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Model Matching Via Stabilizing Static State Feedback ⋆

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Abstract: The problem of stable exact model matching of linear systems by static state feedback is considered. Based on the effect that a static state feedback produces on the unobservable/controllable part of the closed-loop system, we provide the set of all static state feedback control laws that solve the exact model matching. Moreover, necessary and sufficient conditions for the existence of static state feedback that solves the model matching with stability are presented, along with a method to determine such a required state feedback law.

Keywords: Linear multivariable systems, State feedback, Stabilization, Model matching.

1. INTRODUCTION

A control scheme widely used in the Automatic Control field is the static state feedback. This is used to tackle a great variety of problems, including (exact) model matching.

The model matching problem consists in compensating of a given dynamic, linear, time-invariant, and finite-dimensional system with either static or dynamic state feedback so that the resulting system has a prespecified transfer function matrix, usually called the model transfer function matrix. Adequately, the model transfer function matrix is specified as a proper rational matrix.

The model matching problem is of both theoretical and practical importance since a number of control problems can be related to it, see Ichikawa (1985). Among those are the decoupling problem, the model following problem, the model tracking problem, the servomechanisms problem, and the model reference adaptive control.

The model matching problem by static state feedback has been studied for many authors, however the general case has not been solved yet. A meticulous formulation and the first solution of the static state feedback problem were given by Wolovich (1972). His solution to the problem made use of a set of feedback invariants along with a transformation of coordinates derived from the controllability matrix of the system, and this leads to determining if a certain set of polynomial equations is solvable. An analogous solution was given later by Wang and Desoer (1972). Although their method utilizes essentially the same feedback invariants and coordinates transformation, they reduced the problem to one of solving a system of linear algebraic equations.

A different approach was presented by Moore and Silverman (1972). Their approach was based on an algorithm for characterizing the input-output structural properties of a given linear system. In this way, the state feedback model matching problem is solved without using any coordinates transformation.

A more general compensation scheme to achieve exact model matching is dynamic compensation. Necessary and sufficient conditions for the existence of such a dynamical model matching solution were given by Moore and Silverman (1972) by using the structure algorithm.

An important result, which affected the study of exact model matching problem, was published by Hautus and Heymann (1978). The contribution concerns necessary and sufficient conditions under which the action of a cascade compensator on a given system can be realized by a static state feedback law applied to the system. This result revived the interest in transfer function methods in exact model matching. In this context, exact model matching was studied, among others, by Kučera (1981), Kaczorek (1982), and Kimura et al. (1982). Torres et al. (2005) considered the problem using the Hermite normal form for proper rational matrices.

If the model matching problem has a solution, then stability of the closed-loop model system is an important issue. Morse (1973) described the possible distributions of system eigenvalues that can be achieved while maintaining a model matching configuration. This leads to necessary and sufficient conditions for the existence of a matching solution which results in a stable compensated system.

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Anderson and Scott (1977) obtained a parametric solution to the stable exact model matching problem using an algebraic approach. Pernebo (1981), Vardulakis and Karcanias (1985), Antsaklis (1986), and Gao and Antsaklis (1989) proved a number of equivalent necessary and sufficient conditions for the existence of stable solutions to exact model matching. Marro and Zattoni (2002) provided further insight into the exact model matching problem with stability using the geometric approach.

The literature relating to the model matching problem is extensive. However, a complete solution to the problem has been obtained only when dynamic compensation is used, while the model matching problem by static state feedback is an open problem yet.

In this work, we focus on the internal stability of the closed-loop system having as transfer function matrix the model transfer function matrix. We suppose that there already exists a static state feedback law that solves the problem without stability. Then, taking that as a starting point, we completely characterize the set of static state feedback laws that solve the model matching problem. Based on it, we present necessary and sufficient conditions for the existence of a stabilizing state feedback law, and a method to find it.

This paper is organized as follows. The problem statement is presented in Section 2, and the relevant previous results in Section 3. The main results of this work are presented in Section 4 along with an illustrative example.

2. PROBLEM FORMULATION

Throughout the paper, $\mathbb{R}$ denotes the field of real numbers. Accordingly, $\mathbb{R}^n$ stands for the $n-$vector space over $\mathbb{R}$ and $\mathbb{R}^{m\times r}$ stands for the set of $m\times r$ matrices with entries in $\mathbb{R}$. Finally, $\mathbb{R}_p(s)$ denotes the ring of proper rational functions over $\mathbb{R}$ and $\mathbb{R}_p^{m\times r}(s)$ denotes the set of $m\times r$ matrices with entries in $\mathbb{R}_p(s)$.

Let (A, B, C) be a state space representation of a linear time-invariant differential system described by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are, respectively, the state, input and output vectors of the system.

Let us consider a static state feedback control law (F, G) of the form

$$u(t) = Fx(t) + Gv(t), \tag{2}$$

where $F \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{m\times r}$ are constant matrices, with rank $G = r$. When (2) is applied to the system (1) then the closed-loop system becomes

$$\begin{aligned} \dot{x}(t) &= (A + BF)x(t) + BGv(t) \\ y(t) &= Cx(t). \end{aligned} \tag{3}$$

The transfer function matrix of the closed-loop system, from the new input $v(t)$ to the output $y(t)$ is then given by

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG. \tag{4}$$

If $r = m$ (matrix G is square and nonsingular), then (2) is said to be a regular state feedback, and if $r < m$ then it is said to be a non-regular state feedback.

In this work the exact model matching problem with stability is defined as follows:

Given a linear multivariable system (A, B, C) and a desired proper and stable rational matrix $T_m(s)$ of size $p \times m$, the exact model matching problem with stability consists in finding a static state feedback law (F, G), with G nonsingular, such that the transfer function matrix of the closed-loop system is

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG = T_m(s),$$

furthermore the matrix $A + BF$ is asymptotically stable (i.e. has all its eigenvalues in the open left half plane).

In this work, we are interested in studying the properties of the closed-loop model system, and the properties of F and G such that the closed-loop system is asymptotically stable, without affecting the model transfer function matrix $T_m(s)$.

3. PRELIMINARY RESULTS

The closed-loop transfer matrix $T_{F,G}(s)$ given in (4) also can be written as

$$T_{F,G}(s) = T(s)W(s),$$

where

$$W(s) = [I - F(sI - A)^{-1}B]^{-1}G \tag{5}$$

is a proper rational matrix. In the case of regular state feedback, (5) is a biproper matrix, i.e. a proper rational matrix whose inverse exists and is also proper rational. In the case of non-regular state feedback, (5) is column biproper, i.e.

$$\text{rank} \lim_{s\to\infty} W(s) = r.$$

Thus, the action of a state feedback law (F, G) on the system (A, B, C) can be represented in transfer function matrix form as the post-multiplication of the system transfer function matrix $T(s)$ by a proper rational matrix $W(s)$.

The converse problem, i.e. under which conditions a proper rational matrix that post-multiplies $T(s)$ can be realized using a state feedback law applied to the system, is known as the feedback realizability of compensators. Then, a given proper compensator $W(s)$ is said to be feedback realizable if there exists a static state feedback control law (F, G) such that

$$W(s) = [I - F(sI - A)^{-1}B]^{-1}G.$$

Conditions for static state feedback realization of dynamic compensators are well known. The following result applies to square nonsingular compensators.

Lemma 1. Hautus and Heymann (1978) . Given a system (A, B, C), let $N_1(s)$, $D(s)$ be right coprime polynomial matrices such that

$$(sI - A)^{-1}B = N_1(s)D^{-1}(s)$$

and let $W(s) \in \mathbb{R}_p^{m\times m}(s)$ be a nonsingular compensator. Then $W(s)$ is feedback realizable if and only if

a) $W(s)$ is biproper, and

b) $W^{-1}(s)D(s)$ is a polynomial matrix. ∎

The required regular static state feedback is linked to a constant matrix solution (M, E) to the polynomial matrix equation, see Kučera and Zagalak (1991),

$$W^{-1}(s)D(s) = ED(s) + MN_1(s),$$

where E is a nonsingular constant matrix. Then, the regular static state feedback realizing $W(s)$ is given by

$$F = -E^{-1}M, \quad G = E^{-1}.$$

When a compensator $W(s)$ is feedback realizable by static state feedback law, the stability of the closed-loop system is an important issue. The following result applies to nonsingular compensators.

Lemma 2. Kučera (1990). Let $W(s) \in \mathbb{R}_p^{m\times m}(s)$ be a nonsingular compensator that is feedback realizable on a stabilizable system (A, B, C). Let $N_1(s)$, $D(s)$ be right coprime polynomial matrices such that

$$(sI - A)^{-1}B = N_1(s)D^{-1}(s).$$

Then the resulting closed-loop system (3) is internally stable if and only if $\det W^{-1}(s)D(s)$ has all roots with negative real part. ∎

Lemma 1 is related to the model matching problem as follows.

Lemma 3. Wolovich (1972). Given a system (A, B, C) with transfer function matrix $T(s)$, let $N_1(s)$, $D(s)$ be right coprime polynomial matrices such that

$$(sI - A)^{-1}B = N_1(s)D^{-1}.$$

Let $N_m(s)$, $D_m(s)$ be right coprime polynomial matrices such that $T_m(s) = N_m(s)D_m^{-1}(s)$. Then the exact model matching problem is solvable if and only if there exist a biproper matrix $W(s)$ and a nonsingular polynomial matrix $V(s)$ such that

$$\begin{bmatrix} N(s) \\ W^{-1}(s)D(s) \end{bmatrix} = \begin{bmatrix} N_m(s) \\ D_m(s) \end{bmatrix} V(s), \tag{6}$$

where $N(s) = CN_1(s)$. ∎

Thus, with respect to the model matching problem with stability we have the following proposition.

Proposition 1. Given a stabilizable system (A, B, C) with transfer function matrix $T(s)$, then, the exact model matching with stability is solvable if and only if there exist a biproper matrix $W(s)$ and a nonsingular polynomial matrix $V(s)$, satisfying the conditions of Lemma 3, such that $\det V(s)$ is a stable polynomial.

Proof. *Necessity:* Suppose that a stable match exists. By Lemma 1, there exists a biproper matrix $W(s)$ such that $T(s)W(s) = T_m(s)$ and $W^{-1}(s)D(s)$ is a polynomial matrix, moreover by Lemma 2, $\det W^{-1}(s)D(s)$ is stable. Then $N(s)D^{-1}(s)W(s) = N_m(s)D_m^{-1}(s)$. As $N_m(s)$ and $D_m(s)$ are right coprime, then there exists a nonsingular polynomial matrix $V(s)$ such that (6) holds. As $D_m(s)$ has a stable determinant by assumption, $\det V(s)$ is stable.

Sufficiency: It follows from (6) that

$$\begin{aligned} T(s)W(s) &= N(s)D^{-1}(s)W(s) \\ &= [N_m(s)V(s)]\,[D_m(s)V(s)]^{-1} = T_m(s), \end{aligned}$$

where $D_m(s)V(s) = W^{-1}(s)D(s)$ for some biproper matrix $W(s)$. Then by Lemma 2, a stable match exists. ∎

Note that Proposition 1 provides a complete solution to the model matching problem with stability; however, conditions (6) are implicit. It is also noted that matrix $V(s)$ is not unique when a non-square systems (A, B, C) is considered. Then, if the model matching problem has a solution, a particular $V(s)$ may lead to a matching solution of the problem without stability, while another $V(s)$ may lead to a solution with stability. Indeed, to find such a stabilizing matrix $V(s)$ becomes complicated when large systems are considered.

4. MAIN RESULTS

In this section a different way to obtain a stabilizing regular static state feedback law that solves the model matching problem is provided. Then, necessary and sufficient conditions are presented for the model closed-loop system to be stabilizable by such a regular static state feedback law. The result is based on a modification of the controllable-unobservable part of the closed-loop system.

This section is divided into two parts. In the first part, we characterize the set of static state feedback laws that give rise to the same closed-loop transfer function matrix; this is achieved by analyzing the controllable-unobservable structure of the closed-loop system. Based on this, the second part is devoted to obtain a stabilizing state feedback control law that solves the matching problem.

4.1 The effect of the state feedback on the controllable-unobservable subsystem

Suppose that a particular regular static state feedback law (F_0, G_0) is applied to the system (A, B, C), which produces an unobservable and uncontrollable closed-loop system $(A + BF_0, BG_0, C)$.

Let T be a $n \times n$ similarity transformation matrix which decomposes the closed-loop system $(A+BF_0, BG_0, C)$ into its controllable-observable, controllable-unobservable, and uncontrollable parts, i.e.

$$T^{-1}(A + BF_0)T = \begin{bmatrix} A_{c,o} & 0 & \overline{A}_{13} \\ \overline{A}_{21} & A_{c,no} & \overline{A}_{23} \\ 0 & 0 & A_{nc} \end{bmatrix},$$

$$T^{-1}BG_0 = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} G_0 = \begin{bmatrix} B_{c,o} \\ B_{c,no} \\ 0 \end{bmatrix}, \tag{7}$$

$$CT = \begin{bmatrix} C_{c,o} & 0 & C_{nc} \end{bmatrix}.$$

Then, $(A_{c,o}, B_{c,o}, C_{c,o})$ corresponds to the controllable and observable subsystem, the controllable-unobservable subsystem is represented by $(A_{c,no}, B_{c,no}, 0)$, and $(A_{nc}, 0, C_{nc})$ corresponds to the uncontrollable subsystem, all of them associated with the closed-loop system.

It is noted that

$$T_{F_0,G_0}(s) = C_{c,o}(sI - A_{c,o})^{-1}B_{c,o}.$$

Further, it is denoted

$$\bar{F}_0 = F_0T = [\, F_{c,o} \;\; F_{c,no} \;\; F_{nc} \,]. \tag{8}$$

The equations of the closed-loop system in the coordinates defined by T are given by

$$\begin{bmatrix} \dot{x}_{c,o}(t) \\ \dot{x}_{c,no}(t) \\ \dot{x}_{nc}(t) \end{bmatrix} = \begin{bmatrix} A_{c,o} & 0 & \overline{A}_{13} \\ \overline{A}_{21} & A_{c,no} & \overline{A}_{23} \\ 0 & 0 & A_{nc} \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,no}(t) \\ x_{nc}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} G_0 v(t), \quad (9)$$

and

$$y(t) = \left[\, C_{c,o} \;\; 0 \;\; C_{nc} \,\right] \begin{bmatrix} x_{c,o}(t) \\ x_{c,no}(t) \\ x_{nc}(t) \end{bmatrix}.$$

We are interested in modifying the internal structure of the closed-loop system, without affecting its transfer function matrix. This will be done by analyzing the effect that a different static state feedback law, say (F_1, G_1), produces on the structure of the closed-loop system, without affecting $T_{F_G}(s)$. Thus, let

$$v(t) = F_1 x(t) + G_1 v_1(t), \quad (10)$$

be such static state feedback, where F_1 is defined, in general form, as

$$F_1 = \left[\, \widehat{F}_{c,o} \;\; \widehat{F}_{c,no} \;\; \widehat{F}_{nc} \,\right] T^{-1}.$$

When the static state feedback (F_1, G_1) is applied to the system (9), it is obtained that

$$\dot{x}(t) = \begin{bmatrix} A_{c,o} + B_1 G_0 \widehat{F}_{c,o} & B_1 G_0 \widehat{F}_{c,no} & B_1 G_0 \widehat{F}_{nc} \\ \overline{A}_{21} + B_2 G_0 \widehat{F}_{c,o} & A_{c,no} + B_2 G_0 \widehat{F}_{c,no} & \overline{A}_{23} + B_2 G_0 \widehat{F}_{nc} \\ B_3 G_0 \widehat{F}_{c,o} & B_3 G_0 \widehat{F}_{c,no} & A_{nc} + B_3 G_0 \widehat{F}_{nc} \end{bmatrix}$$

$$\begin{bmatrix} x_{c,o}(t) \\ x_{c,no}(t) \\ x_{nc}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} G_0 G_1 v_1(t),$$

$$y(t) = \left[\, C_{c,o} \;\; 0 \;\; C_{nc} \,\right] \begin{bmatrix} x_{c,o}(t) \\ x_{c,no}(t) \\ x_{nc}(t) \end{bmatrix}.$$

It is seen that $B_3 G_0 = 0$; then $\widehat{F}_{nc}$ does not affect the structure of the closed-loop system, therefore $\widehat{F}_{nc}$ can be any arbitrary real matrix of adequate dimensions.

Now, it is required that

$$T_{F,G}(s) = C_{co}(sI - A_{c,o})^{-1} B_{c,o}$$

for any static state feedback. We arrive to the conclusion that $\widehat{F}_{c,o}$ must be equal to 0, so that $A_{c,o} + B_1 G_0 \widehat{F}_{c,o} = A_{c,o}$. Moreover, in order to preserve $T_{F,G}(s)$ as transfer function matrix of the closed-loop system, it must be also satisfied that $B_1 G_0 \widehat{F}_{c,no} = 0$.

The set of matrices $\widehat{F}_{c,no}$ satisfying that $B_1 G_0 \widehat{F}_{c,no} = 0$ can be parameterized as follows

$$\widehat{F}_{c,no} = KL, \quad (11)$$

where K is a basis for the kernel of $B_1 G_0$ and L is any real matrix of adequate dimensions.

Then, the static state feedback gain (F_1, G_1) is of the form

$$F_1 = \left[\, 0 \;\; KL \;\; \widehat{F}_{nc} \,\right] T^{-1}. \quad (12)$$

With respect to matrix G_1, it is seen that

$$T^{-1} B G_0 G_1 = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} G_0 G_1 = \begin{bmatrix} B_{c,o} \\ B_{c,no} \\ 0 \end{bmatrix} G_1.$$

Note that it is possible to take $G_1 = I$. It follows that matrix G can be parameterized as

$$G = G_0 + \Gamma P, \quad (13)$$

where Γ is a basis for the kernel of B_1 and P is a real matrix of adequate dimensions, such that G is non singular.

Thus, we have arrived at the following result.

Theorem 1. Let (A, B, C) be a linear multivariable system, and let (F_0, G_0) be a regular static state feedback law which is applied to the system (A, B, C) and gives rise to the transfer function matrix $T_{F,G}(s) = C(sI - A - BF_0)^{-1} BG_0$. Then the set of matrices F and G, such that the transfer matrix of the closed loop system is equal to the obtained $T_{F,G}(s)$, can be parameterized as

$$\begin{aligned} F &= F_0 + G_0 F_1 \\ &= \left[\, F_{c,o} \quad F_{c,no} + G_0 KL \quad F_{nc} + G_0 \widehat{F}_{nc} \,\right] T^{-1}, \end{aligned}$$

and

$$G = G_0 + \Gamma P,$$

where K and Γ are obtained, respectively, from (11) and (13). Matrices L and P are arbitrary real matrices of adequate dimensions. ■

4.2 Stabilizing Feedback

In order to obtain a regular static state feedback law that solves the model matching problem with stability, consider a particular static state feedback law (F_0, G_0) that solves the model matching problem, that is, consider that there exists a matrix $V(s)$ such that Lemma 3 holds. Then, by Theorem 1, the set of all static state feedback laws that provide a matching solution can be parameterized as

$$\begin{aligned} F &= \left[\, F_{c,o} \;\; F_{c,no} + G_0 KL \;\; F_{nc} + G_0 \widehat{F}_{nc} \,\right] T^{-1}, \\ G &= G_0 + \Gamma P. \end{aligned} \quad (14)$$

A stable match can be determined as follows. Let δ be the rank of the observability matrix of the subsystem $(A_{c,o}, B_{c,o}, C_{c,o})$ and let q be the rank of the controllability matrix of the system (A, B, C). Then, when the similarity transformation T is applied to the original system (A, B, C), we have

$$T^{-1} A T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad T^{-1} B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \quad (15)$$

$$CT = \left[\, C_1 \;\; C_2 \;\; C_3 \,\right],$$

for some matrices $A_{11} \in \mathbb{R}^{\delta \times \delta}$, $A_{12} \in \mathbb{R}^{\delta \times (q-\delta)}$, $A_{13} \in \mathbb{R}^{\delta \times (n-q)}$, $A_{21} \in \mathbb{R}^{(q-\delta) \times \delta}$, $A_{22} \in \mathbb{R}^{(q-\delta) \times (q-\delta)}$, $A_{23} \in \mathbb{R}^{(q-\delta) \times (n-q)}$, $A_{31} \in \mathbb{R}^{(n-q) \times \delta}$, $A_{32} \in \mathbb{R}^{(n-q) \times (q-\delta)}$, $A_{33} \in \mathbb{R}^{(n-q) \times (n-q)}$ and $B_1 \in \mathbb{R}^{\delta \times m}$, $B_2 \in \mathbb{R}^{(q-\delta) \times m}$, $B_3 \in \mathbb{R}^{(n-q) \times m}$ and $C_1 \in \mathbb{R}^{p \times \delta}$, $C_2 \in \mathbb{R}^{p \times (q-\delta)}$, $C_3 \in \mathbb{R}^{p \times (n-q)}$.

Then, when a static state feedback law (F, G) defined by (14) is applied to solve the model matching problem, in view of (7), (15), and (14), it is obtained that

$$T^{-1}(A+BF)T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \begin{bmatrix} F_{c,o} & F_{c,o}+G_0KL & F_{nc}+G_0\widehat{F}_{nc} \end{bmatrix}$$
$$= \begin{bmatrix} A_{c,o} & 0 & \overline{A}_{13} \\ \overline{A}_{21} & A_{c,o}+B_2G_0KL & \overline{A}_{23} \\ 0 & 0 & A_{nc} \end{bmatrix},$$
$$T^{-1}BG = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} (G_0+\Gamma P) = \begin{bmatrix} B_{c,o} \\ B_2(G_0+\Gamma P) \\ 0 \end{bmatrix},$$
$$CT = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} C_{c,o} & 0 & C_{nc} \end{bmatrix}.$$

The equations of the model closed-loop system in the coordinates defined by T are given by

$$\begin{aligned} \dot{x}_{c,o}(t) &= (A_{11}+B_1F_{c,o})x_{c,o}(t) + \\ &(A_{12}+B_1F_{c,no}+B_1G_0KL)x_{c,no}(t) + \\ &(A_{13}+B_1F_{nc}+B_1G_0\widehat{F}_{nc})x_{nc}(t) + B_1Gv(t) \\ \dot{x}_{c,no}(t) &= (A_{21}+B_2F_{co})x_{c,o}(t) + \\ &(A_{22}+B_2F_{c,no}+B_2G_0KL)x_{c,no}(t) \\ &+(A_{23}+B_2F_{nc}+B_2G_0\widehat{F}_{nc})x_{nc}(t) + B_2Gv(t) \\ \dot{x}_{nc}(t) &= (A_{33}+B_3F_{nc})x_{nc}(t) \\ y(t) &= C_1x_{c,o}(t) + C_3x_{nc}(t). \end{aligned} \quad (16)$$

where, it is noted that $A_{12}+B_1F_{c,no}+B_1G_0KL = 0$ and $B_3G = 0$. It follows that the matrix $A_{11}+B_1F_{c,o}$ is the same for every (F,G) that provides a solution to the model matching problem, while the controllable - unobservable subsystem matrix $A_{c,no} = A_{22}+B_2F_{c,no}+B_2G_0KL$ can be modified through L. In fact, equation (16) can be viewed as an internal feedback of the form

$$\begin{aligned} \dot{x}_{c,no}(t) &= (A_{21}+B_2F_{co})x_{c,o}(t) + \\ &(A_{22}+B_2F_{c,no})x_{c,no}(t) + B_2G_0Ku_{c,no}(t) + \\ &(A_{23}+B_2F_{nc}+B_2G_0\widehat{F}_{nc})x_{nc}(t) + B_2Gv(t) \end{aligned}$$

where

$$u_{c,no}(t) = Lx_{c,no}(t).$$

If the subsystem $(A_{22}+B_2F_{c,no}, B_2G_0K, I)$ is controllable, then L can allocate the eigenvalues of the controllable-unobservable part of the model system at will. However, if the rank of its controllability matrix is $\varepsilon < q-\delta$, then the controllable-unobservable subsystem has $q-\delta-\varepsilon$ fixed modes. Thus, we have arrived at the following result.

Theorem 2. Suppose that the uncontrollable eigenvalues of the given system (A,B,C), if any, are stable. Let (F_0,G_0) be a particular static state feedback law that solves the model matching problem. Then the model matching problem with stability has a solution if and only if

i) the poles of $T_m(s)$ have negative real parts;
ii) the system $(A_{22}+B_2F_{c,no}, B_2G_0K, I)$ is stabilizable.

Proof. The model closed-loop system is asymptotically stable if and only if the eigenvalues of $A_{c,o}$, $A_{c,no}$, and A_{nc} have negative real parts. It is supposed that the non-controllable modes are stable, then the eigenvalues of A_{nc} have negative real parts.

i) Note that

$$T_m(s) = C_{c,o}(sI - A_{c,o})^{-1}B_{c,o}.$$

The subsystem $(A_{c,o}, B_{c,o}, C_{c,o})$ is both controllable and observable. Thus the poles of $T_m(s)$ are the eigenvalues of $A_{c,o}$.

ii) Recall that

$$A_{c,no} = A_{22} + B_2F_{c,no} + B_2G_0KL.$$

The eigenvalues of $A_{c,no}$ can be affected by L and, in particular, can be made to have negative real parts if and only if the subsystem $(A_{22}+B_2F_{c,no}, B_2G_0K, I)$ is stabilizable. Indeed, if L is such a stabilizing feedback gain, then

$$F = \begin{bmatrix} F_{c,o} & F_{c,no}+G_0KL & F_{nc}+G_0\widehat{F}_{nc} \end{bmatrix} T^{-1}.$$

■

If the conditions of Theorem 2 are satisfied, then the required matrix L can be found using the standard techniques of eigenvalue assignment by static state feedback. In particular, if $(A_{22}+B_2F_{c,no}, B_2G_0K, I)$ is controllable, then the controllable-unobservable eigenvalues of the closed-loop system $(A+BF, BG, C)$ can be assigned any desired values. Theorem 2 presents necessary and sufficient conditions for the model matching problem with stability to have a solution. Other advantages of this result with respect to previous solutions on model matching is that we provide a method to obtain a state feedback which solves the problem with stability, and that the conditions of our results can be applied to large systems.

The next example illustrates the main result.

Example. Consider a linear multivariable system (A,B,C) described by

$$A = \begin{bmatrix} -2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 8 & -3 & 5 & 0 & 0 & 3 & 0 \\ 2 & 3 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & -1 & 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 3 & 0 & 1 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 0 & -2 & 0 \end{bmatrix},$$

with transfer function matrix

$$T(s) = \begin{bmatrix} \frac{2s+3}{(s+2)(s+1)} & \frac{1}{s+2} & \frac{2}{s+1} \\ -\frac{2}{s+1} & 0 & -\frac{4}{s+1} \end{bmatrix}$$

and consider the model transfer function $T_m(s) = T(s)$, so that the match already exists. Anyway, a particular static state feedback solving the model matching problem is $F_0 = [0], G_0 = I$.

However, the closed-loop system is not asymptotically stable, it has eigenvalues $\{-1,-2,-3,-4,5,-1,-2\}$. The eigenvalues $\{-1,-2\}$ are controllable and observable, $\{-3,-4,5\}$ are controllable and unobservable, whereas the eigenvalues $\{-1,-2\}$ are uncontrollable.

Can the model matching problem be solved using a different static state feedback law so that the closed-loop system is asymptotically stable? Theorem 2 will be applied to answer the question.

Observe that the uncontrollable eigenvalues are stable and the poles of $T_m(s)$ are $\{-1,-2\}$, therefore condition

i) of Theorem 2 is satisfied. Now, a similarity transformation matrix which decomposes the closed-loop system $(A + BF_0, BG_0, C)$ into its controllable-observable, controllable- unobservable, and uncontrollable parts is given by

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & -\frac{1}{3} & 0 \end{bmatrix}.$$

We have that

$$F_0 T = [\, 0\ 0\ 0 \,] = [\, F_{c,o}\ F_{c,no}\ F_{nc} \,].$$

Thus, given a particular static state feedback law (F_0, G_0) that solves the model matching problem, we infer from (14) that the set of static state gains F that provide a solution to the problem can be parameterized as

$$F = \left[\, F_{c,o}\ \ F_{c,no} + G_0 K L\ \ F_{nc} + G_0 \widehat{F}_{nc} \,\right] T^{-1},$$

where

$$K = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

is a basis for the kernel of

$$B_1 G_0 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

and, L and $\widehat{F}_{nc}$ are arbitrary real matrices of dimension 1×3 and 3×2, respectively. Now, the controllable-unobservable part of the closed-loop system is governed by the equation

$$\begin{aligned} \dot{x}_{c,no}(t) = {} & (A_{21} + B_2 F_{co}) x_{c,o}(t) + \\ & (A_{22} + B_2 F_{c,no}) x_{c,no}(t) + B_2 G_0 K u_{c,no}(t) + \\ & (A_{23} + B_2 F_{nc} + B_2 G_0 \widehat{F}_{nc}) x_{nc}(t) + B_2 G v(t) \end{aligned}$$

where

$$A_{22} + B_2 F_{c,no} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad B_2 G_0 K = \begin{bmatrix} -2 \\ 4 \\ -9 \end{bmatrix}.$$

Note that the subsystem $(A_{22} + B_2 F_{c,no}, B_2 G_0 K, I)$ is controllable, then condition ii) of Theorem 2 is satisfied. Thus, there exists a matrix L such that the control law

$$u_{c,no}(t) = L x_{c,no}(t)$$

can allocate the eigenvalues of the controllable-unobservable part of the closed-loop system at will. A particular control law $u_{c,no}(t)$, which will allocate the controllable-unobservable eigenvalues at $\{-3, -4, -5\}$ is given by

$$u_{c,no}(t) = \left[\, 0\ 0\ \tfrac{10}{9} \,\right] x_{c,no}(t).$$

It follows that

$$KL = \begin{bmatrix} 0 & 0 & -\frac{20}{9} \\ 0 & 0 & \frac{20}{9} \\ 0 & 0 & \frac{10}{9} \end{bmatrix}.$$

Finally, a stabilizing static state feedback law (F, G) that solves the model matching problem is given by

$$F = \begin{bmatrix} -\frac{20}{9} & 0 & -\frac{20}{9} & 0 & 0 & 0 & 0 \\ \frac{20}{9} & 0 & \frac{20}{9} & 0 & 0 & 0 & 0 \\ \frac{10}{9} & 0 & \frac{10}{9} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = G_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, the model system $(A + BF, BG, C)$ with transfer function matrix

$$T_{F,G}(s) = T_m(s) = \begin{bmatrix} \frac{2s+3}{(s+2)(s+1)} & \frac{1}{s+2} & \frac{2}{s+1} \\ -\frac{2}{s+1} & 0 & -\frac{4}{s+1} \end{bmatrix}$$

is asymptotically stable, and it has eigenvalues $-1, -2, -3, -4, -5$.

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