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Factorizing the Monodromy Matrix of Linear Periodic Systems

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Abstract: This note proposes a new approach to computing the Kalman canonical decomposition of finite-dimensional linear periodic continuous-time systems by extending the Floquet theory. Controllable and observable subspaces are characterized by factorizing the monodromy matrix. Then, the conditions for the existence of several periodic Kalman canonical decompositions are extensively studied. The relations to the Floquet factorization, the Floquet-like factorization, and the period-specific realization are also discussed.

Keywords: linear systems; time-varying systems; periodic structures; continuous time systems; controllability; observability; system analysis; decomposition methods; realization theory.

1. INTRODUCTION

In this note, a new approach to computing the periodic Kalman canonical decomposition is presented. Firstly, controllable and observable subspaces are characterized by factorizing the monodromy matrix into a four-by-four upper triangular matrix. Then, the conditions for the existence of several periodic Kalman canonical decompositions are extensively studied. In particular, a new necessary and sufficient condition for the existence of the periodic Kalman canonical decomposition with the same period of the given system is obtained. A key technical tool is computation of matrix logarithms of the upper triangular matrix. The condition is reduced to the positiveness of determinants of the block diagonal elements of the upper triangular matrix; therefore, the computational difficulty has been significantly reduced. The relations to the Floquet-like factorization [9, 10, 11] and the period-specific realization [16] are also discussed; the proposed computation algorithm simultaneously achieves the Floquet-like factorization as well as the periodic Kalman canonical decomposition and shares the common realization technique. Similar arguments are also discussed for the other types of decompositions. Finally, it is shown that it is always possible to construct a periodic coordinate transformation with the double period of the given periodic system. All proofs are omitted due to page limitations.

We use the following notations. $X := Y$ and $Y =: X$ denote that X is defined by Y. $\mathbb{R}$ (respectively, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$) denotes the set of all real numbers (respectively, complex numbers, integers, natural numbers). $\mathbb{R}^n$ denotes the set of all vectors whose elements consist of $\mathbb{R}$ with n-rows. $\mathbb{R}^{n\times m}$ denotes the set of all matrices whose elements consist of $\mathbb{R}$ with n-rows and m-columns. $0_{n\times m} \in \mathbb{R}^{n\times m}$ denotes the zero matrix. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. If the sizes are clear from the context, $0_{n\times m}$ and I_n are simply denoted by 0 and I, respectively. In the case of block matrices, zero matrix components might be omitted for notational simplicity. X^{T} denotes the transpose of $X \in \mathbb{K}^{n\times m}$. $\det X$ (respectively, X^{-1}) denotes the determinant (respectively, inverse) of a matrix $X \in \mathbb{R}^{n\times n}$. $X^{-\mathrm{T}} := (X^{-1})^{\mathrm{T}}$ denotes the transpose of a matrix X^{-1}. $\operatorname{Im} X := \{X\xi : \xi \in \mathbb{K}^m\}$ (respectively, $\operatorname{Ker} X = \{\xi \in \mathbb{K}^m : X\xi = 0\}$) denotes the image (respectively, kernel) of a matrix $X \in \mathbb{R}^{n\times m}$. $\dim \mathcal{V}$ denotes a dimension of a subspace $\mathcal{V} \subset \mathbb{R}^n$, i.e., a number of linearly independent vectors in $\mathcal{V}$. $X\mathcal{V} := \{Xv : v \in \mathcal{V}\}$ denotes a subspace for a matrix $X \in \mathbb{R}^{n\times m}$ and a subspace $\mathcal{V} \subset \mathbb{R}^m$. $\mathcal{V}^{\perp} := \{u : u^{\mathrm{T}}v = 0, \forall v \in \mathcal{V}\}$ denotes the annihilator of a subspace $\mathcal{V} \in \mathbb{R}^n$. $C^k(\mathbb{R}, \mathbb{R}^{n\times m})$ denotes the set of all C^k-functions, i.e., k-times continuously differentiable functions, from $\mathbb{R}$ to $\mathbb{R}^{n\times m}$. $C^k_{\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ denotes the set of all invertible functions in $C^k(\mathbb{R}, \mathbb{R}^{n\times n})$. If the function $P(t)$ is periodic with a period $T > 0$, i.e., $P(t+T) = P(t)$ for $t \in \mathbb{R}$, it is called T-periodic. The set of all T-periodic functions in $C^k(\mathbb{R}, \mathbb{R}^{n\times m})$ is denoted by $C^k_T(\mathbb{R}, \mathbb{R}^{n\times m})$. The set of all invertible T-periodic functions in $C^k(\mathbb{R}, \mathbb{R}^{n\times m})$ is denoted by $C^k_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times m})$.

2. PROBLEM FORMULATION

Consider a linear T-periodic system

$$\dot{x} = A(t)x + B(t)u, \quad \dot{x} := \frac{dx}{dt} \tag{1}$$

$$y = C(t)x \tag{2}$$

where $t \in \mathbb{R}$ is a time, $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is and input, and $y(t) \in \mathbb{R}^p$ is an output for certain nonnegative integers n, m, p. Matrix-valued functions $A \in C^0_T(\mathbb{R}, \mathbb{R}^{n\times n})$, $B \in C^0_T(\mathbb{R}, \mathbb{R}^{n\times m})$, $C \in C^0_T(\mathbb{R}, \mathbb{R}^{p\times n})$ denote the coefficient matrices. In this paper, A, B, and C are supposed to be continuous for simplicity.

An extension to more general class is possible by minor modifications. Let Φ denote the state transition matrix of (1) with $u = 0$, i.e., the unique solution of $\frac{\partial}{\partial s}\Phi(s,t) = A(s)\Phi(s,t)$, $\Phi(t,t) = 0$, $\forall s, t \in \mathbb{R}$.

Consider a kT-periodic coordinate transformation

$$\xi = Z(t)x \tag{3}$$

for certain integer k, where $Z \in C^1_{kT,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ is said to be a kT-periodic coordinate transformation matrix. The reason considering kT-periodic $Z(t)$ instead of T-periodic $Z(t)$ will be clear later. By applying (3), the system (1)-(2) are transformed to another linear kT-peirodic system

$$\dot{\xi} = \tilde{A}(t)\xi + \tilde{B}(t)u, \tag{4}$$

$$y = \tilde{C}(t)\xi, \tag{5}$$

where

$$\tilde{A}(t) := (\dot{Z}(t) + Z(t)A(t))Z(t)^{-1}, \tag{6}$$

$$\tilde{B}(t) := Z(t)B(t), \tag{7}$$

$$\tilde{C}(t) := C(t)Z(t)^{-1}. \tag{8}$$

This problem is to find a kT-periodic coordinate transformation $Z(t)$ such that the triplet $(\tilde{C}, \tilde{A}, \tilde{B})$ has a certain block structure.

Definition 1. A transformed linear kT-periodic system (4)-(5) and the triplet $(\tilde{C}, \tilde{A}, \tilde{B})$ are said to be a kT-periodic Kalman canonical decomposition of (1)-(2) if there exists a kT-periodic coordinate transformation matrix $Z \in C^1_{kT,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ satisfying the following three conditions:

(i) The triplet $(\tilde{C}, \tilde{A}, \tilde{B})$ takes on the following from

$$\tilde{A}(t) = \begin{bmatrix} \tilde{A}_{11}(t) & \tilde{A}_{12}(t) & \tilde{A}_{13}(t) & \tilde{A}_{14}(t) \\ 0 & \tilde{A}_{22}(t) & 0 & \tilde{A}_{24}(t) \\ 0 & 0 & \tilde{A}_{33}(t) & \tilde{A}_{34}(t) \\ 0 & 0 & 0 & \tilde{A}_{44}(t) \end{bmatrix}, \tag{9}$$

$$\tilde{B}(t) = \begin{bmatrix} \tilde{B}_1(t) \\ \tilde{B}_2(t) \\ 0 \\ 0 \end{bmatrix}, \tag{10}$$

$$\tilde{C}(t) = \begin{bmatrix} 0 & \tilde{C}_2(t) & 0 & \tilde{C}_4(t) \end{bmatrix}. \tag{11}$$

(ii) The pair $\left(\tilde{A}_c, \tilde{B}_c\right)$ is controllable, where

$$\tilde{A}_c(t) := \begin{bmatrix} \tilde{A}_{11}(t) & \tilde{A}_{12}(t) \\ 0 & \tilde{A}_{22}(t) \end{bmatrix}, \tag{12}$$

$$\tilde{B}_c(t) := \begin{bmatrix} \tilde{B}_1(t) \\ \tilde{B}_2(t) \end{bmatrix}. \tag{13}$$

(iii) The pair $\left(\tilde{C}_o, \tilde{A}_o\right)$ is observable, where

$$\tilde{C}_o(t) := \begin{bmatrix} \tilde{C}_2(t) & \tilde{C}_4(t) \end{bmatrix}, \tag{14}$$

$$\tilde{A}_o(t) := \begin{bmatrix} \tilde{A}_{22}(t) & \tilde{A}_{24}(t) \\ 0 & \tilde{A}_{44}(t) \end{bmatrix}. \tag{15}$$

3. STRUCTURE OF SUBSPACES

3.1 Controllable Subspace

A set of all controllable state at time t is denoted by

$$\mathcal{C}(t) := \bigcup_{p\in[t,\infty)} \left\{ \int_t^p \Phi(t,\tau)B(\tau)u(\tau)d\tau : u \in \mathbf{U} \right\},$$

where $\mathbf{U}$ denotes the set of all piecewise continuous control inputs on $[t, p]$. $\mathcal{C}(t)$ becomes a subspace of $\mathbb{R}^n$ at each time t and therefore is said to be a controllable subspace at time t [5]. $\mathcal{C}(t)$ satisfies the following conditions for linear T-periodic systems.

Lemma 2. [8, 15] (i) $\mathcal{C}(t)$ is given by

$$\mathcal{C}(t) = \operatorname{Im} W_c(t, t+nT),\ \forall t \in \mathbb{R}, \tag{16}$$

where W_c is the controllability Gramian given by

$$W_c(t,s) := \int_t^s \Phi(t,\tau)B(\tau)B(\tau)^{\mathrm{T}}\Phi(t,\tau)^{\mathrm{T}}d\tau. \tag{17}$$

(ii) $\mathcal{C}(t)$ is Φ-invariant, i.e.,

$$\mathcal{C}(t) = \Phi(t,s)\mathcal{C}(s),\ \forall t, s \in \mathbb{R}. \tag{18}$$

(iii) $\mathcal{C}(t)$ is T-periodic, i.e.,

$$\mathcal{C}(t) = \mathcal{C}(t+T),\ \forall t \in \mathbb{R}. \tag{19}$$

(iv) the dimension of $\mathcal{C}(t)$ is constant, i.e.,

$$\dim \mathcal{C}(t) = \dim \mathcal{C}(0),\ \forall t \in \mathbb{R}. \tag{20}$$

3.2 Observable Subspace

A set of all observable state at time t is denoted by

$$\mathcal{O}(t) := \bigcup_{p\in[t,\infty)} \left\{ \int_t^p \Phi(\tau,t)^{\mathrm{T}}C(\tau)^{\mathrm{T}}y(\tau)d\tau : y \in \mathbf{Y} \right\},$$

where $\mathbf{Y}$ denotes the set of all piecewise continuous control inputs on $[t, p]$. $\mathcal{O}(t)$ becomes a subspace of $\mathbb{R}^n$ at each time t and therefore is said to be a observable subspace at time t [5]. $\mathcal{O}(t)^{\perp}$ satisfies the following conditions for linear T-periodic systems.

Lemma 3. [15] (i) $\mathcal{O}(t)^{\perp}$ is given by

$$\mathcal{O}(t)^{\perp} = \operatorname{Ker} W_o(t, t+nT),\ \forall t \in \mathbb{R}, \tag{21}$$

where W_o is the observability Gramian given by

$$W_o(t,s) := \int_t^s \Phi(\tau,t)^{\mathrm{T}}C(\tau)^{\mathrm{T}}C(\tau)\Phi(\tau,t)d\tau. \tag{22}$$

(ii) $\mathcal{O}(t)^{\perp}$ is Φ-invariant, i.e.,

$$\mathcal{O}(t)^{\perp} = \Phi(t,s)\mathcal{O}(s)^{\perp},\ \forall t, s \in \mathbb{R}. \tag{23}$$

(iii) $\mathcal{O}^{\perp}(t)$ is T-periodic, i.e.,

$$\mathcal{O}(t) = \mathcal{O}(t+T),\ \forall t \in \mathbb{R}. \tag{24}$$

(iv) the dimension of $\mathcal{O}(t)^{\perp}$ is constant, i.e.,

$$\dim \mathcal{O}(t)^{\perp} = \dim \mathcal{O}(0)^{\perp},\ \forall t \in \mathbb{R}. \tag{25}$$

3.3 Intersection Subspace

Among four intersection subspaces, only $\mathcal{C}(t) \cap \mathcal{O}^{\perp}(t)$ is Φ-invariant as well as T-periodic.

Lemma 4. [15] (i) $\mathcal{C}(t) \cap \mathcal{O}(t)^{\perp}$ is Φ-invariant, i.e.,

$$\mathcal{C}(t) \cap \mathcal{O}(t)^{\perp} = \Phi(t,s)\{\mathcal{C}(s) \cap \mathcal{O}(s)^{\perp}\},\ \forall t, s \in \mathbb{R}. \tag{26}$$

(ii) $\mathcal{C}(t) \cap \mathcal{O}(t)^{\perp}$ is T-periodic, i.e.,

$$\mathcal{C}(t) \cap \mathcal{O}(t) = \mathcal{C}(t+T) \cap \mathcal{O}(t+T),\ \forall t \in \mathbb{R}. \tag{27}$$

(iii) the dimension of $\mathcal{C}(t) \cap \mathcal{O}(t)^{\perp}$ is constant, i.e.,

$$\dim\{\mathcal{C}(t) \cap \mathcal{O}(t)^{\perp}\} = \dim\{\mathcal{C}(0) \cap \mathcal{O}(0)^{\perp}\},\ \forall t \in \mathbb{R}. \tag{28}$$

From Lemma 2(iv), Lemma 3(iv), and Lemma 4(iii), the dimensions of all intersection subspaces are constant.

Lemma 5. [15]

$$\dim\{\mathcal{C}(t) \cap \mathcal{O}(t)^{\perp}\} =: n_1,\ \forall t \in \mathbb{R}, \tag{29}$$

$$\dim\{\mathcal{C}(t) \cap \mathcal{O}(t)\} =: n_2,\ \forall t \in \mathbb{R}, \tag{30}$$

$$\dim\{\mathcal{C}(t)^{\perp} \cap \mathcal{O}(t)^{\perp}\} =: n_3,\ \forall t \in \mathbb{R}, \tag{31}$$

$$\dim\{\mathcal{C}(t)^{\perp} \cap \mathcal{O}(t)\} =: n_4,\ \forall t \in \mathbb{R}, \tag{32}$$

where $n_1 + n_2 + n_3 + n_4 = n$.

The conditions (29)-(32) are the necessary and sufficient condition for the existence of the Kalman canonical decomposition for linear time-varying systems [12, 13]. However, the conditions (29)-(32) does not immediately imply the existence of the periodic Kalman canonical decomposition for linear periodic systems [8, 15].

The authors have presented a counterexample to the existence of a T-periodic Kalman canonical decomposition for a given linear T-periodic system [8, 15]. Then, the authors have shown that there is a $2T$-periodic Kalman canonical decomposition for the (A, B)-pair of linear T-periodic system [8]. The authors have shown that there is a $8T$-periodic Kalman canonical decomposition for the (C, A, B)-triple of linear T-periodic system [15]. It is still unclear whether there is a $2T$- or $4T$-periodic Kalman canonical decomposition for the (C, A, B)-triple of linear T-periodic system. We notice that the previous approach [8, 15] is a modification of Weiss [5] from linear time-varying systems to linear periodic systems. Only the periodicity of the system has been utilized to solve the block diagonalization of the controllability Gramian [8] (or the simultaneous block diagonalization of the controllability and the observability Gramians [15]), and therefore, the other detailed structure of the linear periodic system such as the Floquet theory has not been utilized in the previous approach.

This paper presents a new approach, which has been partially presented for the (A, B)-pair in [14], for computing the periodic Kalman canonical decomposition of the (C, A, B)-triple by extending the Floquest theory. Several new conditions for the existence of the periodic Kalman canonical decomposition will be presented. The relations to the Floquet factorization, the Floquet-like factorization and the period-specific realization will be also discussed.

3.4 Factorization of the Monodromy Matrix

The first step in the new approach is the construction of bases of Φ-invariant subspaces. They are collected in accordance with the dimensions of intersection subspaces.

Lemma 6. There exist an orthogonal matrix V whose block components are divided by $V_1 \in \mathbb{R}^{n\times n_1}$, $V_2 \in \mathbb{R}^{n\times n_2}$, $V_3 \in \mathbb{R}^{n\times n_3}$, $V_4 \in \mathbb{R}^{n\times n_4}$ as follows

$$V := [V_1\ V_2\ V_3\ V_4] \tag{33}$$

and satisfy the relations

$$\operatorname{Im} V_1 = \mathcal{C}(0) \cap \mathcal{O}(0)^{\perp}, \tag{34}$$

$$\operatorname{Im} [\ V_1\ V_2\ ] = \mathcal{C}(0), \tag{35}$$

$$\operatorname{Im} [\ V_1\ V_3\ ] = \mathcal{O}(0)^{\perp}, \tag{36}$$

$$\operatorname{Im} V = \mathbb{R}^n. \tag{37}$$

The monodromy matrix $\Phi(T, 0)$ is then transformed to a block upper triangular matrix.

Lemma 7. There exist invertible matrices $\Psi_{11} \in \mathbb{R}^{n_1\times n_1}$, $\Psi_{22} \in \mathbb{R}^{n_2\times n_2}$, $\Psi_{33} \in \mathbb{R}^{n_3\times n_3}$, $\Psi_{44} \in \mathbb{R}^{n_4\times n_4}$, and matrices $\Psi_{12} \in \mathbb{R}^{n_1\times n_2}$, $\Psi_{13} \in \mathbb{R}^{n_1\times n_3}$, $\Psi_{14} \in \mathbb{R}^{n_1\times n_4}$, $\Psi_{24} \in \mathbb{R}^{n_2\times n_4}$, $\Psi_{34} \in \mathbb{R}^{n_3\times n_4}$ such that

$$\Phi(T, 0)V = V \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ 0 & \Psi_{22} & 0 & \Psi_{24} \\ 0 & 0 & \Psi_{33} & \Psi_{34} \\ 0 & 0 & 0 & \Psi_{44} \end{bmatrix}, \tag{38}$$

where V is given in Lemma 6.

4. T-PERIODIC DECOMPOSITION

4.1 T-periodic decomposition with non-constant A-matrix

Let us recall the necessary and sufficient condition of the existence of T-periodic Kalman canonical decomposition.

Theorem 8. [15] Consider the linear T-periodic system (1)-(2). Then there exists a T-periodic coordinate transformation $Z \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ such that the transformed system (4)-(5) takes on the T-periodic Kalman canonical decomposition in $\mathbb{R}$ if and only if there exists a T-periodic $Z(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ satisfying

$$Z(t)W_c(t, t + nT)Z(t)^{\mathrm{T}} = \begin{bmatrix} \tilde{P}_{11}(t) & \tilde{P}_{12}(t) & & \\ \tilde{P}_{12}(t)^{\mathrm{T}} & \tilde{P}_{22}(t) & & \\ & & 0_{n_3\times n_3} & \\ & & & 0_{n_4\times n_4} \end{bmatrix}, \tag{39}$$

$$Z(t)^{-\mathrm{T}}W_o(t, t + nT)Z(t)^{-1} = \begin{bmatrix} 0_{n_1\times n_1} & & & \\ & \tilde{Q}_{22}(t) & & \tilde{Q}_{24}(t) \\ & & 0_{n_3\times n_3} & \\ & \tilde{Q}_{24}(t)^{\mathrm{T}} & & \tilde{Q}_{44}(t) \end{bmatrix}, \tag{40}$$

for certain matrix-valued functions $\tilde{P}_{11} \in C^1(\mathbb{R}, \mathbb{R}^{n_1\times n_1})$, $\tilde{P}_{12} \in C^1(\mathbb{R}, \mathbb{R}^{n_1\times n_2})$, $\tilde{P}_{22} \in C^1(\mathbb{R}, \mathbb{R}^{n_2\times n_2})$, $\tilde{Q}_{22} \in C^1(\mathbb{R}, \mathbb{R}^{n_2\times n_2})$, $\tilde{Q}_{24} \in C^1(\mathbb{R}, \mathbb{R}^{n_2\times n_4})$, $\tilde{Q}_{44} \in C^1(\mathbb{R}, \mathbb{R}^{n_4\times n_4})$, where the submatrices

$$\tilde{P}_{sub} := \begin{bmatrix} \tilde{P}_{11}(t) & \tilde{P}_{12}(t) \\ \tilde{P}_{12}(t)^{\mathrm{T}} & \tilde{P}_{22}(t) \end{bmatrix},$$

$$\tilde{Q}_{sub} := \begin{bmatrix} \tilde{Q}_{22}(t) & \tilde{Q}_{24}(t) \\ \tilde{Q}_{24}(t)^{\mathrm{T}} & \tilde{Q}_{44}(t) \end{bmatrix}$$

are positive definite for $t \in \mathbb{R}$.

The conditions (39)-(40) in Theorem 8 correspond to factorization of matrix-valued functions. These conditions are difficult to compute. This paper proposes an alternative condition for the existence of T-periodic Kalman canonical decomposition.

Theorem 9. Consider the linear T-periodic system (1)-(2). Let $\Psi_{11} \in \mathbb{R}^{n_1\times n_1}$, $\Psi_{22} \in \mathbb{R}^{n_2\times n_2}$, $\Psi_{33} \in \mathbb{R}^{n_3\times n_3}$, $\Psi_{44} \in \mathbb{R}^{n_4\times n_4}$ denote invertible matrices in Lemma 7. Then there exists a T-periodic coordinate transformation $Z \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ such that the transformed system (4)-(5) takes on the T-periodic Kalman canonical decomposition in $\mathbb{R}$ if and only if

$$\det \Psi_{11} > 0, \tag{41}$$

$$\det \Psi_{22} > 0, \tag{42}$$

$$\det \Psi_{33} > 0, \tag{43}$$

$$\det \Psi_{44} > 0. \tag{44}$$

Remark 10. Consider the block diagonal component $\dot{\xi}_1 = \tilde{A}_{11}(t)\xi_1$ in (9). Then, the determinant of its monodromy matrix is positive. The condition (41) is entirely due to this positiveness. The other conditions (42)-(44) can be also obtained from the positiveness of the monodromy matrices of the block diagonal components. Hence, the conditions (41)-(44) are trivial necessary condition.

Remark 11. (38) in Lemma 7 is similar to the monodromy matrix $\Phi(T, 0)$. Because the determinant of the monodromy matrix is positive, the multiplication of all determinants of block components Ψ_{ii} is positive

$$\det \Psi_{11} \det \Psi_{22} \det \Psi_{33} \det \Psi_{44} = \det \Phi(T, 0) > 0.$$

However, we notice that conditions (41)-(44) are not always satisfied; specifically, two or all of $\det \Psi_{ii}$ can be negative. Indeed, the counterexample in [8, 15] satisfies $\det \Psi_{11} = -1$ and $\det \Psi_{22} = -1$. Hence, from the authors point of view, the conditions (41)-(44) are nontrivial sufficient condition.

Remark 12. The conditions (41)-(44) require the computation of the state transition matrix Φ and the integral calculuses of $W_c(t, nT)$ and $\Phi(t, 0)$. These computations are not always analytically tractable but can be reduced to the numerical computations. Hence, from the authors point of view, Theorem 9 is drastically simplified from Theorem 8.

4.2 Relation to the Floquet-like factorization

The periodic Kalman canonical decomposition and the Floquet factorization (or the Floquet-like factorization by the authors) [9, 10, 11] have been independently investigated; specifically, the periodic Kalman canonical decomposition in [8] is based on the factorization of matrix-valued functions due to Sibuya and the Floquet factorization (or the Floquet-like factorization) is based on the computation of matrix logarithms. The proof of Theorem 9 is based on the computation of matrix logarithms and is analogous to the computation procedure of the Floquet-like factorization. Indeed, the Floquet-like factorization as well as the T-periodic Kalman canonical decomposition is simultaneously achieved as follows.

Corollary 13. Consider the linear T-periodic system (1)-(2). Let $\Psi_{11} \in \mathbb{R}^{n_1 \times n_1}$, $\Psi_{22} \in \mathbb{R}^{n_2 \times n_2}$, $\Psi_{33} \in \mathbb{R}^{n_3 \times n_3}$, $\Psi_{44} \in \mathbb{R}^{n_4 \times n_4}$ denote invertible matrices in Lemma 7. Suppose that the conditions (41)-(44) are satisfied. Then, the A-matrix of the transformed system consist of constants and the trigonometric functions with the fundamental frequency $\frac{2\pi}{T}$.

4.3 T-periodic decomposition with constant A-matrix

The A-matrix of the transformed system in (9) becomes non-constant in Theorem 9. The following theorem is a sufficient condition for the existence of T-periodic Kalman canonical decomposition with a constant A-matrix.

Corollary 14. Consider the linear T-periodic system (1)-(2). Suppose that the monodromy matrix $\Phi(T, 0)$ does not have negative real eigenvalues. Then there exists a T-periodic coordinate transformation $Z \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ such that the transformed system (4)-(5) takes on the T-periodic Kalman canonical decomposition and that $\tilde{A}(t)$ in (9) is constant.

4.4 Controllable and/or Observable Decomposition

There does not always exist a T-periodic Kalman canonical decomposition as discussed in Theorem 9. Even if there exists no T-periodic Kalman canonical decomposition, there might be a T-periodic Kalman canonical decomposition partially with respect to the (A, B)-pair or partially with respect to the (C, A)-pair.

Definition 15. A transformed linear kT-periodic system (4)-(5) and the triplet $(\tilde{C}, \tilde{A}, \tilde{B})$ are said to be a controllable kT-periodic Kalman canonical decomposition of (1)-(2) if there exists a kT-periodic coordinate transformation matrix $Z \in C^1_{kT,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ satisfying the following two conditions:
(i) The triplet $(\tilde{C}, \tilde{A}, \tilde{B})$ takes on the following from

$$\tilde{A}(t) = \begin{bmatrix} \tilde{A}_{11}(t) & \tilde{A}_{12}(t) \\ 0 & \tilde{A}_{22}(t) \end{bmatrix}, \tag{45}$$

$$\tilde{B}(t) = \begin{bmatrix} \tilde{B}_1(t) \\ 0 \end{bmatrix}, \tag{46}$$

$$\tilde{C}(t) = \begin{bmatrix} \tilde{C}_1(t) & \tilde{C}_2(t) \end{bmatrix}. \tag{47}$$

(ii) The pair $\left(\tilde{A}_{11}, \tilde{B}_1\right)$ is controllable.

Definition 16. A transformed linear kT-periodic system (4)-(5) and the triplet $(\tilde{C}, \tilde{A}, \tilde{B})$ are said to be an observable kT-periodic Kalman canonical decomposition of (1)-(2) if there exists a kT-periodic coordinate transformation matrix $Z \in C^1_{kT,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ satisfying the following two conditions:
(i) The triplet $(\tilde{C}, \tilde{A}, \tilde{B})$ takes on the following from

$$\tilde{A}(t) = \begin{bmatrix} \tilde{A}_{11}(t) & \tilde{A}_{12}(t) \\ 0 & \tilde{A}_{22}(t) \end{bmatrix}, \tag{48}$$

$$\tilde{B}(t) = \begin{bmatrix} \tilde{B}_1(t) \\ \tilde{B}_2(t) \end{bmatrix}, \tag{49}$$

$$\tilde{C}(t) = \begin{bmatrix} 0 & \tilde{C}_2(t) \end{bmatrix}. \tag{50}$$

(ii) The pair $\left(\tilde{C}_{22}, \tilde{A}_{22}\right)$ is observable.

Depending on the signs of $\det \Psi_{ii}$, the T-periodic system can be transferred to a controllable canonical decomposition or an observable Kalman canonical decomposition.

Corollary 17. Consider the linear T-periodic system (1)-(2). Let $\Psi_{11} \in \mathbb{R}^{n_1 \times n_1}$, $\Psi_{22} \in \mathbb{R}^{n_2 \times n_2}$, $\Psi_{33} \in \mathbb{R}^{n_3 \times n_3}$, $\Psi_{44} \in \mathbb{R}^{n_4 \times n_4}$ denote invertible matrices in Lemma 7. Then there exists a T-periodic coordinate transformation $Z \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ such that the transformed system (4)-(5) takes on the controllable T-periodic Kalman canonical decomposition in $\mathbb{R}$ if and only if

$$\det \Psi_{11} \det \Psi_{22} > 0. \tag{51}$$

Corollary 18. Consider the linear T-periodic system (1)-(2). Let $\Psi_{11} \in \mathbb{R}^{n_1 \times n_1}$, $\Psi_{22} \in \mathbb{R}^{n_2 \times n_2}$, $\Psi_{33} \in \mathbb{R}^{n_3 \times n_3}$, $\Psi_{44} \in \mathbb{R}^{n_4 \times n_4}$ denote invertible matrices in Lemma 7. Then there exists a T-periodic coordinate transformation $Z \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ such that the transformed system (4)-(5) takes on the observable T-periodic Kalman canonical decomposition in $\mathbb{R}$ if and only if

$$\det \Psi_{11} \det \Psi_{33} > 0. \tag{52}$$

Corollary 17 is an alternative condition for the existence of T-periodic decomposition for the (A, B)-pair and Corollary 18 is its dual for the (C, A)-pair. Theorem 9, Corollary 17, and Corollary 18 are summarized as follows.

Corollary 19. Consider the cases I to VIII in Table. 1 depending on the signs of $\det \Psi_{ii}$ $(i = 1, 2, 3, 4)$ in Theorem 9. Case I shows that, if all signs of $\det \Psi_{ii}$ are positive, there exist a T-periodic Kalman canonical decomposition (Can.=+), a controllable T-periodic Kalman canonical decomposition (Con.=+), an observable T-periodic Kalman canonical decomposition(Obs.=+). Case II shows that, if signs of $\det \Psi_{33}$ and $\det \Psi_{44}$ are positive, there exists a controllable T-periodic Kalman canonical decomposition (Con.=+). Similar arguments are valid for the other cases III to VII. Case VIII shows that, if all signs of $\det \Psi_{ii}$ are negative, there exists only one of a controllable T-periodic Kalman canonical decomposition or an observable T-periodic Kalman canonical decomposition (Con.=$-$/+, Obs.=+/$-$).

Case	$\det \Psi_{11}$	$\det \Psi_{22}$	$\det \Psi_{33}$	$\det \Psi_{44}$	Can.	Con.	Obs.
I	+	+	+	+	+	+	+
II	−	−	+	+	−	+	−
III	−	+	−	+	−	−	+
IV	−	+	+	−	−	−	−
V	+	−	−	+	−	−	−
VI	+	−	+	−	−	−	+
VII	+	+	−	−	−	+	−
VIII	−	−	−	−	−	−/+	+/−

Table 1. Possible decompositions for each combination of $\det \Psi_{ii}$.

4.5 Relation to the Period-specific Realization

The proof of Theorem 9 shares the common realization technique in the period-specific realization by the authors [16]. In the period-specific realization problem, the weighting pattern pattern matrix $W(t,p) = C(t)\Phi(t,p)B(p)$ is given and is supposed to be factored as $W(t,p) = L_0(t)R_0(p)$ in the globally reduced form. This means that $L_0(t) = \tilde{C}_2(t)\tilde{\Phi}_{22}(t,0)$ and $R_0(p) = \tilde{\Phi}_{22}(0,p)\tilde{B}_2(p)$ are supposed to be known without knowing the $(\tilde{C}_2, \tilde{A}_{22}, \tilde{B}_2)$-triple of the controllable and observable subsystem. There is a T-periodic controllable and observable subsystem if and only if the index q, which correspond to the monodromy matrix $\det \tilde{\Phi}_{22}(T,0)$ of the unknown subsystem, satisfies $q > 0$. If this condition $q > 0$ is satisfied, the computation procedure in the period-specific realization [16] recovers the unknown $(\tilde{C}_2, \tilde{A}_{22}, \tilde{B}_2)$-triple from $L_0(t)$ and $R_0(p)$. In other words, the dimension of the system is reduced in advance, and then, the minimal realization $(\tilde{C}_2, \tilde{A}_{22}, \tilde{B}_2)$ is computed based on the computation of the matrix logarithms G_{22} and F_{22} in the proof of Theorem 9. If the condition $q < 0$ is satisfied, the computation procedure in the period-specific realization [16] generates the non-minimal (C, A, B)-triple by augmenting the system dimension.

5. $2T$-PERIODIC DECOMPOSITION

Let us recall the existence of periodic Kalman canonical decomposition. Two types of solutions has been presented by the authors. For the (A, B)-pair, it has been shown that $2T$-periodic decomposition is always possible. For the (C, A, B)-triple, it is has been shown that $8T$-periodic decomposition is always possible.

Theorem 20. [8] Consider a linear T-periodic systems in (1). Then, there exists a $2T$-periodic Kalman canonical decomposition.

Theorem 21. [15] Consider a linear T-periodic systems in (1)-(2). Then, there exists a $8T$-periodic Kalman canonical decomposition.

Both solutions have been obtained based on the $2T$-periodic decomposition of the T-periodic matrix-valued function due to Sibuya [17]. One $2T$-periodic decomposition of the controllability Gramian assures $2T$-periodic decomposition for the (A, B)-pair. Triple $2T$-periodic decomposition of the controllability and the observability Gramians assure 2^3T-periodic decomposition for the (C, A, B)-triple. Further investigation is difficult along this approach. This paper proposes an alternative approach based on Theorem 9 and prove that $2T$-periodic decomposition is actually always possible for the (C, A, B)-triple.

Theorem 22. Consider a linear T-periodic systems in (1)-(2). Then, there exists a $2T$-periodic Kalman canonical decomposition.

6. CONCLUSIONS

This note has presented a new approach for computing the periodic Kalman canonical decomposition by extending the Floquet theory. The structure of linear linear periodic systems has been clarified in a unified framework by connecting the periodic Kalman canonical decomposition, the Floqeut factorization, the Floquet-like factorization, and the period-specific realization. Several new conditions for the existence of periodic Kalman canonical decompositions are obtained. The computational procedure is reduced to the factorization of the monodromy matrix, and therefore, is drastically simplified compared with the factorization of the controllability and the observability Gramians in the previous works.

A development of more numerically reliable methods is one of the future topic of this note. An extension to discrete-time systems, which may have a degenerate state transition matrix and require modification from continuous-time systems, is also one of the future topic of this note.

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