Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Optimal design of remote controllers for LTI plants over erasure channels ⋆

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Abstract: In this paper we consider the remote control of a noisy linear time-invariant (LTI) plant over erasure channels located in both the plant-to-controller and controller-to-plant links. We restrict our attention to a class of controllers where all processing is affine, except for some elementary use of sensor-to-controller channel state information. For such controller class, we show that optimal designs separate into an estimation and a state feedback design problem when perfect packet acknowledgements are available at the controller. Interestingly, our results also show that the affine part of the controller converges, as the time goes to infinity, to an LTI filter under the same conditions which guarantee mean-square stability in the well-known LQG control problem over erasure channels. However, our infinite horizon proposal is computationally inexpensive and its steady-state behaviour can be characterized straightforwardly.

Keywords: Networked control systems; erasure channels; optimal control.

1. INTRODUCTION

In recent years, the study of Networked Control Systems (NCSs), i.e., control systems where communication takes place over non-transparent channels, has increased considerably. The understanding of design trade-offs in NCSs has many practical implications and presents several theoretical challenges (Antsaklis and Baillieul [2007]). Of course, the study of NCSs depends on the type of communication channel under analysis. In this paper, we focus on control systems closed over analog erasure channels (Schenato et al. [2007], Silva and Pulgar [2011], Elia [2005]).

In scenarios where the sensor-to-controller link is an erasure channel, the obvious question that arise is: What control signal should be sent to the actuator when sensor data have been missed? In Schenato et al. [2007] the optimal estate estimator is presented in an LQG framework. Thus, the control signals are constructed using an estimation of the plant state based on previous data (see also Sinopoli et al. [2004]). The corresponding estimation error covariance in Schenato et al. [2007], depend explicitly on the sensor-to-controller data-dropout process and, accordingly, do not converge as time grows unbounded. In Schenato [2009] two simpler approaches are studied: hold-input scheme and zero-input scheme. In the first scheme the missed data is replaced with the last control signal sent to the controller, whereas in the second scheme, the control signal is set to zero. A generalization of these two scheme, and other alternative approaches can be found in Moayedi et al. [2013], Tugnait [1981], Liang et al. [2010], Zhang et al. [2011], Silva et al. [2013]. We remark the approach considered in Silva et al. [2013], where a state estimator that embeds a data-dropout compensator is proposed. The class of controller studied in Silva et al. [2013] is such that, beside elementary use of instantaneous channel information, all the processing is affine. The structure of the optimal estimator proposed in Silva et al. [2013] is akin to the one presented in Schenato et al. [2007], however in the former case the estimation error covariance do not depends explicitly on the sensor-to-controller data-dropout process, and thus an easy characterization of the steady state estimator can be made. Moreover, the approach in Silva et al. [2013] generalizes the proposals mentioned above.

In this paper we consider NCSs where both sensor-to-controller and controller-to-actuator links are subject to data-loss simultaneously. Such a setup was considered previously in the Schenato et al. [2007], Moayedi et al. [2013], Garone et al. [2012], Chen et al. [2012]. A remarkable advance was made in Schenato et al. [2007], where it is shown that, if optimal LQG controllers are sought using TCP-like protocols, i.e., assuming the existence of packet acknowledgement from the actuator side to the controller side, then the separation principle holds. In such case, the optimal control is linear and the controller gain converges provided the controller-to-actuator link is sufficiently reliable. However, as was mentioned in the above paragraph, the optimal estimator gain do not converge in this case and thus the expected stationary cost can not be easily characterized. A non trivial extension of the results in Schenato et al. [2007] to the multiple channel case can be found in Garone et al. [2012]. In Moayedi et al. [2013] the problem is addressed considering a generalized hold-input strategy to replace missed data. Thus, the optimal control derived in that work is constrained for that specific class of control strategy. In Chen et al. [2012], the LQG control of system with random input and output gains is addressed. The setup in Chen et al. [2012] consider TCP-like protocols and allows the simultaneous design of both

⋆ This work was supported by CONICYT through grants FONDECYT 1130459, FONDECYT 3140056 and Anillo ACT53. The financial support received from CCTVal (FB0821) of Universidad Técnica Federico Santa María is also acknowledged.

channel and controller. Given that framework, optimal estimator and controller are derived and it is found that separation principle partially holds.

In this paper, we consider the remote control of a noisy multiple-input multiple-output LTI plant where both the sensor-to-controller and the controller-to-actuator links are affected by data dropouts. We constrain ourselves to a class of controllers where all processing is affine, except for the use of sensor-to-controller channel state information to trigger a data dropout compensation mechanism. We give a solution to both finite and infinite horizon quadratic optimal control problems in the considered setup. Our results show that under TCP-like protocols separation principle also holds, and thus the optimal control design consist on solving a state feedback control problem and an estimation problem. We also show that the optimal estimator for this problem coincides with that presented in Silva et al. [2013], where control signals are not subject to data-loss. This allowed us to explore the infinite horizon case and conclude that the affine part of the optimal controller converges to an LTI system and thus a computationally inexpensive steady-state controller can be found. The steady state performance of our proposal is also easily characterized and, interestingly, coincides with the upper bound on the expected performance of the optimal LQG controller presented in Schenato et al. [2007].

The remainder of this paper is organized as follows: Section 2 describes the problem addressed in this paper. Section 3 studies an estimation problem. Section 4 characterizes optimal finite horizon controllers. Section 5 focuses on infinite horizon problems and Section 6 draws conclusions.

Notation: $\mathbb{N}$ denotes the natural numbers, and $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$. For any sequence x, x_k denotes its k^{th} sample, and x^k is used as shorthand for $x_0, \ldots, x_k$. A^T and $\rho(A)$ denote the transpose and the spectral radius of the matrix A. For any real-valued vector x and positive semidefinite matrix M, $||x||_M^2 \triangleq x^T M x$. If x is a second order random variable, then P_x denotes its covariance. Thus, if x is a second order process, then P_{x_k} denotes the covariance of its k^{th} sample. If x is an asymptotically wide-sense stationary second order process, then P_x denotes its steady-state covariance. The cross-covariance between the second order random variables x and y is denoted by P_{xy}. $\hat{\mathcal{E}}\{a|b\}$ denotes the best linear least squares estimator of a, given b [Doob, 1953, p. 155].

2. SETUP AND PROBLEM DEFINITION

Consider a discrete-time LTI plant P modelled by

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad k \in \mathbb{N}, \quad x_0 = x_o, \tag{1a}$$
$$y_k = Cx_k + e_k, \tag{1b}$$

where x is the state, x_o is the initial state, u is the control input, v models process noise, y is a sensor output, e is measurement noise, and (A, B, C) are known matrices of appropriate dimensiones. We assume that x_o is a zero-mean second-order random variable with covariance matrix $P_o \geq 0$, independent of (v, e), and that (v, e) are zero-mean mutually independent i.i.d. sequences, having constant covariance matrices $P_v \geq 0$ and $P_e > 0$. In (1), all signal are allowed to have arbitrary (but compatible) dimensions.

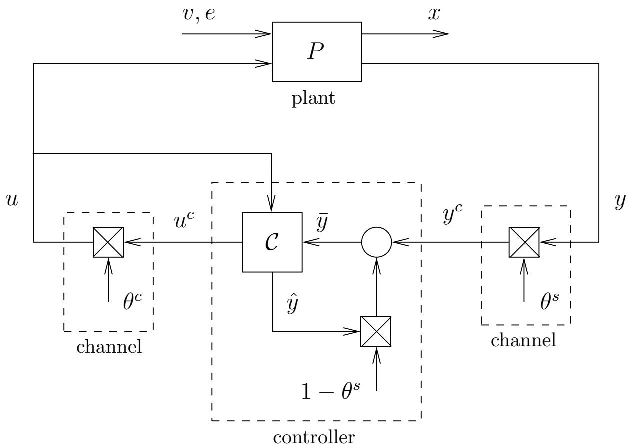

Fig. 1. Considered networked control architecture.

The plant P has to be controlled by a remote controller which communicates with the plant over two erasure channels. The plant input u is assumed to be constructed in a way such that

$$u_k = \theta_k^c u_k^c, \tag{2}$$

where θ^c is a sequence taking values in $\{0, 1\}$ which models data-dropouts in the controller to actuator link, and u^c is the controller output. The controller output, in turn, is constructed via

$$u_k^c = \mathcal{C}_k^u(\bar{y}^{k-1}, u^{k-1}), \quad \bar{y}_k = \theta_k^s y_k + (1 - \theta_k^s)\hat{y}_k, \tag{3a}$$
$$\hat{y}_k = \mathcal{C}_k^{\hat{y}}(\bar{y}^{k-1}, u^{k-1}), \tag{3b}$$

where $\mathcal{C}_k^u$ and $\mathcal{C}_k^{\hat{y}}$ are (possibly time-varying) affine mappings of their arguments, and θ^s is a sequence taking values in $\{0, 1\}$ that models data-dropouts in the sensor to controller link. The networked architecture described above is depicted in Figure 1. Two remarks are in order. First, the considered architecture assumes that the sensor-to-controller channel state is perfectly known at the controller. Such information is used to trigger a data dropout compensator which replaces missing data by suitable estimates in $\hat{y}$ (see also Silva and Solis [2013]). Second, the controller has access to one-step delayed (but otherwise perfect) controller-to-actuator channel state information. That is, the controller-to-actuator link is assumed to use a TCP-like protocol Schenato et al. [2007].

We will assume that the random processes θ^c and θ^s are i.i.d., mutually independent, and independent of (x_o, v, e). We also define $p_s \triangleq \mathcal{P}\{\theta_k^s = 1\}$ and $p_c \triangleq \mathcal{P}\{\theta_k^c = 1\}$. Thus, p_s (resp. p_c) corresponds to the successful transmission probability in the sensor-to-controller (resp. controller-to-actuator) link. We assume that both p_s and p_c are contained in $(0, 1)$.

The goal of this paper is to characterize the (possibly time-varying) mappings $\mathcal{C}_k^u$ and $\mathcal{C}_k^{\hat{y}}$ in (3) which minimize the finite horizon cost function

$$V_N \triangleq \mathcal{E}\left\{||x_N||_{Q_N}^2 + \sum_{i=0}^{N-1}\left(||x_i||_Q^2 + ||u_i||_R^2\right)\right\}, \tag{4}$$

where $||x||_R^2 \triangleq x^T R x$, and $Q \geq 0, Q_N \geq 0$ and $R > 0$ are weighting matrices.

The main result of this paper is to show that the controller with the structure in (3) which minimizes the functional

(4) can be designed by invoking a separation principle. With that objective in mind, we first solve, in Section 3 below, an estimation problem which will be shown to play a key role when characterizing optimal designs.

3. AN OPTIMAL ESTIMATION PROBLEM

Consider the plant P in (1) and a class of state estimators such that the estimate $\hat{x}_k$ of x_k is constructed via

$$\hat{x}_k = \mathcal{F}_k^{\hat{x}}(\bar{y}^{k-1}, u^{k-1}), \quad \bar{y}_k = \theta_k^s y_k + (1-\theta_k^s)\hat{y}_k, \tag{5a}$$

$$\hat{y}_k = \mathcal{F}_k^{\hat{y}}(\bar{y}^{k-1}, u^{k-1}), \tag{5b}$$

where $\mathcal{F}_k^{\hat{x}}$ and $\mathcal{F}_k^{\hat{y}}$ are (possibly time-varying) affine mappings of their arguments and, consistent with the setup described in Section 2, u satisfies (2) with u_k^c being an affine function of $(\bar{y}^{k-1}, u^{k-1})$, and both θ^c in (2) and θ^s in (5a) are as in Section 2. Denote the minimal state estimation error covariance that is achievable with an estimator in the class described by (5a) by $P_{k|k-1}$, and the corresponding optimal estimate of x_k by $\hat{x}_{k|k-1}$. Analogously, $P_{k|k}$ and $\hat{x}_{k|k}$ denote the corresponding minimum variance and optimal estimates when the state estimates are affine functions of $(\bar{y}^k, u^k)$.

Theorem 1. Consider the plant P in (1) under the assumptions of Section 2, and the class of estimators described by (5a) under the assumptions stated above. Then,

$$\hat{x}_{k|k} = (I - J_k C)\hat{x}_{k|k-1} + J_k \bar{y}_k, \tag{6a}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k, \tag{6b}$$

$$P_{k|k} = P_{k|k-1} - p_s J_k C P_{k|k-1}, \tag{6c}$$

$$P_{k+1|k} = AP_{k|k}A^T + P_v, \tag{6d}$$

where $\hat{x}_{0|-1} = 0$, $P_{0|-1} = P_o$, $\hat{y}_k = C\hat{x}_{k|k-1}$, and

$$J_k \triangleq P_{k|k-1}C^T \left(CP_{k|k-1}C^T + P_e\right)^{-1}. \tag{7}$$

Proof. Recall the notation $\hat{\mathcal{E}}\{\cdot|\cdot\}$ introduced in the last paragraph of Section 1 and define

$$\hat{x}_{j|i}^\dagger \triangleq \hat{\mathcal{E}}\left\{x_j|\bar{y}^i, u^i\right\}, \tag{8}$$

$$P_{j|i}^\dagger \triangleq \mathcal{E}\left\{\left(x_j - \hat{x}_{j|i}^\dagger\right)\left(x_j - \hat{x}_{j|i}^\dagger\right)^T\right\}. \tag{9}$$

In general, $\hat{x}_{k|k} \neq \hat{x}_{k|k}^\dagger$. Indeed, $\hat{x}_{k|k}^\dagger$ is a function of any given sequence $\bar{y}$, not necessarily the optimal one. We will first characterize $\hat{x}_{k|k}^\dagger$, $\hat{x}_{k+1|k}^\dagger$, $P_{k|k}^\dagger$ and $P_{k+1|k}^\dagger$. Then, as a second step, we will turn our attention to $\hat{x}_{k|k}$, $\hat{x}_{k+1|k}$, $P_{k|k}$ and $P_{k+1|k}$. We state the following technical lemma in order to prove our result.

Lemma 2. Consider the setup, notation and assumptions of Theorem 1 and its proof. Then,

(a) $\epsilon_k^u = (\theta_k^c - p_c)\, u_k^c$ and $\hat{\mathcal{E}}\{x_k|\epsilon_k^u\} = \mu_{x_k}$.
(b) $\epsilon_k^{\bar{y}} = \theta_k^s \tilde{y}_k - p_s\left(C\hat{x}_{k|k-1} - \hat{y}_k\right)$ with $\tilde{y}_k \triangleq y_k - \hat{y}_k$, and $\hat{\mathcal{E}}\{x_k|\epsilon_k^{\bar{y}}\} = \mu_{x_k} + \bar{J}_k\epsilon_k^{\bar{y}}$ with $\bar{J}_k$ as in (12).
(c) $\mathcal{E}\left\{\left(x_k - \hat{x}_{k|k-1}^\dagger\right)\epsilon_k^{\bar{y}T}\right\} = p_s P_{k|k-1}^\dagger C^T$.

Proof. Omitted due space constraints.

Clearly,

$$\begin{aligned}\hat{x}_{k|k}^\dagger &= \hat{\mathcal{E}}\left\{x_k|\bar{y}^k, u^k\right\} \\ &\overset{(a)}{=} \hat{\mathcal{E}}\left\{x_k|\bar{y}^k, u^{k-1}\right\} + \hat{\mathcal{E}}\left\{x_k|\epsilon_k^u\right\} - \mu_{x_k} \\ &\overset{(b)}{=} \hat{\mathcal{E}}\left\{x_k|\bar{y}^k, u^{k-1}\right\},\end{aligned} \tag{10}$$

where $\epsilon_k^u \triangleq u_k - \hat{\mathcal{E}}\left\{u_k|\bar{y}^k, u^{k-1}\right\}$, (a) follows from elementary properties of $\hat{\mathcal{E}}\{\cdot|\cdot\}$ [Doob, 1953, p. 155], and (b) follows from Lemma 2(a). By using a similar argument, we now have from (10) that

$$\begin{aligned}\hat{x}_{k|k}^\dagger &= \hat{\mathcal{E}}\left\{x_k|\bar{y}^{k-1}, u^{k-1}\right\} + \hat{\mathcal{E}}\left\{x_k|\epsilon_k^{\bar{y}}\right\} - \mu_{x_k}, \\ &\overset{(a)}{=} \hat{x}_{k|k-1}^\dagger + \bar{J}_k\epsilon_k^{\bar{y}},\end{aligned} \tag{11}$$

where $\epsilon_k^{\bar{y}} \triangleq \bar{y}_k - \hat{\mathcal{E}}\left\{\bar{y}_k|\bar{y}^{k-1}, u^{k-1}\right\}$,

$$\begin{aligned}\bar{J}_k &\triangleq p_s P_{k|k-1}^\dagger C^T \\ &\quad \times \left(p_s^2 CP_{k|k-1}^\dagger C^T + p_s^2 P_e + p_s(1-p_s)P_{\tilde{y}_k}\right)^{-1},\end{aligned} \tag{12}$$

and (a) follows from Lemma 2(b). (We also note that our assumptions imply $p_s^2 P_e > 0$.) By using (11) we can write

$$\begin{aligned}P_{k|k}^\dagger &= \mathcal{E}\left\{\left(x_k - \hat{x}_{k|k}^\dagger\right)\left(x_k - \hat{x}_{k|k}^\dagger\right)^T\right\} \\ &= P_{k|k-1}^\dagger + \bar{J}_k P_{\epsilon_k^{\bar{y}}}\bar{J}_k^T - \mathcal{E}\left\{\left(x_k - \hat{x}_{k|k-1}^\dagger\right)\epsilon_k^{\bar{y}T}\right\}\bar{J}_k^T \\ &\quad - \bar{J}_k\mathcal{E}\left\{\epsilon_k^{\bar{y}}\left(x_k - \hat{x}_{k|k-1}^\dagger\right)^T\right\} \\ &= P_{k|k-1}^\dagger - p_s\bar{J}_k CP_{k|k-1}^\dagger,\end{aligned} \tag{13}$$

where we used the definition of $\bar{J}_k$ and Lemma 2(c).

On the other hand, the linearity of $\hat{\mathcal{E}}\{\cdot|\cdot\}$, and the fact that v_k is independent [1] of $(\bar{y}^k, u^k)$ and has zero mean, imply that

$$\begin{aligned}\hat{x}_{k+1|k}^\dagger &= \hat{\mathcal{E}}\left\{Ax_k + Bu_k + v_k|\bar{y}^k, u^k\right\} \\ &= A\hat{x}_{k|k}^\dagger + Bu_k.\end{aligned} \tag{14}$$

Thus,

$$\begin{aligned}P_{k+1|k}^\dagger &= \mathcal{E}\left\{\left(x_{k+1} - \hat{x}_{k+1|k}^\dagger\right)\left(x_{k+1} - \hat{x}_{k+1|k}^\dagger\right)^T\right\} \\ &= AP_{k|k}^\dagger A^T + P_v,\end{aligned} \tag{15}$$

where we used that fact that v_k is independent of $(x_k, \bar{y}^k, u^k)$ and has zero mean.

We will now characterize the choice for $\bar{y}$ which minimizes $P_{k|k}^\dagger$ and $P_{k+1|k}^\dagger$. We use induction. Clearly, $\hat{x}_{0|-1} = \mu_o$ and $P_{0|-1} = P_o$. Assume that $\hat{x}_{k|k-1}$ and $P_{k|k-1}$ are known for some $k \in \mathbb{N}_0$. Such estimate and covariance matrix depend on a specific optimal choice for the mappings $\mathcal{F}_i^{\hat{y}}$, $i \in \{0, \ldots, k-1\}$. For such choice of mappings, $\hat{x}_{k|k-1}^\dagger = \hat{x}_{k|k-1}$ and $P_{k|k-1}^\dagger = P_{k|k-1}$. The question now arises as how to choose $\mathcal{F}_i^{\hat{y}}$, $i \in \{0, \ldots, k\}$ so as to render $P_{k|k}^\dagger$ minimum thus yielding $P_{k|k}^\dagger = P_{k|k}$. It follows from the definition of $\bar{J}_k$ and the properties of Schur complements [Bernstein, 2005, p. 281] that $P_{k|k}^\dagger$ is a nondecreasing

[1] This, and other analog claims made in the paper, follow by inspection from our assumptions, the structure assumed for the plant model and estimators, and from the fact that u_k^c is an affine function of $(\bar{y}^{k-1}, u^{k-1})$.

function of both $P^{\dagger}_{k|k-1}$ and $P_{\tilde{y}_k}$. It is also clear that the mappings $\mathcal{F}^{\hat{y}}_i$, $i \in \{0, \ldots, k-1\}$ affect $P^{\dagger}_{k|k}$ through $P^{\dagger}_{k|k-1}$ only, and that $\mathcal{F}^{\hat{y}}_k$ affects $P^{\dagger}_{k|k}$ through $P_{\tilde{y}_k}$ only.

Given our discussion above, it is immediate to conclude that the choice for $\mathcal{F}^{\hat{y}}_i$, $i \in \{0, \ldots, k-1\}$ that was optimal when calculating $\hat{x}_{k|k-1}$ is also optimal when calculating $\hat{x}_{k|k}$. Since $\hat{y}_k$ is an affine function of $(\bar{y}^{k-1}, u^{k-1})$, it also follows that the new mapping $\mathcal{F}^{\hat{y}}_k$ must be such that (see (1) and recall that e is zero mean) $\hat{y}_k = C\hat{x}_{k|k-1}$. For such choice,

$$
\begin{aligned}
P_{\tilde{y}_k} &= \mathcal{E}\left\{ \left(y_k - C\hat{x}_{k|k-1}\right)\left(y_k - C\hat{x}_{k|k-1}\right)^T \right\} \\
&= CP_{k|k-1}C^T + P_e, \qquad (16)
\end{aligned}
$$

where we used the fact that e_k is independent of $(x_k, \bar{y}^{k-1}, u^{k-1})$ and zero mean. If (16) is replaced in (12), then $\bar{J}_k = J_k$. Hence, once the above described choice for $\mathcal{F}^{\hat{y}}_i$, $i \in \{0, \ldots, k\}$, is made, (6a) and (6c) follow from (11) and (13). The above choice for $\mathcal{F}^{\hat{y}}_i$, $i \in \{0, \cdots k\}$, also minimizes $P^{\dagger}_{k|k}$. Thus, (6b) and (6d) follow from (14) and (15). The proof is thus completed.■

Theorem 1 characterizes the optimal mappings $\mathcal{F}^{\hat{x}}_k$ and $\mathcal{F}^{\hat{y}}_k$ that define the optimal estimator within the considered class. Unsurprisingly, the fact that the controller-to-actuator link uses a TCP-like protocol, makes the optimal filter in Theorem 1 identical to a related optimal filter studied by us in Silva et al. [2013] for a networked control architecture where only the sensor-to-controller link is subject to data dropouts. Furthermore, the structure of the optimal filter is akin to that studied in Schenato et al. [2007]. As foreshadowed in the Introduction, the difference lies in the fact that the covariances $P_{k|k}$ and $P_{k+1|k}$, and the corresponding filter gain J_k, are now deterministic quantities and not random variables. Our filter has a prescribed structure and is hence suboptimal. However, it is interesting to note that the corresponding estimation error covariance $P_{k|k}$ coincides with the upper bound on the expected covariance of the intermittent Kalman filter presented in Sinopoli et al. [2004]. Our results show that such upper bound corresponds to the minimal state estimation error covariance that is achievable when one constrains the estimators to have the structure in (5a).

By construction, our proposal outperforms several filters in the literature, which are special cases of the proposed filter class when the mappings $\mathcal{F}^{\hat{y}}_i$ are suitably chosen (see, e.g., Tugnait [1981], Sun et al. [2008], Liang et al. [2010], Zhang et al. [2011]). We also note that the optimal predictor estimate $\hat{x}_{k+1|k}$ in Theorem 1 is essentially the filter proposed in Zhang et al. [2012]. However, in Zhang et al. [2012], the structure of the filter recursions is *fixed* and not *deduced* as in Theorem 1.

4. OPTIMAL CONTROLLER DESIGNS

In this section we return to the problem of finding the controllers in (3) which minimizes the cost functional in (4). Our main result is stated next.

Theorem 3. Consider the NCS of Figure 1, where the plant P is described by (1), the control input is given by (2), and the controller satisfies (3). If the assumptions of Section 2 hold, then the mappings $\mathcal{C}^u_k$ and $\mathcal{C}^{\hat{y}}_k$ which minimize V_N in (4) are such that, for every $k \in \{0, \cdots, N-1\}$,

$$
u^c_k = -L_k\hat{x}_{k|k-1}, \qquad \hat{y}_k = C\hat{x}_{k|k-1}, \qquad (17)
$$

where $\hat{x}_{k|k-1}$ is as in Theorem 1,

$$
L_k \triangleq \left(R + B^T S_{k+1} B\right)^{-1} B^T S_{k+1} A, \qquad (18)
$$

and S_k satisfies the backwards recursion

$$
\begin{aligned}
S_k &= A^T S_{k+1} A + Q \\
&\quad - p_c A^T S_{k+1} B \left(R + B^T S_{k+1} B\right)^{-1} B^T S_{k+1} A, \qquad (19)
\end{aligned}
$$

$k \in \{N-1, \ldots, 0\}$, with $S_N = Q_N$. In addition, the minimal cost, say V^{opt}_N is given by

$$
\begin{aligned}
V^{\text{opt}}_N &= \mu_o^T S_0 \mu_o + \operatorname{trace}\{S_0 P_o\} + \sum_{k=0}^{N-1} \operatorname{trace}\{S_{k+1} P_v\} \\
&\quad + \sum_{k=0}^{N-1} \operatorname{trace}\left\{\left(A^T S_{k+1} A + Q - S_k\right) P_{k|k}\right\}, \qquad (20)
\end{aligned}
$$

where S_i is above and $P_{k|k}$ is as in Theorem 1.

Proof. We use a standard dynamic programming argument. Given (1), the cost function in (4) can be written as

$$
\begin{aligned}
V_N &\overset{(a)}{=} \mathcal{E}\left\{ ||Ax_{N-1} + Bu_{N-1}||^2_{Q_N} \right\} + \mathcal{E}\left\{ ||v_{N-1}||^2_{Q_N} \right\} \\
&\quad + \mathcal{E}\left\{ ||x_{N-1}||^2_Q + ||u_{N-1}||^2_R \right\} \\
&\quad + \mathcal{E}\left\{ \sum_{i=0}^{N-2} \left(||x_i||^2_Q + ||u_i||^2_R \right) \right\} \\
&\overset{(b)}{=} \mathcal{E}\left\{ ||v_{N-1}||^2_{Q_N} \right\} + \mathcal{E}\Bigg\{ ||x_{N-1}||^2_{S_{N-1}} \\
&\quad + \sum_{i=0}^{N-2} \left(||x_i||^2_Q + ||u_i||^2_R \right) \Bigg\} \\
&\quad + \mathcal{E}\left\{ \left|\left|\left(u^c_{N-1} + L_{N-1}x_{N-1}\right)\right|\right|^2_{p_c(R + B^T Q_N B)} \right\}, \qquad (21)
\end{aligned}
$$

where

$$
\begin{aligned}
L_{N-1} &\triangleq \left(R + B^T Q_N B\right)^{-1} B^T Q_N A, \qquad (22) \\
S_{N-1} &\triangleq A^T Q_N A + Q \\
&\quad - p_c A^T Q_N B \left(R + B^T Q_N B\right)^{-1} B^T Q_N A. \qquad (23)
\end{aligned}
$$

In (21), (a) follows from the fact that v_{N-1} is independent [2] of (x_{N-1}, u_{N-1}) and zero mean. On the other hand (b) can be justified as follows: Since $u_{N-1} = \theta^c_{N-1} u^c_{N-1}$, a straightforward manipulation that exploits the definition of θ^c and the fact that θ^c_{N-1} is independent of (x_{N-1}, u^c_{N-1}) yields

$$
\begin{aligned}
&\mathcal{E}\left\{ ||Ax_{N-1} + Bu_{N-1}||^2_{Q_N} + ||x_{N-1}||^2_Q + ||u_{N-1}||^2_R \right\} \\
&= \mathcal{E}\Big\{ \left|\left|Ax_{N-1} + p_c B u^c_{N-1}\right|\right|^2_{Q_N} + ||x_{N-1}||^2_Q \\
&\quad + \left|\left|u^c_{N-1}\right|\right|^2_{p_c R + p_c(1-p_c) B^T Q_N B} \Big\} \\
&= \mathcal{E}\left\{ ||x_{N-1}||^2_{S_{N-1}} \right\} \\
&\quad + \mathcal{E}\left\{ \left|\left|\left(u^c_{N-1} + L_{N-1}x_{N-1}\right)\right|\right|^2_{p_c(R + B^T Q_N B)} \right\}, \qquad (24)
\end{aligned}
$$

[2] Recall Footnote 1 on page 3.

where the last equality follows from a standard completion-of-squares argument. Equality (b) in (21) thus follows.

Now, if we define $Q_{N-1} \triangleq S_{N-1}$, then (21) and the definition of V_N yields

$$\begin{aligned} V_N &= V_{N-1} + \mathcal{E}\left\{||v_{N-1}||^2_{Q_N}\right\} \\ &\quad + \mathcal{E}\left\{\left|\left|(u^c_{N-1} + L_{N-1}x_{N-1})\right|\right|^2_{p_c(R+B^TQ_NB)}\right\} \\ &= V_{N-1} + \text{trace}\left\{Q_N P_v\right\} \\ &\quad + \text{trace}\left\{p_c\left(R + B^TQ_NB\right)\mathcal{E}\left\{MM^T\right\}\right\} \\ &\geq V_{N-1} + \text{trace}\left\{Q_N P_v\right\} \\ &+ \text{trace}\left\{p_c\left(R + B^TQ_NB\right)L_{N-1}P_{N-1|N-2}L^T_{N-1}\right\}, \end{aligned} \tag{25}$$

where $M \triangleq u^c_{N-1} + L_{N-1}x_{N-1}$ and, consistent with the notation in Section 3 and the proof of Theorem 1 (see (10)), $P_{N-1|N-2}$ denotes the minimum state estimation error variance when one estimates x_{N-1} by using an affine function of either $(\bar{y}^{N-2}, u^{N-2})$. Indeed, the inequality in (25) follows from the fact that u^c_{N-1} is an affine function of $(\bar{y}^{N-2}, u^{N-2})$ and by using well-known linear least squares estimation results [Söderström, 2002, Section 5.3]. Equality in (25) holds if

$$u^c_{N-1} = -L_{N-1}\hat{x}_{N-1|N-2}, \tag{26}$$

where, again consistent with the notation in Section 3, $\hat{x}_{N-1|N-2}$ denotes the best affine least squares estimator of x_{N-1}, given measurements of either $(\bar{y}^{N-2}, u^{N-2})$.

The argument in the above paragraph shows that the optimal choice for u^c_{N-1} is given by (26). For such choice, (see (25))

$$\begin{aligned} V_N &= V_{N-1} + \text{trace}\left\{Q_N P_v\right\} \\ &+ \text{trace}\left\{p_c\left(R + B^TQ_NB\right)L_{N-1}P_{N-1|N-2}L^T_{N-1}\right\}. \end{aligned} \tag{27}$$

Since $P_{N-1|N-2}$ does not depend on u^c (see Theorem 1), it follows that to find the optimal choice for $u^{c,N-2}$, it suffices to minimize V_{N-1}. To do so, it suffices to mimic the argument leading to (25) and (26) and (17)–(19) follows by induction. The expression for the optimal cost also follows by induction from (27), (23) and the definition of L_k.■

Theorem 3 shows that the optimal design of the proposed controllers can be separated into two stages: First, one constructs plant state estimates by using Theorem 1. Then, as the second step, one uses the state estimates so obtained, to feed a static gain which is calculated by solving a modified Ricatti recursion. It follows from a straightforward modification of the results in Schenato [2009], that the optimal controller gain in Theorem 3 defines the (unqualified) optimal control law when perfect communication between the plant and the controller is available, the state is measured without noise, and the link between the controller and the actuators is given by (2). Given our comments after Theorem 1 and the results in Silva et al. [2013], we thus conclude that a complete separation exists between estimation and control in the considered networked architecture. This conclusion hinges on the availability of channel state information at the controller. If either the sensor-to-controller or the controller-to-actuator channel states are unknown to the controller, then separation does not hold (see also Chen et al. [2012]). These observations are consistent with results pertaining to the well-known optimal LQG optimal control problem over erasure channels Schenato et al. [2007]. They apply, however, to a constrained architecture where besides elementary use of channel state information, all processing is constrained to be affine from the onset.

We finally remark that the optimal cost in (20) corresponds to the upper bound, presented in Schenato et al. [2007], on the expected minimal cost achieved by the optimal LQG controller. The latter is consistent with our comments in Section 3, where we noted that the minimum estimation error covariance of the optimal filter in Theorem 1 upper bounds the expected error covariance of the intermittent Kalman filter in Sinopoli et al. [2004].

5. OPTIMAL INFINITE HORIZON ESTIMATORS AND CONTROLLERS

This section presents conditions under which the optimal filter gain J_k in (7) and the optimal controller gain L_k in (18) converge. To that end, we study the convergence properties of the modified Riccati recursions (see (6c), (6d) and (19))

$$P_{k+1|k} = AP_{k|k-1}A^T + P_v - p_sAJ_kCP_{k|k-1}A^T, \tag{28}$$

with J_k as in (7), $k \in \mathbb{N}_0$, $P_{0|-1} = P_o$, and

$$S_k = A^TS_{k+1}A + Q - p_cA^TS_{k+1}BL_k, \tag{29}$$

with L_k as in (18), $k \in \{N-1, \ldots, 0\}$ and $S_N = Q_N$.

Lemma 4. (Schenato [2008, 2009]). Consider both modified Riccati recursions in (28)-(29) under the assumptions of Theorem 1 and Theorem 3. Assume, in addition, that the pairs (A, C) and $(A, Q^{1/2})$ are observable, and that the pairs $(A, P_v^{1/2})$ and (A, B) are controllable. Then:

(1) If $\rho(A) < 1$, then both modified Riccati recursions converge.
(2) If $\rho(A) > 1$, then there exists $p_s^{\inf}$ (resp. $p_c^{\inf}$) such that the modified Riccati recursion in (28) (resp. in (29)) converges if and only if $p_s > p_s^{\inf}$ (resp. $p_c > p_c^{\inf}$).
(3) If the modified Riccati recursions converge, then their limits, say P_∞ and S_∞, are independent of the initial conditions P_o and S_N, and correspond to the unique positive semidefinite solutions of the modified algebraic Riccati equations (MARE) that arise when one sets $P_{k+1|k} = P_{k|k-1} = P_\infty$ and $S_k = S_{k+1} = S_\infty$ in (28)-(29).■

The following corollary is a immediate consequence of Lemma 4:

Corollary 5. Consider the setup and assumptions of Theorem 1 and Theorem 3. Assume, in addition, that the pairs (A, C) and $(A, Q^{1/2})$ are observable, and that the pairs $(A, P_v^{1/2})$ and (A, B) are controllable. The estimator gain J_k in (7) and the controller gain L_k in (18) converge when k tends to infinity if and only if $p_s > p_s^{\inf}$ and $p_c > p_c^{\inf}$, or $\rho(A) < 1$. If that is the case, then:

(1) The limiting optimal choices for the mappings $\mathcal{F}_k^{\hat{x}}$ and $\mathcal{F}_k^{\hat{y}}$ in (5a) are such that

$$\hat{x}_{k|k} = (I - JC)\hat{x}_{k|k-1} + J\bar{y}_k, \tag{30a}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k, \tag{30b}$$

$$\hat{y}_k = C\hat{x}_{k|k-1}, \tag{30c}$$

where $\bar{y}$ is as in (5a), $J \triangleq P_\infty C^T \left(CP_\infty C^T + P_e\right)^{-1}$, and P_∞ is the unique positive semidefinite solution of the MARE

$$P_\infty = AP_\infty A^T + P_v - p_s AJCP_\infty A^T. \quad (31)$$

(2) The limiting optimal choices for the mappings $\mathcal{C}_k^u$ and $C_k^{\hat{y}}$ in (3) admit the state space representation

$$\xi_{k+1} = A(I - JC)\xi_k + AJ\bar{y}_k + Bu_k, \quad (32a)$$
$$u_k^c = -L\xi_k, \quad (32b)$$
$$\hat{y}_k = C\xi_k, \quad (32c)$$

where $L \triangleq \left(B^T S_\infty B + R\right)^{-1} B^T S_\infty A$, and S_∞ is the unique positive semidefinite solution of the MARE:

$$S_\infty = A^T S_\infty A + Q - p_c A^T S_\infty BL. \quad (33)$$

(3) The average cost $\bar{V}_\infty \triangleq \lim_{N\to\infty} \frac{1}{N} V_N$ converges to $\bar{V}_\infty^{\text{opt}}$, with

$$\bar{V}_\infty^{\text{opt}} \triangleq \text{tr}\Big\{S_\infty P_v + \left(A^T S_\infty A + Q - S_\infty\right)\left(P_\infty - p_s JCP_\infty\right)\Big\}, \quad (34)$$

with J, S_∞ and P_∞ as above.■

Corollary 5 provides conditions which guarantee that the derived estimators and controllers in Sections 3 and 4 converge as the horizon N tends to infinity. The fact that there exists a separation between estimation and control in the finite horizon case, allowed us to study convergence of the estimator and the controller in an independent way (see Lemma 4 and Corollary 5). The derived convergence conditions are the same conditions obtained previously in Schenato [2008, 2009] for different setups.

It is also important to highlight that the limiting controller described by Corollary 5 is in fact optimal. That is, if the LTI filter in (32) is used to map $(\bar{y}, u)$ into $(u^c, \hat{y})$ in the NCS of Figure 1, then the steady-state average cost $\bar{V}_\infty$ will be minimal among all LTI filters that map $(\bar{y}, u)$ into $(u^c, \hat{y})$ and render the resulting NCS mean-square stable. [3]

6. CONCLUSIONS

This paper has solved finite and infinite horizon optimal control problems for noisy LTI plants when both sensor-to-controller and controller-to-actuator communication take place over analog erasure channels. We have focused on a class of controllers that embed a data dropout compensator and where, besides elementary use of sensor-to-controller channel state information, all processing is affine. We have shown that separation holds, and that, as the horizon length goes to infinity, the affine part of the optimal controller converges to an LTI system and thus a computationally inexpensive controller can be found. Moreover, the steady-state performance of that filter can be easily characterized. Future work should focus on multichannel architectures and on architectures without feedback in the controller-to-actuator link.

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[3] We will include a complete proof of this fact in a future paper.