Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

# Control over Additive White Gaussian Noise Channels: Bode-Type Integrals, Channel Blurredness, Negentropy Rate, and Beyond ⋆

**Song FANG** * **Hideaki ISHII** ** **Jie CHEN** ***

* *Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China, (e-mail: fang.song@my.cityu.edu.hk)*
** *Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, 4259 Nagatsuta-cho, Midori-ku, Yokohama 226-8502, Japan, (e-mail: ishii@dis.titech.ac.jp)*
*** *Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China, and Department of Electrical Engineering, University of California, Riverside, CA 92521 USA (e-mail: jichen@cityu.edu.hk)*

**Abstract:** This paper aims at developing Bode-type integrals for control systems over additive white Gaussian noise channels, by way of deriving information theoretic equalities and inequalities. The integrals characterize the fundamental performance trade-offs of such networked feedback systems with linear time-invariant plants and causal stabilizing controllers. We propose two new notions to facilitate our development: channel blurredness and negentropy rate. The channel blurredness provides an alternative measure for the quality of communication channels to the conventional notion of channel capacity. The negentropy rate, on the other hand, relates the entropy rate of a stochastic process to its power spectrum. Both notions are shown to be closely relevant to networked feedback systems. Indeed, the Bode-type integrals developed herein are seen to depend on the channel blurredness of the communication channel, as well as the negentropy of the exogenous disturbances.

*Keywords:* Bode-type integrals, information theoretic analysis, channel blurredness, negentropy rate, networked feedback systems.

## 1. INTRODUCTION

Networked control [Antsaklis and Baillieul (2007)] refers to such a setting in which the measurement and control signals are transmitted over certain communication channels (uplink channel and downlink channel, respectively) to and from the plant's sensors and actuators. Figure 1 shows a typical configuration of networked feedback systems. The sheer nature of networked feedback systems thus requires the reconciliation and integration of communications and control technologies. Accordingly, analysis and design of networked feedback systems call for understanding of the interplay between information and control theories.

Bode sensitivity integral relation [Bode (1945)] is one of the most important results on the fundamental limitations of control systems. While various extensions have been found for different systems [Sung and Hara (1988); Freudenberg and Looze (1988); Chen (1995); Chen and Nett (1995); Zang and Iglesias (2003)], this result cannot be readily applied to networked feedback systems due to inherent communication constraints on channel capacity, data rate, signal-to-noise ratio (SNR), etc. That a networked feedback system often exhibits nonlinear, time-varying behavior also poses a formidable barrier.

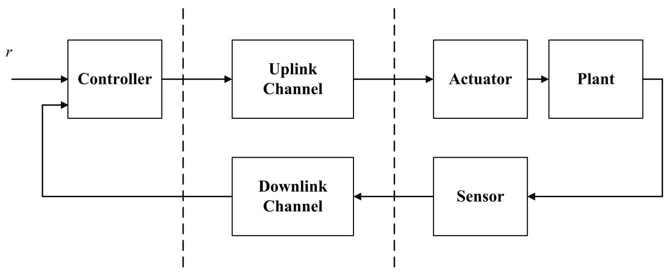


Fig. 1. A networked feedback system

Martins and co-workers [Martins et al. (2007); Martins and Dahleh (2008)] recently conducted an information-theoretic study into single-input and single-output (SISO) networked feedback systems, which has led to Bode-type integral inequalities incorporating channel capacity. Subsequently, [Okano et al. (2008); Ishii et al. (2009)] obtained similar results for multiple-input and multiple-output systems, and for more general cases, allowing the

⋆ This research was supported in part by the Hong Kong RGC under the project CityU 111810, CityU 111511, by the City University of Hong Kong under Project 9380054, and by the Aihara Project, the FIRST program from JSPS, initiated by CSTP.

presence of uplink and downlink simultaneously. Ensuing works sought after extensions to nonlinear systems [Yu and Mehta (2010)] and stochastic switched systems [Li and Hovakimyan (2013)]. Applications of these integrals were also made to molecular fluctuations analysis [Lestas et al. (2010)] and vehicle platoon control systems [Zhao et al. (2014)]. In all these works, concepts and results from information theory have played a central role, which shed new light on the study of networked feedback systems.

This paper seeks to develop Bode-type integrals with respect to channels of more specific structures of which we consider additive white Gaussian noise (AWGN) channels. Section 2 introduces the necessary notations and preliminaries. Section 3 gives the definitions of two newly proposed notions: channel blurredness and negentropy rate. Necessary interpretations and specifications are also provided. In Section 4, Bode-type integrals are obtained for control systems over AWGN channels with linear time-invariant (LTI) plants and causal stabilizing controllers. Our results differ from those of [Martins and Dahleh (2008)] in the following aspects. First, by introducing the notion of negentropy rate, we are able to accommodate non-Gaussian disturbances. Second, we use our newly proposed concept of channel blurredness to characterize the channel quality and prove that it is indeed relevant. Finally, our integrals concern the entire frequency range, and thus are more appropriate for addressing performance trade-off issues.

## 2. NOTATIONS AND PRELIMINARIES

In this section we collect some key definitions and preliminary results from information theory ([Pinsker (1964); Papoulis and Pillai (2002); Cover and Thomas (2006)]). We consider real-valued continuous random variables and discrete-time stochastic processes. The logarithm used in this paper is that with base 2, and all the integrals herein are over appropriate sets of the variables.

**Definition 2.1**. *The differential entropy of a random variable $a$ with density function $f(x)$ is defined as*

$$h(a) = -\int f(x)\log f(x)\,\mathrm{d}x.$$

**Definition 2.2**. *The differential entropy of a set of random variables $a_1, a_2, \ldots, a_k$ with joint density function $f(x_1, x_2, \ldots, x_k)$ is defined as*

$$h(a_1, a_2, \ldots, a_k) = -\int f(x_1, x_2, \ldots, x_k)\log f(x_1, x_2, \ldots, x_k)\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_k.$$

**Definition 2.3**. *The conditional differential entropy of two random variables $a, b$ with joint density $f(x, y)$ and conditional density function $f(x|y)$ is defined as*

$$h(a|b) = -\int f(x, y)\log f(x|y)\,\mathrm{d}x\,\mathrm{d}y.$$

**Definition 2.4**. *The mutual information between two random variables $a, b$ with joint density function $f(x, y)$ is defined as*

$$I(a; b) = \int f(x, y)\log\frac{f(x, y)}{f(x)f(y)}\,\mathrm{d}x\,\mathrm{d}y.$$

**Definition 2.5**. *The entropy rate of a stochastic process $\{a_k\}$ is defined as*

$$h_\infty(a) = \limsup_{k\to\infty}\frac{h(a_{1,\ldots,k})}{k},$$

*where $a_{1,\ldots,k}$ is the abbreviated notation of $a_1, a_2 \ldots, a_k$, and this abbreviation will be adopted throughout this paper.*

**Definition 2.6**. *The (mutual) information rate between two stochastic processes $\{a_k\}, \{b_k\}$ is defined as*

$$I_\infty(a; b) = \limsup_{k\to\infty}\frac{I(a_{1,\ldots,k}; b_{1,\ldots,k})}{k}.$$

**Definition 2.7**. *A zero-mean stochastic process $\{a_k\}$ is asymptotically stationary if the following limit exists for every $k$:*

$$R_a(k) = \lim_{l\to\infty}\mathrm{E}\left[a_{l+k}a_l\right].$$

*For an asymptotically stationary $\{a_k\}$, its asymptotic power spectrum is defined as*

$$S_a(\omega) = \sum_{k=-\infty}^{\infty} R_a(k)\mathrm{e}^{-jk\omega}.$$

**Definition 2.8** ([Hyvärinen and Oja (2000)]). *The negentropy (or negative entropy) of a random variable $a$ is defined as*

$$J(a) = h(a_G) - h(a),$$

*where $a_G$ is a Gaussian variable with the same variance as $a$.*

The following lemma lists the key properties of entropy and mutual information relevant to our subsequent development:

**Lemma 2.1** ([Pinsker (1964); Cover and Thomas (2006)]).
(1) $I(a; b) = I(b; a) = h(a) - h(a|b) = h(b) - h(b|a) \geq 0$, *in which equality holds if and only if $a$ and $b$ are independent.*
(2) $h(a|b) \leq h(a)$, *in which equality holds if and only if $a$ and $b$ are independent.*
(3) $h(a, b) = h(a) + h(b|a)$.
(4) *Suppose that $f$ is a causal function, then $h(a|b) \leq h(a|f(b))$, in which equality holds if and only if $f$ is invertible.*
(5) *Suppose that $f$ is a causal function, then $h(a+f(b)|b) = h(a|b)$, $h(a|b) = h(a|b, f(b))$, and $I(a; b|c) = I(a; b+f(c)|c)$.*
(6) $I(a; b|c) = I(b; a|c) = h(a|c) - h(a|b, c) = h(b|c) - h(b|a, c) \geq 0$, *in which equality holds if and only if $a$ and $b$ are independent given $c$.*
(7) $I(a; b, c) = I(a; b) + I(a; c|b)$, *and* $I(a; b, c|d) = I(a; b|d) + I(a; c|b, d)$.
(8) $I(a; b) \leq I(a; b, c)$, *and* $h(a|b) \geq h(a|b, c)$. *For both, equality holds if and only if $a$ and $c$ are independent given $b$.*
(9) $h(a_{1,\ldots,k}) = \sum_{i=1}^{k} h(a_i|a_{1,\ldots,i-1})$, *and* $h(a_{1,\ldots,k}|b) = \sum_{i=1}^{k} h(a_i|a_{1,\ldots,i-1}, b)$.
(10) $I(a_{1,\ldots,k}; b) = \sum_{i=1}^{k} I(a_i; b|a_{1,\ldots,i-1})$, *and* $I(a_{1,\ldots,k}; b|c) = \sum_{i=1}^{k} I(a_i; b|a_{1,\ldots,i-1}, c)$.

The next lemma introduces the entropy power inequality, which will play an important role in the sequel.

**Lemma 2.2** ([Cover and Thomas (2006)]). *Let $a$ and $b$ be independent random variables, then*

$$2^{2h(a+b)} \geq 2^{2h(a)} + 2^{2h(b)},$$

*in which equality holds if and only if $a$ and $b$ are both Gaussian.*

## 3. CHANNEL BLURREDNESS AND NEGENTROPY RATE

### *3.1 Channel Blurredness*

We consider a general channel as depicted in Figure 2, in which $v$ is the channel input, $u$ the output, and $n$ the noise process. We assume that $n$ does not depend on $v$.

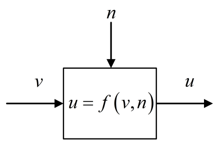


Fig. 2. A channel with input $v$, noise $n$, and output $u$

**Definition 3.1.** *Consider the channel given in Figure 2. The blurredness of the channel (or* ***channel blurredness****) is given by*

$$B \triangleq \min_{v} I\left(n;u\right), \tag{1}$$

*which is measured in bits. The minimum is to be taken over classes of inputs of interest.*

If $u = f(v,n) = v+n$, then the channel in Figure 2 reduces to an additive channel, as shown in Figure 3. An additive channel with $n$ being an white Gaussian noise independent of the channel input $v$ is said to be an AWGN channel.

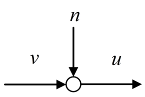


Fig. 3. An additive channel

**Theorem 3.1.** *Consider the AWGN channel depicted in Figure 3 with power constraint $\mathrm{E}v^2 \leq P$. Let $n$ be zero-mean with variance $\sigma_n^2$. Then its channel blurredness is given by*

$$B = \min_{p(v)\colon \mathrm{E}v^2 \leq P} I\left(n;u\right)$$
$$= \frac{1}{2}\log\left(1+\frac{\sigma_n^2}{P}\right) \text{ bits per channel use.} \tag{2}$$

**Proof.** Since $n$ does not depend on $v$, we have

$$I\left(n;u\right) = h(u) - h(u|n) = h(u) - h(v|n) = h(u) - h(v),$$

and

$$I\left(v;u\right) = h(u) - h(u|v) = h(u) - h(n|v) = h(u) - h(n).$$

Using the entropy power inequality,

$$2^{2h(u)} \geq 2^{2h(v)} + 2^{2h(n)},$$

it follows that

$$2^{2[h(v)-h(u)]} + 2^{2[h(n)-h(u)]} \leq 1,$$

and

$$I\left(n;u\right) = h(u) - h(v) \geq -\frac{1}{2}\log\left\{1 - 2^{2[h(n)-h(u)]}\right\}$$
$$= -\frac{1}{2}\log\left[1 - 2^{-2I(v;u)}\right].$$

It is known from [Cover and Thomas (2006)] that $I\left(v;u\right) = h(u) - h(n)$ reaches its maximum

$$C = \max_{p(v)\colon \mathrm{E}v^2 \leq P} I\left(v;u\right) = \frac{1}{2}\log\left(1+\frac{P}{\sigma_n^2}\right),$$

when $v$ is Gaussian and $\mathrm{E}v^2 = P$. In this case,

$$I\left(n;u\right) = h(u) - h(v) = -\frac{1}{2}\log\left\{1 - 2^{2[h(n)-h(u)]}\right\}$$
$$= -\frac{1}{2}\log\left[1 - 2^{-2I(v;u)}\right].$$

Thus, $I\left(n;u\right)$ reaches its minimum. Consequently,

$$B = \min_{p(v)\colon \mathrm{E}v^2 \leq P} I\left(n;u\right)$$
$$= -\frac{1}{2}\log\left[1 - 2^{-2\max_{p(v)\colon \mathrm{E}v^2 \leq P} I(v;u)}\right] = \frac{1}{2}\log\left(1+\frac{\sigma_n^2}{P}\right).$$

□

**Remark 3.1.** It is well-known that the channel capacity of the AWGN channel is given by [Cover and Thomas (2006)]

$$C = \max_{p(v)\colon \mathrm{E}v^2 \leq P} I\left(v;u\right) = \frac{1}{2}\log\left(1+\frac{P}{\sigma_n^2}\right).$$

Using Theorem 3.1, the relationship between the channel blurredness and channel capacity of the AWGN channel can be found as

$$B = -\frac{1}{2}\log\left(1-2^{-2C}\right) = \frac{1}{2}\log\left(1+\frac{1}{2^{2C}-1}\right), \tag{3}$$

or alternatively,

$$2^{-2B} + 2^{-2C} = 1, (2^{2B}-1)(2^{2C}-1) = 1. \tag{4}$$

Clearly, the larger $C$ is, the smaller $B$ is. Hence, the channel blurredness serves as a measure of poorness on the channel's quality.

### *3.2 Negentropy Rate*

**Definition 3.2.** *The* ***negentropy rate*** *of an asymptotically stationary stochastic process $\{a_k\}$ is defined as*

$$J_\infty\left(a\right) \triangleq h_\infty\left(a_G\right) - h_\infty\left(a\right), \tag{5}$$

*where $\{a_G\left(k\right)\}$ is a Gaussian process with the same asymptotic power spectrum as $\{a_k\}$.*

**Proposition 3.1.** *Suppose that $\{a_k\}$ is asymptotically stationary with asymptotic power spectrum $S_a\left(\omega\right)$, then*

$$J_\infty\left(a\right) = h_\infty\left(a_G\right) - h_\infty\left(a\right)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{2\pi \mathrm{e} S_a\left(\omega\right)}\mathrm{d}\omega - h_\infty\left(a\right). \tag{6}$$

*Furthermore, $J_\infty\left(a\right) \geq 0$, and the equality holds if and only if $\{a_k\}$ is Gaussian.*

**Proof.** It is known from [Martins and Dahleh (2008)] that

$$h_\infty\left(a\right) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{2\pi \mathrm{e} S_a\left(\omega\right)}\mathrm{d}\omega,$$

in which equality holds if and only if $\{a_k\}$ is Gaussian. Since $S_{a_G}\left(\omega\right) = S_a\left(\omega\right)$, we have

$$h_{\infty}(a_G) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{2\pi \mathrm{e} S_{a_G}(\omega)}\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{2\pi \mathrm{e} S_{a}(\omega)}\mathrm{d}\omega.$$

As a result,

$$J_{\infty}(x) = h_{\infty}(a_G) - h_{\infty}(a)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{2\pi \mathrm{e} S_{a}(\omega)}\mathrm{d}\omega - h_{\infty}(a).$$

Furthermore, $J_{\infty}(a) \geq 0$, and the equality holds if and only if $\{a_k\}$ is Gaussian. □

**Remark 3.2.** Negentropy rate generalizes the concept of negentropy to asymptotically stationary stochastic processes, and it provides a measure of non-Gaussianity for such processes.

## 4. INFORMATION THEORETIC (IN)EQUALITIES AND BODE-TYPE INTEGRALS

In this section, we derive Bode-type integrals for control systems over AWGN channels with LTI plants and causal stabilizing controllers, by obtaining information theoretic equalities and inequalities for such systems first.

### *4.1 Networked Feedback System*

Consider the SISO networked feedback system with an uplink AWGN channel, as depicted in Figure 4. In this setup, the reference signal $r$ is assumed to be known, and $y$ denotes the plant's output.

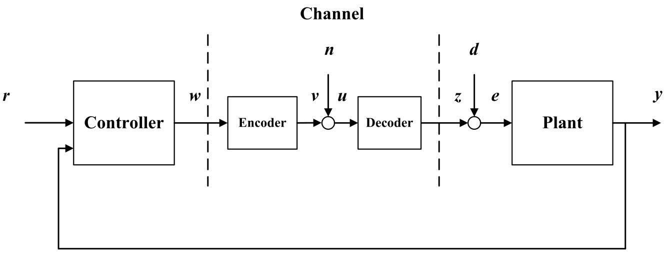


Fig. 4. A control system over an AWGN channel

Let the plant $\mathcal{P}$ be an LTI system, with the state-space model given by

$$\begin{bmatrix} x_{k+1} \\ y_k \end{bmatrix} = \begin{bmatrix} A & b \\ c & 0 \end{bmatrix}\begin{bmatrix} x_k \\ e_k \end{bmatrix},$$

where $x_k \in \mathbb{R}^m, e_k \in \mathbb{R}, y_k \in \mathbb{R}, A \in \mathbb{R}^{m\times m}, b \in \mathbb{R}^{m\times 1}, c \in \mathbb{R}^{1\times m}$. Suppose that the initial state $x_0$ is a random vector with a finite entropy $h(x_0)$.

The controller $\mathcal{K}$ is assumed to be causal. That is, at any time constant $k$, $w_k = \mathcal{K}_k(r_{0,\ldots,k}, y_{0,\ldots,k})$.

The channel noise $\{n_k\}$ is assumed to be an independent and identically distributed (i.i.d.) Gaussian process with zero mean and variance $\sigma_n^2$. The channel input $\{v_k\}$ is subject to a power constraint $\mathrm{E}v^2 \leq P$. Besides, $\{n_k\}$ does not depend on $\{v_k\}$. The disturbance $\{d_k\}$ is assumed to be additive and asymptotically stationary, which does not depend on $\{z_k\}$. We assume that $\{n_k\}, \{d_k\}, x_0$ are independent of each other.

The encoder and the decoder are assumed to be causal. According to the standardized interfaces and layering principles [Gallager (2008)], the encoder may include the quantizer, the source encoder, and the channel encoder, etc., while the decoder may include the channel decoder, the source decoder and the table lookup, etc. Mathematically, the causalities of the encoder $\mathcal{E}$ and decoder $\mathcal{D}$ translate into $v_k = \mathcal{E}_k(w_{0,\ldots,k})$ and $z_k = \mathcal{D}_k(u_{0,\ldots,k})$.

### *4.2 Information Theoretic Equalities and Inequalities*

The information theoretic equalities and inequalities given below provide the necessary foundations for the rest of this paper, which relate the system's input and output signals using information measures. The proofs of Lemma 4.1 and Lemma 4.2 are omitted due to lack of space.

**Lemma 4.1.** *For the system given in Figure 4, the following equality holds:*

$$h(e_{0,\ldots,K}) = h(d_{0,\ldots,K}) + I(e_{0,\ldots,K}; x_0) + I(n_{0,\ldots,K}; e_{0,\ldots,K}, x_0). \quad (7)$$

**Lemma 4.2.**

$$h_{\infty}(e) \geq h_{\infty}(d). \quad (8)$$

*Furthermore,*

$$h_{\infty}(e) = h_{\infty}(d) + I_{\infty}(n; e) + \limsup_{K\to\infty}\frac{I(n_{0,\ldots,K}, e_{0,\ldots,K}; x_0)}{K+1}. \quad (9)$$

### *4.3 Bode-Type Integrals*

As one of the main results of this paper, the following Bode-type integral characterizes the fundamental performance trade-off of the system given in Figure 4.

**Theorem 4.1.** *Suppose that $\mathcal{K}$ stabilizes the system in the sense that $\sup_k \mathrm{E}\left(x_k^T x_k\right) < \infty$, and $\{e_k\}$ is asymptotically stationary. Then the following integral holds:*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{\frac{S_e(\omega)}{S_d(\omega)}}\mathrm{d}\omega = J_{\infty}(e) - J_{\infty}(d) + I_{\infty}(n; e) + \limsup_{K\to\infty}\frac{I(n_{0,\ldots,K}, e_{0,\ldots,K}; x_0)}{K+1}, \quad (10)$$

*where $S_e(\omega)$ and $S_d(\omega)$ are the asymptotic power spectrum of $\{e_k\}$ and $\{d_k\}$ respectively.*

**Proof.** By Proposition 3.1, we have

$$J_{\infty}(d) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{2\pi \mathrm{e} S_d(\omega)}\mathrm{d}\omega - h_{\infty}(d),$$

and

$$J_{\infty}(e) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{2\pi \mathrm{e} S_e(\omega)}\mathrm{d}\omega - h_{\infty}(e).$$

From (9), it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{2\pi\mathrm{e}S_e(\omega)}\mathrm{d}\omega-\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{2\pi\mathrm{e}S_d(\omega)}\mathrm{d}\omega$$
$$-J_\infty(e)+J_\infty(d)$$
$$=h_\infty(e)-h_\infty(d)$$
$$=I_\infty(n;e)+\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K},e_{0,\ldots,K};x_0)}{K+1}.$$

This can be rewritten as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{\frac{S_e(\omega)}{S_d(\omega)}}\mathrm{d}\omega=J_\infty(e)-J_\infty(d)+I_\infty(n;e)+\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K},e_{0,\ldots,K};x_0)}{K+1}.$$

□

**Remark 4.1.** Of the four terms on the right-hand side of (10), the first two terms $J_\infty(d)$ and $J_\infty(e)$ depend directly upon the non-Gaussianity of $\{d_k\}$ and $\{e_k\}$ respectively. The third term $I_\infty(n;e)$, as to be seen shortly, is determined by the channel's properties. The last term $\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K},e_{0,\ldots,K};x_0)}{K+1}$ is intimately related to the properties of the plant $\mathcal{P}$. Based on the results in [Martins et al. (2007)], it can be obtained that

$$\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K},e_{0,\ldots,K};x_0)}{K+1}$$
$$\geq\liminf_{K\to\infty}\frac{I(n_{0,\ldots,K},e_{0,\ldots,K};x_0)}{K+1}$$
$$\geq\liminf_{K\to\infty}\frac{I(e_{0,\ldots,K};x_0)}{K+1}\geq\sum_{i=1}^{m}\max\{0,\log|\lambda_i(A)|\}.$$

**Remark 4.2.** When $\{n_k\}=0$, and $\{d_k\}$ is Gaussian (so $J_\infty(d)=0$), then the equality in (10) reduces to [Martins et al. (2007); Martins and Dahleh (2008)]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{\frac{S_e(\omega)}{S_d(\omega)}}\mathrm{d}\omega$$
$$=J_\infty(e)+\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K},e_{0,\ldots,K};x_0)}{K+1}$$
$$\geq J_\infty(e)+\liminf_{K\to\infty}\frac{I(e_{0,\ldots,K};x_0)}{K+1}$$
$$\geq\sum_{i=1}^{m}\max\{0,\log|\lambda_i(A)|\}.$$

The following theorem relates the term $I_\infty(n;e)$ to the blurredness of the channel, and the relationship characterizes the influence of the channel's properties in the integral.

**Theorem 4.2.** *Consider the system in Figure 4. Given $\mathcal{P}$,*

$$\min_{\mathcal{K}}\max_{\mathcal{E},\mathcal{D},d}I_\infty(n;e)$$
$$\geq B=\frac{1}{2}\log\left(1+\frac{\sigma_n^2}{P}\right)=\frac{1}{2}\log\left(1+\frac{1}{2^{2C}-1}\right),\tag{11}$$

*in which the maximum is to be taken over all $\mathcal{E},\mathcal{D},d$ for which the system can be stabilized by a causal controller; and the minimum is to be taken over all causal $\mathcal{K}$ such that the system is stabilized. Herein, stability means $\sup_k\mathrm{E}\left(x_k^Tx_k\right)<\infty$ and $\{e_k\}$ is asymptotically stationary.*

**Proof.** Since $n_{0,\ldots,K}$ and $d_{0,\ldots,K}$ are independent of each other, we have

$$I(n_{0,\ldots,K};e_{0,\ldots,K})=h(n_{0,\ldots,K})-h(n_{0,\ldots,K}|e_{0,\ldots,K})$$
$$=h(n_{0,\ldots,K}|d_{0,\ldots,K})-h(n_{0,\ldots,K}|e_{0,\ldots,K})$$
$$\leq h(n_{0,\ldots,K}|d_{0,\ldots,K})-h(n_{0,\ldots,K}|e_{0,\ldots,K},d_{0,\ldots,K})$$
$$=I(n_{0,\ldots,K};e_{0,\ldots,K}|d_{0,\ldots,K}).$$

On the other hand, if $d_{0,\ldots,K}$ is known, it can be viewed equivalently as part of the reference $r$, which is also assumed to be known. Thus

$$I(n_{0,\ldots,K};e_{0,\ldots,K}|d_{0,\ldots,K})=I(n_{0,\ldots,K};e_{0,\ldots,K})|_{d_{0,\ldots,K}=0}.$$

So

$$\max_{\mathcal{E},\mathcal{D},d}\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K};e_{0,\ldots,K})}{K+1}$$
$$=\max_{\mathcal{E},\mathcal{D},d=0}\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K};e_{0,\ldots,K})}{K+1}$$
$$=\max_{\mathcal{E},\mathcal{D},d=0}\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K};z_{0,\ldots,K})}{K+1}.$$

Next, as $z_k=\mathcal{D}_k(u_{0,\ldots,k})$ is causal, we have

$$I(n_{0,\ldots,K};z_{0,\ldots,K})\leq I(n_{0,\ldots,K};u_{0,\ldots,K}),$$

and thus

$$\max_{\mathcal{E},\mathcal{D},d=0}\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K};z_{0,\ldots,K})}{K+1}$$
$$=\max_{\mathcal{E},\mathcal{D},d=0}\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K};u_{0,\ldots,K})}{K+1}.$$

Then it can be obtained that

$$\max_{\mathcal{E},\mathcal{D},d}I_\infty(n;e)=\max_{\mathcal{E},\mathcal{D},d=0}\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K};u_{0,\ldots,K})}{K+1}.$$

Thus

$$\min_{\mathcal{K}}\max_{\mathcal{E},\mathcal{D},d}I_\infty(n;e)\geq\min_{\mathcal{K}'}\max_{\mathcal{E},\mathcal{D},d}I_\infty(n;e)$$
$$=\min_{\mathcal{K}'}\max_{\mathcal{E},\mathcal{D},d=0}\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K};u_{0,\ldots,K})}{K+1}$$
$$=\min_{p(v):\,\mathrm{E}v^2\leq P}\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K};u_{0,\ldots,K})}{K+1},$$

in which $\mathcal{K}$ is to be taken from the set of causal stabilizing (in the sense that $\sup_k\mathrm{E}\left(x_k^Tx_k\right)<\infty$, and $\{e_k\}$ is asymptotically stationary) controllers, while $\mathcal{K}'$ is from the set of causal controllers, and the inequality holds as $\mathcal{K}$ is a subset of $\mathcal{K}'$.

It is worth mentioning that as there exists an implicit feedback from the channel's output to the encoder, the channel in the networked feedback system is no longer a memoryless channel. Let $\bar{v},\bar{n},\bar{u}$ denote the input, noise, and output of the memoryless version of the channel. As

$$I(n_{0,\ldots,K};u_{0,\ldots,K})=h(n_{0,\ldots,K})-h(n_{0,\ldots,K}|u_{0,\ldots,K})$$
$$=h(n_{0,\ldots,K})-h(n_0|u_{0,\ldots,K})-h(n_1|n_0,u_{0,\ldots,K})$$
$$-\ldots-h(n_K|n_{0,\ldots,K-1},u_{0,\ldots,K}),$$

and

$$I(\bar{n}_{0,\ldots,K};\bar{u}_{0,\ldots,K})=h(\bar{n}_{0,\ldots,K})-h(\bar{n}_{0,\ldots,K}|\bar{u}_{0,\ldots,K})$$
$$=h(\bar{n}_{0,\ldots,K})-h(\bar{n}_0|\bar{u}_{0,\ldots,K})-h(\bar{n}_1|\bar{n}_0,\bar{u}_{0,\ldots,K})$$
$$-\ldots-h(\bar{n}_K|\bar{n}_{0,\ldots,K-1},\bar{u}_{0,\ldots,K})$$
$$=h(\bar{n}_{0,\ldots,K})-h(\bar{n}_0|\bar{u}_0)-h(\bar{n}_1|\bar{u}_1)-\ldots-h(\bar{n}_K|\bar{u}_K),$$

we have

$$\min_{p(v):\,\mathrm{E}v^2\leq P}\limsup_{K\to\infty}\frac{I(n_{0,\ldots,K};u_{0,\ldots,K})}{K+1}$$
$$\geq\min_{p(\bar{v}):\,\mathrm{E}\bar{v}^2\leq P}\limsup_{K\to\infty}\frac{I(\bar{n}_{0,\ldots,K};\bar{u}_{0,\ldots,K})}{K+1}.$$

So

$$\begin{aligned}
&\min_{\mathcal{K}} \max_{\mathcal{E},\mathcal{D},d} I_{\infty}(n;e) \\
&\geq \min_{p(v):\,\mathrm{E}v^2 \leq P} \limsup_{K\to\infty} \frac{I(n_{0,\ldots,K}; u_{0,\ldots,K})}{K+1} \\
&\geq \min_{p(\bar{v}):\,\mathrm{E}\bar{v}^2 \leq P} \limsup_{K\to\infty} \frac{I(\bar{n}_{0,\ldots,K}; \bar{u}_{0,\ldots,K})}{K+1} \\
&= \min_{p(\bar{v}):\,\mathrm{E}\bar{v}^2 \leq P} I(\bar{n};\bar{u}) = B = \frac{1}{2}\log\left(1+\frac{\sigma_n^2}{P}\right) \\
&= \frac{1}{2}\log\left(1+\frac{1}{2^{2C}-1}\right),
\end{aligned}$$

where the first equality holds as the channel is memoryless. □

**Remark 4.3.** The term $\min_{\mathcal{K}} \max_{\mathcal{E},\mathcal{D},d} I_{\infty}(n;e)$ can be interpreted as designing the controller to minimize the worst-case information rate over all encoders, decoders, and disturbances, which is similar in a way to the min-max optimization in $H_{\infty}$ control.

**Remark 4.4.** In [Martins and Dahleh (2008)], for a control system with LTI plants and causal stabilizing controllers over a communication channel with feedback capacity $C_f$ (for AWGN channels, feedback capacity $C_f$ is equal to channel capacity $C$), it is proven that if $\{d_k\}$ is asymptotically stationary and Gaussian auto-regressive, and the system is stabilized in the sense that $\sup_k \mathrm{E}\left(x_k^T x_k\right) < \infty$, as well as that $\{e_k\}$ is asymptotically stationary, then

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \min\left\{0, \log\sqrt{\frac{S_e(\omega)}{S_d(\omega)}}\right\} \mathrm{d}\omega \\
&\geq \sum_{i=1}^{m} \max\left\{0, \log|\lambda_i(A)|\right\} - C_f.
\end{aligned}$$

Comparison of this result with those obtained by us can bring about deeper insights into this problem.

## 5. CONCLUSION

In this paper, we have developed Bode-type integrals for control systems over AWGN channels with LTI plants and causal stabilizing controllers via an information theoretic approach. Two new notions are proposed to facilitate our development: channel blurredness and negentropy rate, and the integrals are shown to depend on the channel blurredness of the communication channel, as well as the negentropy of the exogenous disturbances.

## ACKNOWLEDGEMENTS

The authors would like to thank Professor Shinji Hara (The University of Tokyo) for the insightful discussions.

## REFERENCES

Antsaklis, P. and Baillieul, J. (2007). Special issue on the technology of networked control systems. *Proceedings of the IEEE*, 95, 5–8.

Bode, H. (1945). *Network Analysis and Feedback Amplier Design.* D.Van Nostrand.

Chen, J. (1995). Sensitivity integral relations and design trade-offs in linear multivariable feedback systems. *IEEE Transactions on Automatic Control*, 40, 1700–1716.

Chen, J. and Nett, C. (1995). Sensitivity integrals for multivariable discrete-time systems. *Automatica*, 31, 1113–1124.

Cover, T. and Thomas, J. (2006). *Elements of Information Theory.* Wiley-Interscience.

Freudenberg, J. and Looze, D. (1988). *Frequency Domain Properties of Scalar and Multivariable Feedback Systems.* Springer-Verlag, Berlin.

Gallager, R. (2008). *Principles of Digital Communications.* Cambridge University Press, Cambridge, U.K.

Hyvärinen, A. and Oja, E. (2000). Independent component analysis: algorithms and applications. *Neural Networks*, 13, 411–430.

Ishii, H., Okano, K., and Hara, S. (2009). Achievable sensitivity bounds for MIMO control systems via an information theoretic approach. *System & Control Letters*, 60, 111–118.

Lestas, I., Vinnicombe, G., and Paulsson, J. (2010). Fundamental limits on the suppression of molecular fluctuations. *Nature*, 467, 174–178.

Li, D. and Hovakimyan, N. (2013). Bode-like integral for stochastic switched systems in the presence of limited information. *Automatica*, 49, 1–8.

Martins, N. and Dahleh, M. (2008). Feedback control in the presence of noisy channels: 'Bode-like' fundamental limitations of performance. *IEEE Transactions on Automatic Control*, 53, 1604–1615.

Martins, N., Dahleh, M., and Doyle, J. (2007). Fundamental limitations of disturbance attenuation in the presence of side information. *IEEE Transactions on Automatic Control*, 52, 56–66.

Okano, K., Hara, S., and Ishii, H. (2008). Characterization of a complementary sensitivity property in feedback control: an information theoretic approach. *Automatica*, 45, 504–509.

Papoulis, A. and Pillai, S. (2002). *Probability, Random Variables and Stochastic Processes.* McGraw-Hill, New York.

Pinsker, M. (1964). *Information and Information Stability of Random Variables and Processes.* Holden Day, San Francisco, CA.

Sung, H. and Hara, S. (1988). Properties of sensitivity and complementary sensitivity functions in single input single output digital control systems. *International Journal of Control*, 48, 2429–2439.

Yu, S. and Mehta, P. (2010). Bode-like fundamental performance limitations in control of nonlinear systems. *IEEE Transactions on Automatic Control*, 55, 1390–1405.

Zang, G. and Iglesias, P. (2003). Nonlinear extension of Bode's integral based on an information-theoretic interpretation. *System & Control Letters*, 50, 11–19.

Zhao, Y., Minero, P., and Gupta, V. (2014). On disturbance propagation in leader-follower systems with limited leader information. *Automatica*, 50, 591–598.