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Discrete-Time Linear Quadratic Optimal Control with Fixed and Free Terminal State via Double Generating Functions

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Abstract: A double generating functions based method for the discrete-time linear quadratic optimal control problem with fixed and free terminal state, i.e., hard constraint problem and soft constraint problem, is developed in this paper. By this method, the optimal state and input can be expressed by functions only in terms of the boundary conditions and pre-computed coefficients. Accordingly, the whole optimal computation can be divided into two parts in which the off-line part calculates the coefficients and the on-line part generates the optimal solutions by these proposed functions. In view of this, the developed method is useful in the on-line iterative computation for optimal control problem with a number of different boundary conditions. Examples demonstrate the effectiveness of the double generating functions method for the both hard and soft constraint problems with different boundary conditions.

Keywords: Double generating functions; Discrete-time; Linear quadratic; Optimal control.

1. INTRODUCTION

The theory of linear quadratic (LQ) optimal control is concerned with operating a linear dynamic system at quadratic minimum cost. Generally, the initial time, terminal time, and initial state in the finite-horizon optimal control problem are fixed. According to the terminal state, there exist two representative problem formulations. One is called the hard constraint problem (HCP) that denotes the problem with fixed terminal state and the other one is called the soft constraint problem (SCP) that with free terminal state [Park and Scheeres, 2006].

In on-line practice, HCP usually requires the efficient generation of the optimal trajectories for different sets of initial and terminal boundary conditions (BCs). For example, the on-demand control of biped walking robot in the complex environment needs to adjust the robot step length and walking speed for each step [Hao et al., 2013b]. When referring to the SCP, engineer in the real application needs to specify the weighting factors of the cost function and compare the optimal results with the designated design goals, especially the factor of the terminal cost. This means it will be an iterative process that the engineer adjusts the weighting factor to get a controller more in line with the design goals.

Dynamic programming [Bellman, 2003] and Riccati framework [Kucera, 1972, Imura, 2004] are the famous approaches to solve the optimal control problem. But both of them have to implement the whole computation repetitively for each different set of BCs [1] of the HCP and SCP. This yields the heavy burden for practice.

Recently, the single generating function method for the optimal control problem with hard and soft constraint in the continuous-time case is presented [Park and Scheeres, 2006]. Since this approach does not require one to implement the whole computation repetitively for different BCs in the on-line computation, it provides an advantage over methods rooted in the conventional dynamic programming and Riccati framework. In order to further decrease the on-line computational burden, the double generating functions method is proposed in the continuous-time case for the HCP [Hao et al., 2013a,c]. This approach enables one to obtain the optimal solutions (state and input) only by partial differentiations and algebraic manipulations of the double generating functions in the on-line computation.

Interesting characteristics of the generating function theory in the continuous-time case also attract researchers to study the analogue in the discrete-time case. Based on the discrete Hamiltonian mechanics [Lall and West, 2006], Ohsawa et al. [2011] develops a discrete analogue of Hamilton–Jacobi theory which provides an appropriate way to study the discrete-time LQ optimal control

[1] The terminal cost factor of the SCP in the transversality condition can indirectly adjust the terminal state, we also treat it as the BC.

problem via generating functions. Forward and backward generating functions specify the family of the canonical transformations from the initial BC to the current state-costate and the terminal BC to the current state-costate, respectively [Hao et al., 2013a,c]. Both of these two kinds of generating functions can be obtained by solving either the right discrete Hamilton–Jacobi equation (HJE) or the left HJE. Lee [2012] applies the backward generating function approach to the discrete-time LQ optimal control problem with a hybrid system by solving the right discrete HJE.

This paper develops the double generating functions method for the HCP and SCP in the discrete-time case. By selecting any two different generating functions from the forward and/or backward types, we construct the double generating functions for optimal control problems. In the light of this, the optimal solutions can be expressed by the functions only in terms of the BCs and pre-computed coefficients. Accordingly, the whole computation can be divided into two parts in which the off-line part calculates the coefficients and the on-line part generates the optimal solutions satisfying the particular BCs. This enables us to efficiently get the optimal solutions for fundamentally different sets of BCs in the on-line computation. The discrete double generating functions possess theoretical significance for discrete-time optimal control that is equivalent to that of the continuous ones for the continuous-time optimal control which can generate the optimal solutions precisely and efficiently in the on-line computation.

This paper is organized as follows. The discrete-time LQ optimal control problem with fixed and free state is formulated in Section 2. Section 3 gives the first order necessary conditions for optimizing the HCP and SCP. Based on this, the forward and backward generating functions are derived in Section 4. Section 5, which is the main part of this paper, develops the double generating functions method for the HCP and SCP. Examples are implemented in Section 6 to illustrate the effectiveness of the developed approach. Section 7 concludes the paper.

2. PROBLEM FORMULATION

Consider to find a sequence of the input minimizing a discrete-time quadratic cost function

$$J = \phi(x_N) + \frac{1}{2}\sum_{k=0}^{N-1}(x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k) \tag{1}$$

subject to the following discrete-time linear system with boundary constraints

$$x_{k+1} = f_{\mathrm{d}}(x_k, u_k) = Ax_k + Bu_k \tag{2}$$

$$x_0 = x^0, \quad \psi(x_N) = 0 \tag{3}$$

over a fixed number of time steps N. Here, $k \in \mathbb{N}$ is the discrete time. The vectors $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the state and input of the problem, respectively. The constant matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are both symmetric positive definite, $A \in \mathbb{R}^{n\times n}$ is invertible, and $B \in \mathbb{R}^{n\times m}$. The vector $x^0 \in \mathbb{R}^n$ is the value of the fixed initial state. The terminal cost and terminal boundary constraint are defined by $\phi(x_N) : \mathbb{R}^n \to \mathbb{R}$ and $\psi(x_N) : \mathbb{R}^n \to \mathbb{R}^{l\leqslant n}$, respectively.

We consider two representative problem formulations characterized by the types of the terminal BCs [Park and Scheeres, 2006]:

- HCP: The above problem with $\phi(x_N) \equiv 0$, $\psi(x_N) = x_N - x^N$ for the given terminal state value $x^N \in \mathbb{R}^n$.
- SCP: The above problem with $\phi(x_N) = x_N^{\mathrm{T}} P x_N \in \mathbb{R}$ for the constant matrix $P \in \mathbb{R}^{n\times n}$ (symmetric positive definite), $\psi(x_N)$ does not exist.

As the initial time, terminal time, and initial state are fixed in the original problem formulation, the above two types of the constraints are placed here to represent two cases of the terminal state. The terminal states are fixed and free in the HCP and SCP, respectively. Theory of double generating functions will be developed in the remainder to solve these two typical problems. Further, we present that the double generating functions method is useful for on-line iterative computation of the optimal solutions corresponding to different (initial and terminal) BCs in HCP, and different terminal weighting factor P in SCP.

3. NECESSARY CONDITIONS FOR OPTIMALITY

In this section, we present the first order necessary conditions for optimality according to the discrete minimum principle. Section 3.1 gives the necessary conditions for optimizing HCP, and Section 3.2 for optimizing SCP.

3.1 Necessary Conditions for Optimizing HCP

The discrete minimum principle [Sage, 1968] gives the first order necessary conditions for optimizing HCP

$$x_{k+1} = D_2\mathcal{H}_{\mathrm{d}}^{+}(x_k, \lambda_{k+1}) \tag{4}$$

$$\lambda_k = D_1\mathcal{H}_{\mathrm{d}}^{+}(x_k, \lambda_{k+1}) \tag{5}$$

and the optimal input

$$u_k = -M\lambda_{k+1} \tag{6}$$

where $\lambda_k \in \mathbb{R}^n$ is introduced as the costate, $M = R^{-1}B^{\mathrm{T}} \in \mathbb{R}^{m\times n}$, and D_i stands for the partial derivative of a function with respect to its i-th argument. Here, (4) and (5) are the right discrete Hamilton's equations [Lall and West, 2006], and $\mathcal{H}_{\mathrm{d}}^{+}(x_k, \lambda_{k+1})$ is the right discrete Hamiltonian which has the expression

$$\mathcal{H}_{\mathrm{d}}^{+}(x_k, \lambda_{k+1}) = \frac{1}{2}x_k^{\mathrm{T}} Q x_k + \lambda_{k+1}^{\mathrm{T}} A x_k - \frac{1}{2}\lambda_{k+1}^{\mathrm{T}} G \lambda_{k+1} \tag{7}$$

where $G = BR^{-1}B^{\mathrm{T}} \in \mathbb{R}^{n\times n}$.

The necessary conditions for optimality can also be expressed by the left discrete Hamilton's equations [Lall and West, 2006]

$$x_k = -D_1\mathcal{H}_{\mathrm{d}}^{-}(\lambda_k, x_{k+1}) \tag{8}$$

$$\lambda_{k+1} = -D_2\mathcal{H}_{\mathrm{d}}^{-}(\lambda_k, x_{k+1}) \tag{9}$$

where the left discrete Hamiltonian is defined by the following [Lall and West, 2006]

$$\mathcal{H}_{\mathrm{d}}^{-}(\lambda_k, x_{k+1}) = \mathcal{H}_{\mathrm{d}}^{+}(x_k, \lambda_{k+1}) - \lambda_k^{\mathrm{T}} x_k - \lambda_{k+1}^{\mathrm{T}} x_{k+1}. \tag{10}$$

After substitution of (8) and (9) into (10), we obtain

$$\mathcal{H}_{\mathrm{d}}^{-}(\lambda_k, x_{k+1}) = -\frac{1}{2}x_{k+1}^{\mathrm{T}} C x_{k+1} - \lambda_k^{\mathrm{T}} D x_{k+1} + \frac{1}{2}\lambda_k^{\mathrm{T}} E \lambda_k \tag{11}$$

where the coefficient matrices C, D, and $E \in \mathbb{R}^{n\times n}$ are

$$C = -\,(G + AQ^{-1}A^{\mathrm{T}})^{-1} \tag{12}$$

$$D = (A + GA^{-\mathrm{T}}Q)^{-1} \tag{13}$$

$$E = Q^{-1}A^{\mathrm{T}}(G + AQ^{-1}A^{\mathrm{T}})^{-1}AQ^{-1} - Q^{-1}. \tag{14}$$

Remark 1. The coefficient C is symmetric negative definite because A is invertible, Q is symmetric positive definite, and G is symmetric positive semi-definite in (12). We can find the item $A + GA^{-\mathrm{T}}Q$ in (13) is invertible by rewriting it as $(AQ^{-1}A^{\mathrm{T}} + G)A^{-\mathrm{T}}Q$. Furthermore, it can be found that the coefficient E is symmetric negative semi-definite by rewriting the right hand side of (14) as $-(A + GA^{-\mathrm{T}}Q)^{-1}(GA^{-\mathrm{T}}QA^{-1}G + G)(A^{\mathrm{T}} + QA^{-1}G)^{-1}$.

The necessary conditions for optimality can be represented by the right Hamiltonian system (4), (5), and (7) or the left Hamiltonian system (8), (9), and (11), evaluating the optimal trajectory of HCP corresponds to solving either of these two systems satisfying the fixed (initial and terminal) BCs $x_0 = x^0$ and $x_N = x^N$. Hence the HCP is reduced to a two point boundary value problem (TPBVP).

3.2 Necessary Conditions for Optimizing SCP

According to the discrete minimum principle, the additional condition (transversality condition)

$$\lambda_N = D_1\phi(x_N) = Px_N \tag{15}$$

is given with (4) and (5) to constitute the first order necessary conditions for optimizing the SCP in terms of the right Hamiltonian. Naturally, (8) and (9) with (15) constitute the necessary conditions for optimizing the SCP in terms of the left Hamiltonian.

Accordingly, evaluating the optimal trajectory of the SCP corresponds to solving the right or left Hamilton's equations satisfying (15) and $x_0 = x^0$. Except the initial BC, (15) which relates x_N and λ_N provides another BC for the SCP, so the SCP is also reduced to a TPBVP.

4. FORWARD AND BACKWARD GENERATING FUNCTIONS

In this section, after a general description of the generating function in Section 4.1, we develop the forward generating function in Section 4.2 and re-derive the backward generating function [Lee, 2012] in Section 4.3. Both of these are the bases to construct the double generating functions.

4.1 General Description

In the TPBVP, both the right and the left discrete Hamilton's equations define the same dynamics $\{(x_k, \lambda_k)\}_{k=0}^N$ which is the sequence of the optimal state-costate. A generating function specifies the family of canonical transformations that describe the dynamics $\{(x_k, \lambda_k)\}_{k=0}^N$ under the condition that if this generating function is a solution of the HJE [Ohsawa et al., 2011]. The coordinate transformation from the initial to the current, $(x_0, \lambda_0) \mapsto (x_k, \lambda_k)$, is the canonical transformation [Goldstein et al., 2001]. The type II and III [2] forward generating functions $F_{2\mathrm{f}}(x_k, \lambda_0, k)$ and $F_{3\mathrm{f}}(\lambda_k, x_0, k)$ are the right tools to specify the family of these canonical transformations $(0 \leqslant k \leqslant N)$ by the relations

$$\lambda_k = D_1F_{2\mathrm{f}}(x_k, \lambda_0, k) \tag{16}$$

$$x_0 = D_2F_{2\mathrm{f}}(x_k, \lambda_0, k) \tag{17}$$

and

$$x_k = -\,D_1F_{3\mathrm{f}}(\lambda_k, x_0, k) \tag{18}$$

$$\lambda_0 = -\,D_2F_{3\mathrm{f}}(\lambda_k, x_0, k) \tag{19}$$

respectively. Furthermore, $F_{2\mathrm{f}}(x_k, \lambda_0, k)$ and $F_{3\mathrm{f}}(\lambda_k, x_0, k)$ satisfy the corresponding left and right discrete HJEs [3] [Ohsawa et al., 2011]

$$\begin{aligned} F_{2\mathrm{f}}(x_{k+1}, \lambda_0, k+1) =& F_{2\mathrm{f}}(x_k, \lambda_0, k) - D_1F_{2\mathrm{f}}(x_k, \lambda_0, k)x_k \\ &- \mathcal{H}_{\mathrm{d}}^{-}\big(D_1F_{2\mathrm{f}}(x_k, \lambda_0, k), x_{k+1}\big) \end{aligned} \tag{20}$$

$$\begin{aligned} F_{3\mathrm{f}}(\lambda_{k+1}, x_0, k+1) =& F_{3\mathrm{f}}(\lambda_k, x_0, k) - \lambda_k^{\mathrm{T}}D_1F_{3\mathrm{f}}(\lambda_k, x_0, k) \\ &- \mathcal{H}_{\mathrm{d}}^{+}\big(-D_1F_{3\mathrm{f}}(\lambda_k, x_0, k), \lambda_{k+1}\big) \end{aligned} \tag{21}$$

respectively.

The type II and III backward generating functions $F_{2\mathrm{b}}(x_k, \lambda_N, k)$ and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$ are the tools to specify the family of the canonical coordinate transformation from the terminal to the current, $(x_N, \lambda_N) \mapsto (x_k, \lambda_k)$ $(0 \leqslant k \leqslant N)$ by the relations

$$\lambda_k = D_1F_{2\mathrm{b}}(x_k, \lambda_N, k) \tag{22}$$

$$x_N = D_2F_{2\mathrm{b}}(x_k, \lambda_N, k) \tag{23}$$

and

$$x_k = -\,D_1F_{3\mathrm{b}}(\lambda_k, x_N, k) \tag{24}$$

$$\lambda_N = -\,D_2F_{3\mathrm{b}}(\lambda_k, x_N, k) \tag{25}$$

respectively. Moreover, $F_{2\mathrm{b}}(x_k, \lambda_N, k)$ and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$ satisfy the corresponding right and left discrete HJEs

$$\begin{aligned} F_{2\mathrm{b}}(x_{k-1}, \lambda_N, k-1) =& F_{2\mathrm{b}}(x_k, \lambda_N, k) - D_1F_{2\mathrm{b}}(x_k, \lambda_N, k)x_k \\ &+ \mathcal{H}_{\mathrm{d}}^{+}\big(x_{k-1}, D_1F_{2\mathrm{b}}(x_k, \lambda_N, k)\big) \end{aligned} \tag{26}$$

$$\begin{aligned} F_{3\mathrm{b}}(\lambda_{k-1}, x_N, k-1) =& F_{3\mathrm{b}}(\lambda_k, x_N, k) - \lambda_k^{\mathrm{T}}D_1F_{3\mathrm{b}}(\lambda_k, x_N, k) \\ &+ \mathcal{H}_{\mathrm{d}}^{-}\big(\lambda_{k-1}, -D_1F_{3\mathrm{b}}(\lambda_k, x_N, k)\big) \end{aligned} \tag{27}$$

respectively.

4.2 Forward Generating Functions

We give the following proposition to present the explicit expressions of $F_{2\mathrm{f}}(x_k, \lambda_0, k)$ and $F_{3\mathrm{f}}(\lambda_k, x_0, k)$ by solving the corresponding HJEs (20) and (21), respectively.

Proposition 2.

(i) The type II forward generating function has the expression of

$$\begin{aligned} F_{2\mathrm{f}}(x_k, \lambda_0, k) =& \frac{1}{2}x_k^{\mathrm{T}}X_{2\mathrm{f},k}x_k + \lambda_0^{\mathrm{T}}Y_{2\mathrm{f},k}x_k \\ &+ \frac{1}{2}\lambda_0^{\mathrm{T}}Z_{2\mathrm{f},k}\lambda_0 \end{aligned} \tag{28}$$

where the coefficient matrices $X_{2\mathrm{f},k} = X_{2\mathrm{f},k}^{\mathrm{T}}$, $Y_{2\mathrm{f},k}$, and $Z_{2\mathrm{f},k} = Z_{2\mathrm{f},k}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ $(0 \leqslant k \leqslant N)$, satisfy the following forward recurrence relations [4]

[2] The definition of the type of generating functions is given by Goldstein et al. [2001].

[3] The generating function satisfies both the right and left discrete HJEs which are in terms of the right and left discrete Hamiltonian, respectively. It is free of us to obtain the generating function by solving any kind of HJE.

[4] It should be noted that all the inverse items in this paper are invertible. This will be presented in the further work.

$$X_{2\mathrm{f},k+1} = D^{\mathrm{T}}(I + X_{2\mathrm{f},k}E)^{-1}X_{2\mathrm{f},k}D + C \tag{29}$$
$$Y_{2\mathrm{f},k+1} = Y_{2\mathrm{f},k}(I + EX_{2\mathrm{f},k})^{-1}D \tag{30}$$
$$Z_{2\mathrm{f},k+1} = Z_{2\mathrm{f},k} - Y_{2\mathrm{f},k}(I + EX_{2\mathrm{f},k})^{-1}EY_{2\mathrm{f},k}^{\mathrm{T}} \tag{31}$$

with the initial conditions $X_{2\mathrm{f},0} = 0$, $Y_{2\mathrm{f},0} = I$, and $Z_{2\mathrm{f},0} = 0$, respectively, where $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix.

(ii) The type III forward generating function has the expression of

$$F_{3\mathrm{f}}(\lambda_k, x_0, k) = \frac{1}{2}\lambda_k^{\mathrm{T}}X_{3\mathrm{f},k}\lambda_k + x_0^{\mathrm{T}}Y_{3\mathrm{f},k}\lambda_k + \frac{1}{2}x_0^{\mathrm{T}}Z_{3\mathrm{f},k}x_0 \tag{32}$$

where the coefficient matrices $X_{3\mathrm{f},k} = X_{3\mathrm{f},k}^{\mathrm{T}}$, $Y_{3\mathrm{f},k}$, and $Z_{3\mathrm{f},k} = Z_{3\mathrm{f},k}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ $(0 \leqslant k \leqslant N)$, satisfy the following forward recurrence relations

$$X_{3\mathrm{f},k+1} = A(I + X_{3\mathrm{f},k}Q)^{-1}X_{3\mathrm{f},k}A^{\mathrm{T}} + G \tag{33}$$
$$Y_{3\mathrm{f},k+1} = Y_{3\mathrm{f},k}(I + QX_{3\mathrm{f},k})^{-1}A^{\mathrm{T}} \tag{34}$$
$$Z_{3\mathrm{f},k+1} = Z_{3\mathrm{f},k} - Y_{3\mathrm{f},k}(I + QX_{3\mathrm{f},k})^{-1}QY_{3\mathrm{f},k}^{\mathrm{T}} \tag{35}$$

with the initial conditions $X_{3\mathrm{f},0} = 0$, $Y_{3\mathrm{f},0} = -I$, and $Z_{3\mathrm{f},0} = 0$, respectively.

Proof.

(i) Type II $F_{2\mathrm{f}}(x_k, \lambda_0, k)$ takes the form of quadratic as (28) [Ohsawa et al., 2011]. From (16) and (17), we have $F_{2\mathrm{f}}(x_k, \lambda_0, k)|_{k=0} = \lambda_0^{\mathrm{T}}x_0$, which yields the given initial conditions $X_{2\mathrm{f},0} = 0$, $Y_{2\mathrm{f},0} = I$, and $Z_{2\mathrm{f},0} = 0$. Then, we solve the left discrete HJE (20) to obtain the explicit expression of $F_{2\mathrm{f}}(x_k, \lambda_0, k)$. It is clear that (20) is a function in terms of three variables x_k, x_{k+1}, and λ_0. Thus, we first eliminate λ_k in (8) by (16) so that x_k can be expressed by a function in terms of x_{k+1} and λ_0. Then, replacing the x_k in (20) by this function, we obtain a new quadratic equation only in terms of x_{k+1} and λ_0. Since this new quadratic equation should be satisfied for any x_{k+1} and λ_0, we get the forward recurrence relations (29)–(31). Further, since (29) is a discrete-time Riccati equation, by considering the initial $X_{2\mathrm{f},0} = 0$ and $Z_{2\mathrm{f},0} = 0$, we know that both $X_{2\mathrm{f},k}$ and $Z_{2\mathrm{f},k}$ are symmetric.

(ii) The proof here is similar to the part (i) by solving the corresponding right discrete HJE (21).

4.3 Backward Generating Functions

We give the following proposition without proof to present the expressions of $F_{2\mathrm{b}}(x_k, \lambda_N, k)$ and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$ by solving the corresponding HJEs (26) and (27), respectively.

Proposition 3.

(i) The type II backward generating function has the expression of

$$F_{2\mathrm{b}}(x_k, \lambda_N, k) = \frac{1}{2}x_k^{\mathrm{T}}X_{2\mathrm{b},k}x_k + \lambda_N^{\mathrm{T}}Y_{2\mathrm{b},k}x_k + \frac{1}{2}\lambda_N^{\mathrm{T}}Z_{2\mathrm{b},k}\lambda_N \tag{36}$$

where the coefficient matrices $X_{2\mathrm{b},k} = X_{2\mathrm{b},k}^{\mathrm{T}}$, $Y_{2\mathrm{b},k}$, and $Z_{2\mathrm{b},k} = Z_{2\mathrm{b},k}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ $(0 \leqslant k \leqslant N)$, satisfy the following backward recurrence relations

$$X_{2\mathrm{b},k-1} = A^{\mathrm{T}}(I + X_{2\mathrm{b},k}G)^{-1}X_{2\mathrm{b},k}A + Q \tag{37}$$
$$Y_{2\mathrm{b},k-1} = Y_{2\mathrm{b},k}(I + GX_{2\mathrm{b},k})^{-1}A \tag{38}$$
$$Z_{2\mathrm{b},k-1} = Z_{2\mathrm{b},k} - Y_{2\mathrm{b},k}(I + GX_{2\mathrm{b},k})^{-1}GY_{2\mathrm{b},k}^{\mathrm{T}} \tag{39}$$

with the terminal conditions $X_{2\mathrm{b},N} = 0$, $Y_{2\mathrm{b},N} = I$, and $Z_{2\mathrm{b},N} = 0$, respectively.

(ii) The type III backward generating function has the expression of

$$F_{3\mathrm{b}}(\lambda_k, x_N, k) = \frac{1}{2}\lambda_k^{\mathrm{T}}X_{3\mathrm{b},k}\lambda_k + x_N^{\mathrm{T}}Y_{3\mathrm{b},k}\lambda_k + \frac{1}{2}x_N^{\mathrm{T}}Z_{3\mathrm{b},k}x_N \tag{40}$$

where the coefficient matrices $X_{3\mathrm{b},k} = X_{3\mathrm{b},k}^{\mathrm{T}}$, $Y_{3\mathrm{b},k}$, and $Z_{3\mathrm{b},k} = Z_{3\mathrm{b},k}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ $(0 \leqslant k \leqslant N)$, satisfy the following backward recurrence relations

$$X_{3\mathrm{b},k-1} = D(I + X_{3\mathrm{b},k}C)^{-1}X_{3\mathrm{b},k}D^{\mathrm{T}} + E \tag{41}$$
$$Y_{3\mathrm{b},k-1} = Y_{3\mathrm{b},k}(I + CX_{3\mathrm{b},k})^{-1}D^{\mathrm{T}} \tag{42}$$
$$Z_{3\mathrm{b},k-1} = Z_{3\mathrm{b},k} - Y_{3\mathrm{b},k}(I + CX_{3\mathrm{b},k})^{-1}CY_{3\mathrm{b},k}^{\mathrm{T}} \tag{43}$$

with the terminal conditions $X_{3\mathrm{b},N} = 0$, $Y_{3\mathrm{b},N} = -I$, and $Z_{3\mathrm{b},N} = 0$, respectively.

5. OPTIMAL SOLUTIONS VIA DOUBLE GENERATING FUNCTIONS

This section gives the main results of this paper. Based on the preceding single generating functions, we develop the double generating functions method for the HCP and SCP in this section. In light of this approach, the optimal solutions for the HCP and SCP can be expressed as the functions only in terms of the BCs and the pre-computed coefficients, which are exhibited in Section 5.1 and 5.2, respectively. In view of this, the optimal trajectories can be efficiently generated for the iterative on-line computation.

5.1 Optimal Solutions for HCP

Six kinds of double generating functions can be constructed by the choice of any two different single generating functions among $F_{2\mathrm{f}}(x_k, \lambda_0, k)$, $F_{3\mathrm{f}}(\lambda_k, x_0, k)$, $F_{2\mathrm{b}}(x_k, \lambda_N, k)$, and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$. Since double generating functions with same time directions ($F_{2\mathrm{f}}(x_k, \lambda_0, k)$ and $F_{3\mathrm{f}}(\lambda_k, x_0, k)$, $F_{2\mathrm{b}}(x_k, \lambda_N, k)$ and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$) will cause instabilities when generate optimal trajectories as the time steps increase [Hao et al., 2013c]. Therefore, we here only use double generating functions with different time directions to obtain the optimal solutions by the following theorem.

Theorem 4. Suppose the following

(i) The matrices $X_{2\mathrm{f},k}$, $Y_{2\mathrm{f},k}$, and $Z_{2\mathrm{f},k} \in \mathbb{R}^{n\times n}$ $(0 \leqslant k \leqslant N)$, satisfy (29)–(31) with the initial conditions $X_{2\mathrm{f},0} = 0$, $Y_{2\mathrm{f},0} = I$, and $Z_{2\mathrm{f},0} = 0$, respectively.

(ii) The matrices $X_{3\mathrm{f},k}$, $Y_{3\mathrm{f},k}$, and $Z_{3\mathrm{f},k} \in \mathbb{R}^{n\times n}$ $(0 \leqslant k \leqslant N)$, satisfy Eqs. (33)–(35) with the initial conditions $X_{3\mathrm{f},0} = 0$, $Y_{3\mathrm{f},0} = -I$, and $Z_{3\mathrm{f},0} = 0$, respectively.

(iii) The matrices $X_{2\mathrm{b},k}$, $Y_{2\mathrm{b},k}$, and $Z_{2\mathrm{b},k} \in \mathbb{R}^{n\times n}$ $(0 \leqslant k \leqslant N)$, satisfy (37)–(39) with the terminal conditions $X_{2\mathrm{b},N} = 0$, $Y_{2\mathrm{b},N} = I$, and $Z_{2\mathrm{b},N} = 0$, respectively.

(iv) The matrices $X_{3\mathrm{b},k}$, $Y_{3\mathrm{b},k}$, and $Z_{3\mathrm{b},k} \in \mathbb{R}^{n\times n}$ $(0 \leqslant k \leqslant N)$, satisfy (41)–(43) with the terminal conditions $X_{3\mathrm{b},N} = 0$, $Y_{3\mathrm{b},N} = -I$, and $Z_{3\mathrm{b},N} = 0$, respectively.

Then with the fixed initial and terminal BCs, the optimal state x_k^* $(1 \leqslant k \leqslant N-1)$ and input u_k^* $(0 \leqslant k \leqslant N-1)$ for the HCP can be generated by

$$\begin{bmatrix} x_k^* \\ u_k^* \end{bmatrix} = \begin{bmatrix} X_{3\mathrm{b},k}(X_{3\mathrm{f},k} - X_{3\mathrm{b},k})^{-1}Y_{3\mathrm{f},k}^{\mathrm{T}}, & -X_{3\mathrm{f},k}(X_{3\mathrm{f},k} - X_{3\mathrm{b},k})^{-1}Y_{3\mathrm{b},k}^{\mathrm{T}} \\ M(X_{3\mathrm{f},k+1} - X_{3\mathrm{b},k+1})^{-1}Y_{3\mathrm{f},k+1}^{\mathrm{T}}, & -M(X_{3\mathrm{f},k+1} - X_{3\mathrm{b},k+1})^{-1}Y_{3\mathrm{b},k+1}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_0 \\ x_N \end{bmatrix} \tag{44}$$

or

$$\begin{bmatrix} x_k^* \\ u_k^* \end{bmatrix} = \begin{bmatrix} -(I + X_{3\mathrm{f},k}X_{2\mathrm{b},k})^{-1}Y_{3\mathrm{f},k}^{\mathrm{T}}, & -(I + X_{3\mathrm{f},k}X_{2\mathrm{b},k})^{-1}X_{3\mathrm{f},k}Y_{2\mathrm{b},k}^{\mathrm{T}} \\ M(I + X_{2\mathrm{b},k+1}X_{3\mathrm{f},k+1})^{-1}X_{2\mathrm{b},k+1}Y_{3\mathrm{f},k+1}^{\mathrm{T}}, & -M(I + X_{2\mathrm{b},k+1}X_{3\mathrm{f},k+1})^{-1}Y_{2\mathrm{b},k+1}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda_N \end{bmatrix} \tag{45}$$

where $\lambda_N = -X_{3\mathrm{f},N}^{-1}(x_N + Y_{3\mathrm{f},N}^{\mathrm{T}}x_0)$, or

$$\begin{bmatrix} x_k^* \\ u_k^* \end{bmatrix} = \begin{bmatrix} -(I + X_{3\mathrm{b},k}X_{2\mathrm{f},k})^{-1}X_{3\mathrm{b},k}Y_{2\mathrm{f},k}^{\mathrm{T}}, & -(I + X_{3\mathrm{b},k}X_{2\mathrm{f},k})^{-1}Y_{3\mathrm{b},k}^{\mathrm{T}} \\ -M(I + X_{2\mathrm{f},k+1}X_{3\mathrm{b},k+1})^{-1}Y_{2\mathrm{f},k+1}^{\mathrm{T}}, & M(I + X_{2\mathrm{f},k+1}X_{3\mathrm{b},k+1})^{-1}X_{2\mathrm{f},k+1}Y_{3\mathrm{b},k+1}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \lambda_0 \\ x_N \end{bmatrix} \tag{46}$$

where $\lambda_0 = -X_{3\mathrm{b},0}^{-1}(x_0 + Y_{3\mathrm{b},0}^{\mathrm{T}}x_N)$, or

$$\begin{bmatrix} x_k^* \\ u_k^* \end{bmatrix} = \begin{bmatrix} (X_{2\mathrm{b},k} - X_{2\mathrm{f},k})^{-1}Y_{2\mathrm{f},k}^{\mathrm{T}}, & -(X_{2\mathrm{b},k} - X_{2\mathrm{f},k})^{-1}Y_{2\mathrm{b},k}^{\mathrm{T}} \\ -MX_{2\mathrm{b},k+1}(X_{2\mathrm{b},k+1} - X_{2\mathrm{f},k+1})^{-1}Y_{2\mathrm{f},k+1}^{\mathrm{T}}, & MX_{2\mathrm{f},k+1}(X_{2\mathrm{b},k+1} - X_{2\mathrm{f},k+1})^{-1}Y_{2\mathrm{b},k+1}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_N \end{bmatrix} \tag{47}$$

where $\lambda_0 = Z_{2\mathrm{f},N}^{-1}(x_0 - Y_{2\mathrm{f},N}x_N)$ and $\lambda_N = Z_{2\mathrm{b},0}^{-1}(x_N - Y_{2\mathrm{b},0}x_0)$.

Proof. The optimal solutions can be generated via the double generating functions constructed by $F_{3\mathrm{f}}(\lambda_k, x_0, k)$ and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$. Eliminating λ_k in (18) by (24) yields the x_k^* $(1 \leqslant k \leqslant N-1)$ in (44) for the HCP. Moreover, eliminating x_k in (18) by (24) and substituting this new equation (changing the indices from k to $k+1$) into (6) leads to the u_k^* $(0 \leqslant k \leqslant N-1)$ in (44) for the HCP. We can also derive the optimal solutions in (45)–(47) based on the other three double generating functions by the similar way. In addition, we get the calculation for λ_0 by letting $k=0$ in (24) and λ_N by letting $k=N$ in (18), or λ_0 by letting $k=N$ in (17) and λ_N by letting $k=0$ in (23).

5.2 Optimal Solutions for SCP

Since x_0 and the transversality condition (15) which relates x_N and λ_N are given in the SCP, single ones $F_{3\mathrm{f}}(\lambda_k, x_0, k)$, $F_{2\mathrm{b}}(x_k, \lambda_N, k)$, and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$ are the most appropriate candidates to construct double generating functions for the SCP. Again, we only consider double generating functions constructed by the single ones with different time directions, i.e., $F_{3\mathrm{f}}(\lambda_k, x_0, k)$ and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$, and $F_{3\mathrm{f}}(\lambda_k, x_0, k)$ and $F_{2\mathrm{b}}(x_k, \lambda_N, k)$, to generate the optimal solutions for the SCP by the following theorem.

Theorem 5. Suppose that the coefficients of $F_{3\mathrm{f}}(\lambda_k, x_0, k)$, $F_{2\mathrm{b}}(x_k, \lambda_N, k)$, and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$ $(0 \leqslant k \leqslant N)$ satisfy the corresponding recurrence relations with the known BCs, respectively. Then with the fixed initial BC, the optimal state x_k^* $(1 \leqslant k \leqslant N)$ and input u_k^* $(0 \leqslant k \leqslant N-1)$ for the SCP can be generated by

$$x_k^* = -V_k(V_k + X_{3\mathrm{f},k})^{-1}Y_{3\mathrm{f},k}^{\mathrm{T}}x_0 \tag{48}$$

$$u_k^* = M(V_{k+1} + X_{3\mathrm{f},k+1})^{-1}Y_{3\mathrm{f},k+1}^{\mathrm{T}}x_0 \tag{49}$$

where $V_k = Y_{3\mathrm{b},k}^{\mathrm{T}}(P + Z_{3\mathrm{b},k})^{-1}Y_{3\mathrm{b},k} - X_{3\mathrm{b},k}$, or

$$x_k^* = -(I + X_{3\mathrm{f},k}W_k)^{-1}Y_{3\mathrm{f},k}^{\mathrm{T}}x_0 \tag{50}$$

$$u_k^* = MW_{k+1}(I + X_{3\mathrm{f},k+1}W_{k+1})^{-1}Y_{3\mathrm{f},k+1}^{\mathrm{T}}x_0 \tag{51}$$

where $W_k = X_{2\mathrm{b},k} + Y_{2\mathrm{b},k}^{\mathrm{T}}P(I - Z_{2\mathrm{b},k}P)^{-1}Y_{2\mathrm{b},k}$.

Proof. The optimal solutions can be generated based on double generating functions constructed by $F_{3\mathrm{f}}(\lambda_k, x_0, k)$ and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$. First, we rewrite (24) as $x_k = V_k\lambda_k$ by the help of (15) and (25), where $V_k = Y_{3\mathrm{b},k}^{\mathrm{T}}(P + Z_{3\mathrm{b},k})^{-1}Y_{3\mathrm{b},k} - X_{3\mathrm{b},k}$. Then based on this new relation and (18), eliminating the λ_k yields x_k^* (48), and eliminating the x_k (changing the indices from k to $k+1$) and substituting into (6) leads to u_k^* (49). This can also be achieved by selecting $F_{3\mathrm{f}}(\lambda_k, x_0, k)$ and $F_{2\mathrm{b}}(x_k, \lambda_N, k)$. Similarly, first rewrite (22) as $\lambda_k = W_kx_k$, where $W_k = X_{2\mathrm{b},k} + Y_{2\mathrm{b},k}^{\mathrm{T}}P(I - Z_{2\mathrm{b},k}P)^{-1}Y_{2\mathrm{b},k}$. Then based on this new equation, (18), and (6), we obtain (50) and (51).

6. EXAMPLES

To present the effectiveness of the developed method for the on-line iterative computation, we give two examples, one for the HCP and the other for the SCP.

Example 6. Consider the problem (1)–(3) in the case of HCP with

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix},\ Q = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix},\ R = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

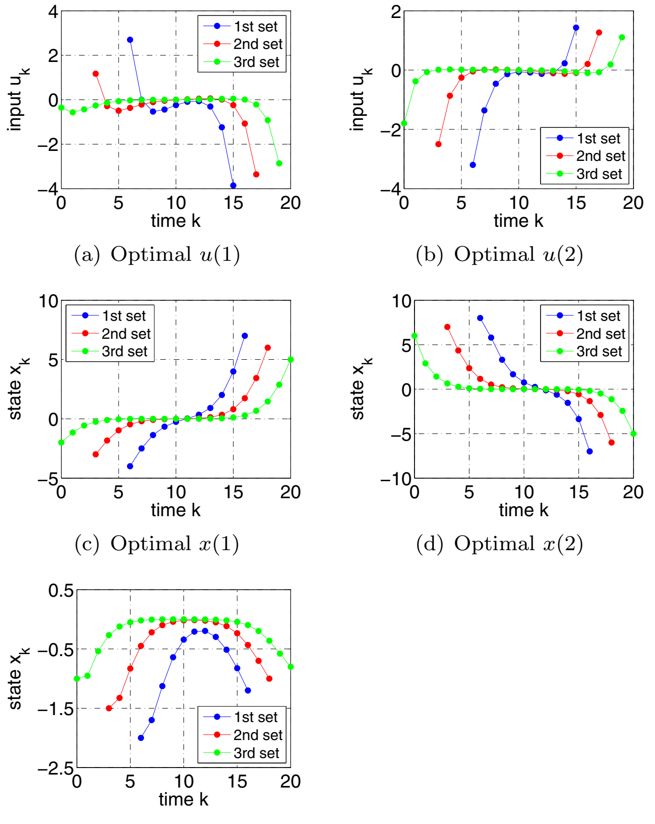
Let the initial and terminal BCs, (k_0, x_0) and (k_N, x_N), jointly be three different sets as in Table 1, where k_0 and $k_N \in \mathbb{N}$ $(0 \leqslant k_0 \leqslant k_N \leqslant N)$ denote the initial and terminal time, respectively. The objective is to generate the optimal input and state satisfying each set of BCs in Table 1.

Table 1. Three sets of BCs

	Initial (k_0, x_0)	Terminal (k_N, x_N)
1st set	$(6, [-4, 8, -2]^{\mathrm{T}})$	$(16, [7, -7, -1.2]^{\mathrm{T}})$
2nd set	$(3, [-3, 7, -1.5]^{\mathrm{T}})$	$(18, [6, -6, -1]^{\mathrm{T}})$
3rd set	$(0, [-2, 6, -1]^{\mathrm{T}})$	$(20, [5, -5, -0.8]^{\mathrm{T}})$

The generation equations (44)–(47) are all in terms of the BCs and the pre-computed coefficients. Therefore, we can divide the whole computation into two parts, i.e., off-line part and on-line part. Here, we take the generation equation (44) as example. First, in the off-line part, we choose the time interval as the maximum $[0, 20]$ according to Table 1 to calculate the coefficients of the $F_{3\mathrm{f}}(\lambda_k, x_0, k)$ and $F_{3\mathrm{b}}(\lambda_k, x_N, k)$. Then, in the on-line part, we can efficiently generate the optimal input and state satisfying each different set of the BCs in Table 1 by (44). Fig. 1 shows the result in which the optimal inputs corresponding to the three different sets of BCs are presented by Fig. 1(a) (first element $u(1)$) and 1(b) (second element $u(2)$), and the optimal states corresponding to the three sets of BCs are presented by Fig. 1(c) (first element $x(1)$), 1(d) (second element $x(2)$), and 1(e) (third element $x(3)$). It is clear that the trajectory of the optimal state $\big(x(1), x(2), x(3)\big)$ satisfy the BCs in Table 1.

Example 7. Consider the problem (1)–(3) in the case of SCP with the same A, B, Q, and R in Example 6 and the

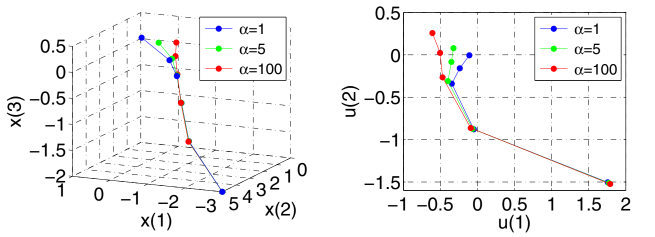


(a) Optimal $u(1)$ (b) Optimal $u(2)$

(c) Optimal $x(1)$ (d) Optimal $x(2)$

(e) Optimal $x(3)$

Fig. 1. Optimal input and state for the three sets of BCs



(a) Optimal state (b) Optimal input

Fig. 2. Optimal input and state for the three values of P

fixed initial state $x_0 = (-3, 5, -2)^{\mathrm{T}}$ over a fixed number of time steps $N = 5$. To indirectly adjust the terminal state to approach the origin, we usually try a number of different terminal weighting factors by experience and compare the optimal results to the designated goal. Based on this, we can choose the most satisfactory factor P. Here, we set the factor P as three different values: $P = \alpha P_{\mathrm{v}}$, $\alpha = 1, 5, 100$,

$$P_{\mathrm{v}} = \begin{bmatrix} 0.1 & 0.2 & -0.3 \\ 0.2 & 1 & -1.6 \\ -0.3 & -1.6 & 2.8 \end{bmatrix}.$$

Similarly, after off-line computation for the coefficients of the double generating functions, we can efficiently generate the optimal state and input corresponding to different terminal factors by (48) and (49), or (50) and (51).

The result is presented in Fig. 2. We can see that the trajectories of the optimal state satisfy the fixed initial BC and have different terminal values corresponding to different weighting factors. It is clear that the terminal state of the third (red) trajectory, which approaches the origin of the frame, corresponds to the most satisfactory factor P.

By these two examples, it is shown that the developed double generating functions method decreases the on-line iterative computational burden and can be useful in practice for the HCP and SCP, especially when the number of iterations is large.

7. CONCLUSION

This paper develops the double generating functions method for the HCP and SCP in the discrete-time case. Since the optimal solutions can be expressed only in terms of the BCs and pre-computed coefficients by this method, we can compute the coefficients in the off-line part and generate the optimal solutions in the on-line part by simple algebraic manipulations. In view of this, the developed method is useful in the on-line iterative computation for optimal control problem with a number of different BCs. The prospect of the double generating functions method for the discrete-time nonlinear optimal control is also bright.

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