock ridge in front of the blade) is constant along the blade and identical for each blade. Furthermore, d_n is related to the vertical position of the bit U according to $d_n(t) = U(t) - U(t - t_n)$, where t_n is the time required for the bit to rotate by an angle $2\pi/n$ to its current position at time t. The delay $t_n(t)$ is solution of $\Phi(t) - \Phi(t - t_n) = 2\pi/n$. The combined depth of cut of the bit is simply $d = nd_n$ or $d = n\,[U(t) - U(t - t_n)]$. The torque function T and the weight on bit W are functions of the history of Φ (denoted by ${}^t_0\Phi$) and the history of U (denoted by ${}^t_0 U$), indeed the cutting components of T and W are proportional to the depth of cut d. Φ_0, T_0 and W_0 are the stationary quantities associated with the trivial solution of (11). Notice that (11) constitutes a retarded dynamical model characterized by a state-dependent delay t_n.

The damped harmonic oscillator equation:

$$m_0\ddot{U} + c_0\dot{U} + k_0(U - ROPt) = -\mu_1 T\left(\dot{\Phi}_b(t)\right), \tag{12}$$

is another modeling technique to represent axial vibrations [8]. U, $\dot{U}$ and $\ddot{U}$ stand for the axial variables: position, velocity and acceleration respectively, ROP is the rate of penetration and m_0, c_0, and k_0 represent the mass, damping and spring constant, μ_1 is a constant depending on the bit geometry.

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