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Event-Triggered Control Over Noisy Feedback Channels ⋆

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Abstract: In this paper, the stability of a discrete-time event-triggered control system over noisy feedback channels is analyzed. The transmission between the controller and the actuator is triggered by an event involving estimated control, desirable control, and a trigger threshold. It is revealed that if the trigger threshold is less than a bound determined by the system and controller matrices, then the closed-loop system is mean square stable. In addition, two equivalent conditions based on algebraic Riccati inequality/equation (ARI/ARE) are proposed to facilitate further analysis.

Keywords: Event-triggered control; Noisy channels; Bounded Real Lemma; ARI; ARE.

1. INTRODUCTION

In the increasingly popular networked control systems (NCS) (Xiong and Lam [2007]), the sensors, controllers, and actuators are usually geographically dispersed, and the entire system is implemented with limited energy and communication resources, which poses a big challenge to feedback control design. In order to reduce the demand on energy consumption and communication while maintaining satisfactory closed-loop stability and performance, developing new control theory and technology would be necessary. One possible solution to the problem is to use the so-called event-trigger strategy (Tabuada [2007]), which originates from the research on aperiodic sampling. The study of aperiodic sampling can be traced back to the 1960s (Gupta [1963]). However, the technique had not received enough attention for many years until the late 1990s (Åström and Bernhardsson [1999, 2002]). In Åström and Bernhardsson's work, it has been revealed that better performance is likely to be achieved with aperiodic sampling. Furthermore, a special Lebesgue sampling based on hysteretic quantization, called the level-crossing sampling, has recently been proposed in Kofman [2003], Kofman and Braslavsky [2006].

The resurgence of research on event-based sampling has led to the gradually forming event-triggered control (ETC) in the past decade. It has been verified by researchers that this new control technology can prevent unnecessary samplings as well as information transmissions and require less control updates than the traditional periodic control. Consequently, the implementation cost can be reduced for the control of NCSs. The ETC theory is first systematically studied in Tabuada [2007]. It is based on the Liapunov stability theory so that the control renders the closed system input-to-state (ISS) stable (Sontag [2008]). With this stabilizing ETC law, the event rule is guaranteed to be legitimate, meaning that the inter-sampling time is lower bounded and only a finite number of events can occur in a finite interval of time. In other words, accumulative events known as the Zeno behavior (Ames *et al.* [2006]) will not happen. Such kind of "legitimacy", which is also mentioned in Kofman [2003], depends on the assumption that the state estimation error is zero at the sampling instant. The event-trigger strategy is applied to some wireless sensor/actuator networks and generalized to a decentralized form in Mazo and Tabuada [2011]. The local triggering conditions are introduced while the assumption that the state estimation error is zero at the sampling instant does not hold anymore. To ensure the "legitimacy", a minimum time $\tau_{\min}$ is set instead of being provided by the local triggering rule itself. If a new event occurs within the minimum time $\tau_{\min}$, it is ignored and the next updating time can be excited only after time interval $\tau_{\min}$. Meanwhile, the distributed ETC of NCSs is systematically analyzed in Wang and Lemmon [2011]. Besides, the ETC under data-rate communication constraints is studied in Li *et al.* [2012], relating ETC to the data rate problem (Nair and Evans [2004], Tatikonda and Mitter [2004]).

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Recently, the periodic event-triggered control (PETC) has been proposed in Heemels and Donkers [2013], Heemels *et al.* [2013]. This scheme combines the advantages of both ETC and traditional periodic control. In ETC, "legitimacy" has to be ensured either inherently, or by forcing a $\tau_{\min}$ value when system disturbance or the decentralization is considered. Besides, the event-triggering conditions need to be checked all the time. In PETC, traditional periodic sampling is preserved while the transmission of feedback information is event-triggered. The inter-update time is naturally lower bounded by sampling period and the event-triggering conditions only need to be checked at sampling instants. Between two consecutive feedback transmissions, such open loop schemes as constant control and model-based control can be adopted. Then, the PETC system is modeled as the discrete-time ETC. Related results on nonlinear systems can be found in Eqtami *et al.* [2010].

In this paper, we investigate the closed-loop stability of a discrete-time event-triggered control system over noisy channels. In the presence of channel noise, the error between estimated control and desirable control is not zero at the updating instant if the control is the channel input and constant control law is adopted when there is no transmission. We will show that if proper event-trigger is designed, the closed-loop stability can still hold. Based on the robust control theory, a sufficient condition for closed-loop stability in a mean square sense is established. A highlight of the obtained results is that the tuning rule for parameters can be identified in a straightforward manner.

The rest of the paper is organized as follows. In Section 2, the event-trigger strategy, together with an perturbed linear model, for the control over noisy channels is formulated. In Section 3, the perturbed linear model is modified to an augmented perturbed linear model with some scalar conditions. Then, based on the robust control theory, a sufficient condition is given for the existence of a Liapunov function. Subsequently, the closed-loop stability is established. In addition, two equivalent ARI/ARE conditions are proposed in Section 4 to facilitate further analysis. Finally, conclusions are drawn in Section 5.

Nomenclature: Throughout this paper, the common probability space for all random variables is denoted by $(\Omega, \mathscr{F}, \mathbb{P})$; the expectation operator is denoted by $E\{\cdot\}$; for a sequence $\{s_m\}$, s_m is also denoted by $s(m)$; $\mathbb{R}^p$ and $\mathbb{R}^{p\times q}$ represent the p-dimensional real vector space and the set of all $p \times q$ real matrices, respectively; for a general complex matrix M, M^* and M^T denote the conjugate transpose and the transpose of M, respectively; a square matrix X is Hermitian if $X = X^*$; the largest and the smallest eigenvalues of Hermitian X are denoted by $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$, respectively; $\|M\|$ denotes the spectral norm, *i.e.*, $\sqrt{\lambda_{\max}(M^*M)}$, of a general matrix M; I (respectively, 0) denotes the identity (respectively, zero) matrix with compatible dimension; for two Hermitian matrices X and Y, the notation $X \succeq Y$ (respectively, $X \succ Y$) means that $X - Y$ is positive semi-definite (respectively, positive definite); a square matrix A is defined to be Schur if all its eigenvalues locate within the open unit disc; for a discrete-time system transfer matrix $G(z)$, the H_∞ norm is denoted by $\|G\|_\infty$ satisfying $\|G\|_\infty = \sup_{\omega\in[-\pi,\pi]} \|G(e^{j\omega})\|$.

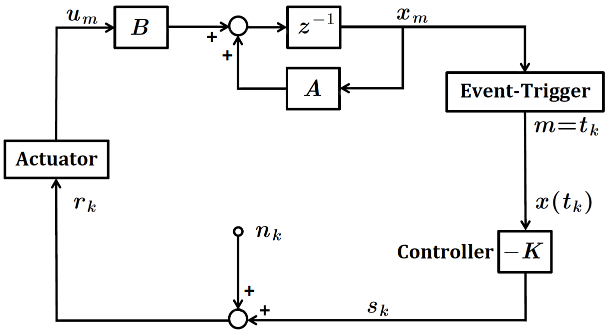

Fig. 1. Event-triggered control over noisy channels.

2. PROBLEM FORMULATION

2.1 Event-Trigger Strategy

The event-triggered feedback configuration considered in this paper is depicted in Fig. 1, which is the following discrete-time system:

$$\begin{aligned} &x_{m+1} = Ax_m + Bu_m; \quad u_m = r_k,\ t_k \le m < t_{k+1}; \\ &r_k = s_k + n_k;\ s_k = -Kx(t_k),\ (A - BK) \text{ is Schur.} \end{aligned} \tag{1}$$

Here, $A \in \mathbb{R}^{n_x\times n_x}$, $B \in \mathbb{R}^{n_x\times n_c}$, $K \in \mathbb{R}^{n_c\times n_x}$; $x \in \mathbb{R}^{n_x}$, $u, r, s, n \in \mathbb{R}^{n_c}$; $m = 0, 1, 2, ...$ are the discrete-time steps; and $\{t_k\} \subset \{0, 1, 2, ...\}$ with $t_0 = 0$ is the sequence of the updating time, which is generated by the event-trigger. The initial state x_0 can either be treated as a random vector with bounded $E\{\|x_0\|^2\}$; or as a deterministic real vector, with the expectation $E\{\|x_0\|^2\}$ being $\|x_0\|^2$ itself. At each updating time $m = t_k$, the control signal is transmitted by the controller as the sending message s_k, and disturbed by the additive channel noise n_k, then received by the actuator as r_k. Before the next updating time, the received signal r_k is stored in the actuator and the constant control law is adopted.

Assumption 1. The pair (A, B) is assumed stabilizable such that K exists to render $(A - BK)$ Schur.

Assumption 2. The additive channel noises $\{n_k\}$ are independent and identically distributed, zero mean, and with covariance matrix being diagonal and satisfying that $\sigma^2 I \succeq E\{n_k \cdot n_k^T\}$ for some constant $\sigma > 0$.

Definition 1. [Nair and Evans, 2004, (2.5)] System (1) is said to be stabilized in the mean square sense by the event-triggered mechanism with the controller $-K$ if

$$\sup_{m\in\mathbb{N}} E\{\|x_m\|^2\} < +\infty.$$

The event-trigger strategy considered in this paper is based on the *estimated control* $\{\hat{u}_m\}$ and the *desirable control* $\{\tilde{u}_m\}$, which will be defined in the following.

We start at $t_0 \triangleq 0$ and set the initial *estimated control* as $\hat{u}_0 \triangleq 0$. For $m \in \mathbb{N}$, we denote $\xi_m \triangleq [x_m^T \ \ (\hat{u}_m)^T]^T$, define the *desirable control* as $\tilde{u}_m \triangleq -Kx_m$, and the triggering function as $\mathcal{C}(\xi_m) \triangleq \|\hat{u}_m - \tilde{u}_m\|^2 - \lambda^2\|\tilde{u}_m\|^2 = \xi_m^T Q \xi_m$, where $\lambda > 0$ is the *trigger threshold*, and

$$Q \triangleq \begin{bmatrix} (1-\lambda^2)K^TK & K^T \\ K & I \end{bmatrix}.$$

Then, the *event-triggering condition* is designed as

$$\|\hat{u}_m - \tilde{u}_m\|^2 \ge \lambda^2\|\tilde{u}_m\|^2, \quad i.e., \quad \mathcal{C}(\xi_m) \ge 0. \tag{2}$$

The event-trigger strategy is then formulated in the following algorithm.

Algorithm 1:

Step 1. At the beginning of each updating process, $m = t_k$, $k \geq 0$, $\mathcal{C}(\xi(t_k)) = \|\hat{u}(t_k) - \tilde{u}(t_k)\|^2 - \lambda^2\|\tilde{u}(t_k)\|^2 \geq 0$, the event-triggering condition is satisfied, so the desirable control $\tilde{u}(t_k)$ is transmitted as the channel input s_k, and received by the actuator as $r_k = s_k + n_k$. Then, r_k is adopted as the true control u_m for the plant. The next estimated control $\hat{u}(t_k + 1)$ is set as

$$\hat{u}(t_k + 1) \triangleq -Kx(t_k) \ .$$

Step 2. If $\mathcal{C}(\xi(t_k + 1)) \geq 0$, then set $t_{k+1} \triangleq t_k + 1$.

Step 3. If $\mathcal{C}(\xi_m) < 0$ for $m \geq t_k + 1$ before the next triggering time, then keep both the true control and the estimated control unchanged,

$$u_m = u_{m-1} = r_k, \quad \hat{u}_{m+1} \triangleq \hat{u}_m = -Kx(t_k).$$

When for some $m > t_k + 1$, $\mathcal{C}(\xi_m) \geq 0$, the trigger event occurs, and this m is detected as t_{k+1}.

Step 4. A new updating cycle begins, go to Step 1. If finite t_{k+1} does not exist, we denote $t_{k+1} = +\infty$.

2.2 "Perturbed Linear Model"

Define the noise sequence $\{d_m\}$ as

$$d_m = n_k, \ t_k \leq m < t_{k+1} \ ; \quad E\{\|d_m\|^2\} = n_c\sigma^2. \tag{3}$$

Following the perturbed linear model in Heemels and Donkers [2013], which ignores the channel noise, we can rephrase the event-trigger strategy for control over noisy channels described in Section 2.1 as:

$$x_{m+1} = (A - BK)x_m + Bd_m + Bw_m, \tag{4}$$

where

$$d_m = \begin{cases} d_{m-1}, & \text{if } \ \|\hat{u}_m - \tilde{u}_m\| < \lambda\|\tilde{u}_m\|; \\ \text{a new random variable}, & \text{otherwise.} \end{cases}$$

$\tilde{u}_m = -Kx_m$, $\hat{u}_0 = 0$, and

$$\hat{u}_{m+1} = \begin{cases} \tilde{u}_m, \text{ if } \|\hat{u}_m - \tilde{u}_m\| \geq \lambda\|\tilde{u}_m\|; \\ \hat{u}_m, \text{ if } \|\hat{u}_m - \tilde{u}_m\| < \lambda\|\tilde{u}_m\|. \end{cases}$$

and the perturbed variable w_m is introduced as

$$w_m = \begin{cases} 0, & \text{if } \|\hat{u}_m - \tilde{u}_m\| \geq \lambda\|\tilde{u}_m\|; \\ \hat{u}_m - \tilde{u}_m, & \text{if } \|\hat{u}_m - \tilde{u}_m\| < \lambda\|\tilde{u}_m\|. \end{cases}$$

Thus, in *perturbed linear model* (4) of the event-triggered control, the perturbed term w_m satisfies

$$\|w_m\| \leq \lambda\|\tilde{u}_m\| \leq (\lambda\|K\|)\|x_m\|. \tag{5}$$

3. STOCHASTIC STABILITY RESULTS

In this section, the perturbed linear model is modified to an augmented perturbed linear model with some scalar conditions, which plays a vital role in demonstrating the existence of Liapunov functions in Theorem 2. Based on the result, the closed-loop stability is established in Theorem 3.

3.1 Augmented Perturbed Linear Model

Before proceeding, we introduce a lemma which will be used in the sequel.

Lemma 1. (Special case of Bounded Real Lemma (BRL), see Ionescu *et al.* [1999].) For matrices $\mathcal{A}$ and $\mathcal{B}$ with compatible dimensions, let $\gamma > 0$ and $\mathcal{G}(z) \triangleq (zI - \mathcal{A})^{-1}\mathcal{B}$ be given. Then, the following three statements are equivalent:

(BRLs1). $\mathcal{A}$ is Schur and $\|\mathcal{G}\|_\infty < \gamma$.

(BRLs2). The discrete-time definite constrained algebraic Riccati equation (ARE)

$$\mathcal{A}^T X\mathcal{A} - X + I - \mathcal{A}^T X\mathcal{B}(-\gamma^2 I + \mathcal{B}^T X\mathcal{B})^{-1}\mathcal{B}^T X\mathcal{A} = 0 \tag{6}$$

has a unique stabilizing solution $X \succ 0$ under the definite constraint $\gamma^2 I - \mathcal{B}^T X\mathcal{B} \succ 0$, that is, $(\mathcal{A} - \mathcal{B}(-\gamma^2 I + \mathcal{B}^T X\mathcal{B})^{-1}\mathcal{B}^T X\mathcal{A})$ is Schur, and there exist V, W such that

$$\begin{bmatrix} X - I - \mathcal{A}^T X\mathcal{A} & -\mathcal{A}^T X\mathcal{B} \\ -\mathcal{B}^T X\mathcal{A} & \gamma^2 I - \mathcal{B}^T X\mathcal{B} \end{bmatrix} = \begin{bmatrix} W^T \\ V^T \end{bmatrix} [W \ \ V] \succeq 0.$$

(BRLs3). There exists $X \succ 0$ solving the linear matrix inequality (LMI)

$$\begin{bmatrix} X - I - \mathcal{A}^T X\mathcal{A} & -\mathcal{A}^T X\mathcal{B} \\ -\mathcal{B}^T X\mathcal{A} & \gamma^2 I - \mathcal{B}^T X\mathcal{B} \end{bmatrix} \succ 0.$$

In the following theorem, we present a sufficient condition for the existence of quadratic Liapunov functions.

Theorem 2. Consider system (4) subject to Assumption 1. For any K such that $A_p \triangleq A - BK$ is Schur, if the trigger threshold λ satisfies

$$0 < \lambda < \frac{1}{\|K\| \cdot \|G\|_\infty}, \tag{7}$$

where $G(z) \triangleq (zI - A_p)^{-1}B$, then there exist parameters $\gamma, \theta > 0$ such that

$$\lambda\theta\|K\| < 1, \qquad \frac{1}{\theta^2} + \frac{1}{\gamma^2} < \frac{1}{\|G\|_\infty^2}. \tag{8}$$

Furthermore, there exist a positive definite matrix P and parameters $\varepsilon > 0$,

$$c_2 \triangleq \lambda_{\max}(P), \text{ and } c_1 \triangleq \lambda_{\min}(P) > 0, \tag{9}$$

such that for the Liapunov function $V(x) \triangleq x^T Px$, the following inequalities hold

$$V(x_{m+1}) - V(x_m) \leq -\varepsilon\|x_m\|^2 + \varepsilon\theta^2\|w_m\|^2 + \varepsilon\gamma^2\|d_m\|^2, \tag{10}$$

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2. \tag{11}$$

Proof. First, the *augmented perturbed linear model* is presented as follows. We introduce an auxiliary parameter $\gamma_p > 0$, and denote

$$A_p \triangleq A - BK, B_p \triangleq \begin{bmatrix} \frac{\gamma_p}{\gamma}B & \frac{\gamma_p}{\theta}B \end{bmatrix}, v_m \triangleq \begin{bmatrix} \frac{\gamma}{\gamma_p}d_m \\ \frac{\theta}{\gamma_p}w_m \end{bmatrix}. \tag{12}$$

Then, system (4) of the perturbed linear model of the discrete-time event-triggered control over noisy channels can be modified as an *augmented perturbed linear model*

$$x_{m+1} = A_p x_m + B_p v_m. \tag{13}$$

Next, we study $\|G_p\|_\infty$, which is the following finite H_∞ norm of $G_p(z) \triangleq (zI - A_p)^{-1}B_p$,

$$\|G_p\|_\infty = \|G_p^*\|_\infty = \sup_\omega \left\| B_p^T \left(e^{-j\omega} I - A_p^T\right)^{-1} \right\|$$
$$= \sup_\omega \sqrt{\lambda_1 \left((e^{j\omega} I - A_p)^{-1} B_p B_p^T \left(e^{-j\omega} I - A_p^T\right)^{-1} \right)}$$
$$= \gamma_p \sqrt{\frac{1}{\theta^2} + \frac{1}{\gamma^2}} \cdot \|G\|_\infty,$$

where $\|G\|_\infty$ is the finite H_∞ norm of $G(z) = (zI - A_p)^{-1}B$, and the last equality results from the fact that

$$B_p B_p^T = \begin{bmatrix} \frac{\gamma_p}{\theta} B & \frac{\gamma_p}{\gamma} B \end{bmatrix} \cdot \begin{bmatrix} \frac{\gamma_p}{\theta} B^T \\ \frac{\gamma_p}{\gamma} B^T \end{bmatrix} = \gamma_p^2 \left(\frac{1}{\theta^2} + \frac{1}{\gamma^2} \right) BB^T. \quad (14)$$

Since

$$\gamma^2 \|d_m\|^2 + \theta^2 \|w_m\|^2 = \gamma_p^2 \|v_m\|^2,$$

inequality (10) is equivalent to

$$\begin{aligned} &\varepsilon \left(\gamma_p^2 \|v_m\|^2 - \|x_m\|^2\right) \geq V(x_{m+1}) - V(x_m) \\ &= (A_p x_m + B_p v_m)^T P (A_p x_m + B_p v_m) - x_m^T P x_m, \end{aligned} \quad (15)$$

It is straightforward to verify that condition (15) is guaranteed by the following matrix inequality:

$$\begin{bmatrix} X - I - A_p^T X A_p & -A_p^T X B_p \\ -B_p^T X A_p & \gamma_p^2 I - B_p^T X B_p \end{bmatrix} \succeq 0, \quad (16)$$

where $X = \frac{1}{\varepsilon} P \succ 0$. Furthermore, (16) can be ensured by the following strict inequality:

$$\begin{bmatrix} X - I - A_p^T X A_p & -A_p^T X B_p \\ -B_p^T X A_p & \gamma_p^2 I - B_p^T X B_p \end{bmatrix} \succ 0. \quad (17)$$

By Lemma 1, the feasibility of (17) is equivalent to that

$$\gamma_p > \|\tilde{G}\|_\infty = \gamma_p \sqrt{\frac{1}{\theta^2} + \frac{1}{\gamma^2}} \cdot \|G\|_\infty,$$

which is the scalar condition

$$\frac{1}{\theta^2} + \frac{1}{\gamma^2} < \frac{1}{\|G\|_\infty^2}.$$

Next, by condition (7), we can choose any θ and γ such that

$$\theta \in \left(\|G\|_\infty, \frac{1}{\lambda \|K\|} \right), \qquad \gamma > \frac{\theta \|G\|_\infty}{\sqrt{\theta^2 - \|G\|_\infty^2}}. \quad (18)$$

Thus, we have verified the existence of $\gamma, \theta > 0$ satisfying condition (8), which guarantees the existence of the matrix P such that inequalities (10) and (11) hold. The proof of Theorem 2 is completed. ■

Remark 1. (i) From (8), it follows that a small γ, which intuitively means that the influence of channel noise on system trajectories is weak, will lead to a large θ, which in turn results in small value of λ and, consequently, more frequent updating.

(ii) To find desirable P, λ, θ, and ε satisfying the conditions in Theorem 2, one may resort to solving the following matrix inequality

$$\begin{bmatrix} P - \varepsilon I - A_p^T P A_p & -A_p^T PB & -A_p^T PB \\ -B^T P A_p & \varepsilon\gamma^2 I - B^T PB & -B^T PB \\ -B^T P A_p & -B^T PB & \varepsilon\theta^2 I - B^T PB \end{bmatrix} \succeq 0, \quad (19)$$

which is equivalent to (16). When γ and θ are fixed, (19) becomes an LMI, which can be solved efficiently and reliably by existing algorithms (Boyd and Vandenberghe [2004]). As discussed in Heemels and Donkers [2013], there may exist a Pareto optimal curve (Boyd and Vandenberghe [2004]) of (γ, θ) for which LMI (19) is feasible, and a heuristic method may be used to find the curve. However, tuning the two parameters γ and θ to ensure the feasibility of (19) may not be an easy task. While in Theorem 2, a simple scalar tuning rule for (γ, θ) is proposed in condition (8), or equivalently (18), which can guide the design of the value of the trigger threshold λ.

3.2 Stochastic Stability

We are now in a position to present the main stability result.

Theorem 3. Consider system (1) subject to Assumptions 1 and 2. For any K making $(A - BK)$ Schur, if the trigger threshold λ satisfies condition (7), then the closed-loop system is stable in the sense of Definition 1, that is,

$$\sup_{m \in \mathbb{N}} E\{\|x_m\|^2\} < +\infty.$$

Proof. By Theorem 2, we can choose parameters $\theta, \gamma > 0$ satisfying condition (8), and let $P \succ 0$ and $\varepsilon > 0$ such that inequalities (11), (10), and (16) hold. Combining inequalities (5), (11), and (10), we obtain that $c_1 \|x_m\|^2 \leq V(x_m)$, and

$$\begin{aligned} &V(x_{m+1}) \\ &\leq V(x_m) - \varepsilon \left(1 - \lambda^2 \theta^2 \|K\|^2\right) \|x_m\|^2 + \varepsilon \gamma^2 \|d_m\|^2, \\ &\leq \left(1 - \frac{\varepsilon \left(1 - \lambda^2 \theta^2 \|K\|^2\right)}{c_2}\right) \cdot V(x_m) + \varepsilon \gamma^2 \|d_m\|^2. \end{aligned}$$

Taking expectation, and using Assumption 2 as well as (3) and (8), we obtain that $E\{\|x_m\|^2\} \leq E\{V(x_m)\}/c_1$, and

$$E\{V(x_{m+1})\} \leq \alpha E\{V(x_m)\} + n_c \sigma^2 \varepsilon \gamma^2, \ \forall m \geq 0,$$

where $\alpha \triangleq 1 - \varepsilon \left(1 - \lambda^2 \theta^2 \|K\|^2\right)/c_2 < 1$.

By (16) and (9), $P \succeq \varepsilon I + A_p^T P A_p \succeq \varepsilon I, \quad c_2 \geq c_1 \geq \varepsilon$. Therefore, $\alpha \geq \varepsilon \lambda^2 \theta^2 \|K\|^2 / c_2 > 0$. Defining a sequence $\{E_m\}$ as

$$E_{m+1} = \alpha \cdot E_m + n_c \sigma^2 \varepsilon \gamma^2, \ E_0 = E\{V(x_0)\},$$

and applying the comparison principle (Gil' [2007]), we obtain that

$$E\{V(x_m)\} \leq E_m = \alpha^m E_0 + \frac{1 - \alpha^m}{1 - \alpha} \cdot n_c \sigma^2 \varepsilon \gamma^2.$$

Consequently, for $\forall \ m \geq 0$,

$$\begin{aligned} E\{\|x_m\|^2\} &\leq \frac{E\{V(x_m)\}}{c_1} \\ &\leq \frac{c_2}{c_1} \alpha^m E\{\|x_0\|^2\} + \frac{c_2 n_c \sigma^2 \gamma^2}{c_1 \left(1 - \lambda^2 \theta^2 \|K\|^2\right)}, \end{aligned} \quad (20)$$

and $\sup_{m \geq 0} E\{\|x_m\|^2\}$ is upper-bounded by

$$\frac{c_2}{c_1} E\{\|x_0\|^2\} + \frac{c_2 n_c \sigma^2 \gamma^2}{c_1 \left(1 - \lambda^2 \theta^2 \|K\|^2\right)} < +\infty,$$

which completes the proof. ■

Remark 2. (i) ISS stability (Sontag [2008]) has been established in (20). If the ISS gain $(\sup E\{\|x_m\|^2\}/(n_c\sigma^2))$ is considered as the event-triggered control performance, the important role of the extremal eigenvalues c_1 and c_2 of matrix P has also been demonstrated. Although the desirable parameters θ and γ can be found from condition (18), they are not sufficient to determine the control performance.

(ii) It is noted from (20) that the role of tuning rule (8) or (18) is not only for theoretical analysis but also for implementation. When designing the event-triggered control, one cannot make the trigger threshold λ approach the upper bound in (7), otherwise the upper bound of $E\{\|x_m\|^2\}$ in (20) will increase to infinity and the closed-loop stability may not be guaranteed.

4. EQUIVALENT ARI/ARE

In this section, the matrix P in the quadratic Liapunov function will be obtained by means of ARI/ARE, which will be simplified to equivalent ARI/ARE in Theorem 6. These results can facilitate further analysis of control performance.

From the analysis in Section 3, the matrix P and the parameter ε can be scaled. Without loss of generality, we set $\varepsilon = 1$. Then, under the constraints that $X = \frac{1}{\varepsilon}P = P \succ 0$ and $\gamma_p^2 I - B_p^T X B_p \succ 0$, applying the Schur complement (Boyd and Vandenberghe [2004]), we obtain that (16) as well as (19) can be guaranteed by the following algebraic Riccati inequality (ARI) with definite constraint (Damm [2004]):

$$\begin{aligned} \mathscr{R}(P) \triangleq\, & P - I - A_p^T P A_p - \\ & A_p^T P B_p \left(\gamma_p^2 I - B_p^T P B_p\right)^{-1} B_p^T P A_p \succeq 0\ , \qquad (21) \\ & \gamma_p^2 I - B_p^T P B_p \succ 0, \end{aligned}$$

where $\mathscr{R}(\cdot)$ is the Riccati operator (Damm [2004]).

By Section 3.1 and Lemma 1, if condition (8) is satisfied, there exists P being the unique positive definite stabilizing solution to the following ARE with definite constraint:

$$\begin{aligned} & P - I - A_p^T P A_p = \\ & \quad A_p^T P B_p \left(\gamma_p^2 I - B_p^T P B_p\right)^{-1} B_p^T P A_p, \qquad (22) \\ & \gamma_p^2 I - B_p^T P B_p \succ 0. \end{aligned}$$

Although ARI (21) and ARE (22) have provided P, they involve the auxiliary parameter γ_p and the auxiliary matrix B_p. We will derive equivalent ARI/ARE independent of γ_p and B_p as follows. Denote

$$\beta \triangleq \frac{1}{\sqrt{\frac{1}{\theta^2} + \frac{1}{\gamma^2}}} > \|G\|_\infty, \quad \beta\lambda\|K\| < 1. \qquad (23)$$

We have the following proposition.

Proposition 4. The following equivalence of positive definiteness holds

$$\gamma_p^2 I - B_p^T P B_p \succ 0 \quad \Leftrightarrow \quad \beta^2 I - B^T P B \succ 0.$$

Proof. If $\gamma_p^2 I - B_p^T P B_p \succ 0$,

$$\begin{aligned} & \gamma_p^2 I - \begin{bmatrix} \frac{\gamma_p}{\gamma} B^T \\ \frac{\gamma_p}{\theta} B^T \end{bmatrix} \cdot P \cdot \begin{bmatrix} \frac{\gamma_p}{\gamma} B & \frac{\gamma_p}{\theta} B \end{bmatrix} \\ & = \gamma_p^2 \cdot \begin{bmatrix} I - \frac{1}{\gamma^2} B^T P B & \frac{1}{\gamma\theta} B^T P B \\ -\frac{1}{\gamma\theta} B^T P B & I - \frac{1}{\theta^2} B^T P B \end{bmatrix} \succ 0\ . \qquad (24) \end{aligned}$$

Then, applying the Schur complement, we obtain that

$$\begin{aligned} & I - \frac{1}{\gamma^2} B^T P B \\ & \succ \frac{1}{\gamma^2\theta^2} B^T P B \left(I - \frac{1}{\theta^2} B^T P B\right)^{-1} B^T P B. \end{aligned} \qquad (25)$$

Note that $B^T P B = \theta^2 I - \left(\theta^2 I - B^T P B\right)$, then

$$\begin{aligned} & \frac{1}{\theta^2} B^T P B \left(I - \frac{1}{\theta^2} B^T P B\right)^{-1} B^T P B \\ & = \left(I - \left(I - \frac{1}{\theta^2} B^T P B\right)\right) \left(I - \frac{1}{\theta^2} B^T P B\right)^{-1} B^T P B \\ & = \left(I - \frac{1}{\theta^2} B^T P B\right)^{-1} \left(\theta^2 I - \left(\theta^2 I - B^T P B\right)\right) - B^T P B \\ & = \theta^4 \left(\theta^2 I - B^T P B\right)^{-1} - \theta^2 I - B^T P B. \qquad (26) \end{aligned}$$

Combining (25), (26), and that $\theta^2 I - B^T P B \succ 0$ resulting from (24), we obtain that

$$\begin{aligned} & \gamma^2 I - B^T P B \succ \theta^4 \left(\theta^2 I - B^T P B\right)^{-1} - \theta^2 I - B^T P B; \\ & \theta^2 I - B^T P B \succ \frac{\theta^4}{\gamma^2 + \theta^2} I \;\Rightarrow\; \beta^2 I - B^T P B \succ 0. \qquad (27) \end{aligned}$$

Conversely, if $\beta^2 I - B^T P B \succ 0$, then as the implication in (27) is in fact an equivalence relation, we obtain that matrix inequality (25) holds. And similarly, $I - \frac{1}{\theta^2} B^T P B \succ \frac{1}{\gamma^2\theta^2} B^T P B (I - \frac{1}{\gamma^2} B^T P B)^{-1} B^T P B$.

Applying the Schur complement, we obtain the positive definiteness of $\left(\gamma_p^2 I - B_p^T P B_p\right)$, which completes the proof. ■

Let $\gamma_p^2 I - B_p^T P B_p \succ 0$, and denote $M_p \triangleq B_p(\gamma_p^2 I - B_p^T P B_p)^{-1} B_p^T$, $M \triangleq B(\beta^2 I - B^T P B)^{-1} B^T$. The following proposition will be obtained.

Proposition 5. The matrices M_p and M are equal.

Proof. By the Matrix Inversion Formulas (Zhou *et al.* [1996]), $(\beta^2 I - B^T P B)^{-1} = \frac{1}{\beta^2} I + \frac{1}{\beta^2} B^T P (\beta^2 I - B B^T P)^{-1} B$. Then,

$$\begin{aligned} M & = \frac{1}{\beta^2} B B^T + \frac{1}{\beta^2} B B^T P \left(\beta^2 I - B B^T P\right)^{-1} B B^T \\ & = \frac{1}{\beta^2} B B^T + \frac{1}{\beta^2} B B^T \left(P^{-1} - \frac{1}{\beta^2} B B^T\right)^{-1} \frac{1}{\beta^2} B B^T. \end{aligned}$$

Similarly,

$$\begin{aligned} M_p = & \frac{1}{\gamma_p^2} B_p B_p^T + \\ & \frac{1}{\gamma_p^2} B_p B_p^T \left(P^{-1} - \frac{1}{\gamma_p^2} B B^T\right)^{-1} \frac{1}{\gamma_p^2} B_p B_p^T. \end{aligned}$$

By (14),

$$B_p B_p^T = \frac{\gamma_p^2}{\beta^2} B B^T, \quad \frac{1}{\gamma_p^2} B_p B_p^T = \frac{1}{\beta^2} B B^T.$$

Therefore, $M_p = M$, which completes the proof. ■

Combining Proposition 5 and Proposition 4, we obtain the following result.

Theorem 6. The following equivalences between ARIs and between AREs hold:

- ARI (21) is equivalent to the following ARI

$$\begin{aligned} &\mathscr{R}(P) \triangleq P - I - A_p^T P A_p - \\ &\quad A_p^T P B \left(\beta^2 I - B^T P B\right)^{-1} B^T P A_p \succeq 0 \ , \\ &\beta^2 I - B^T P B \succ 0 \ ; \end{aligned} \tag{28}$$

- ARE (22) is equivalent to the following ARE

$$\begin{aligned} &P - I - A_p^T P A_p = \\ &\quad A_p^T P B \left(\beta^2 I - B^T P B\right)^{-1} B^T P A_p \ , \\ &\beta^2 I - B^T P B \succ 0. \end{aligned} \tag{29}$$

Remark 3. The obtained ARI/ARE conditions only involve one single parameter, and there are many algorithms in the literature that can solve ARE efficiently (Bini *et al.* [2012]). More importantly, the ARI/ARE conditions can be used to derive some analytic relationships among stability/performance and channel noise as well as the event-trigger parameters.

5. CONCLUSIONS

In this paper, the stochastic stability of discrete-time event-triggered control system over noisy channels has been demonstrated. Based on the perturbed linear model of periodic event-triggered control proposed in Heemels and Donkers [2013], a simple scalar tuning rule for the event-trigger parameters is established to ensure closed-loop stability. Equivalences between ARIs/AREs are revealed as well. These new results will be generalized by utilizing non-quadratic Liapunov functions (Bacciotti and Rosier [2005]), and applied to the analysis of H_∞ and H_2 performances of the proposed event-trigger strategy in future study. Besides, the model-based event-triggered control over noisy channels based on the constant control case will be studied in the near future.

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