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Switched Linear Systems Control Design: A Transfer Function Approach ⋆

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Abstract: In this paper we develop a new control design procedure for continuous-time switched linear systems. Beyond the global stability, two performance indexes based on $\mathcal{H}_\infty$ theory and passivity are considered. The proposed switching control design is entirely based on convex combinations of subsystems transfer functions. In this precise context a new min-type switching function depending on the state and input variables is introduced which opens the possibility to generalize the same ideas to obtain less conservative solutions to other control design problems of switched systems appearing in the literature. An illustrative example is solved and discussed.

Keywords: Switched Systems, Continuous-time Systems, Hybrid systems, $\mathcal{H}_\infty$ and Passivity.

1. INTRODUCTION

Switched systems constitute an important subclass of hybrid systems characterized by presenting several subsystems and a switching rule that selects, at each time instant, one of them to be connected. The switching rule can act in two different ways. First, it can be arbitrary and may present an unbounded switching frequency playing the role of a severe external perturbation, or may respect a pre-specified interval of time in which the switching rule remains unchanged and a subsystem is switched on by preserving a given dwell time. In this first case, the control goal is to assure stability and performance even in the presence of the external perturbation or to determine the minimum dwell time in order to accomplish the same goals. In the second case, the switching rule can be a control variable to be determined in order to preserve stability and impose to the overall switched system a performance as good as possible. In this paper, we treat the second case where the switching rule is the control variable. The books Liberzon [2003], Sun & Ge [2005] and the papers Decarlo et al. [2000], Liberzon & Morse [1999], Lin & Antsaklis [2009], Shorten et al. [2007] are useful references for early theoretical developments on these topics.

The stability analysis of continuous-time switched linear systems has been treated by several authors, as for instance, Branicky [1998], Decarlo et al. [2000], Geromel & Colaneri [2006], Hespanha [2004] and Liberzon & Morse [1999]. The increasing interest on this subclass of systems is motivated by some of their important features which can not be directly analyzed by the classical methods available in the literature. As for instance, if all subsystems are stable, an unappropriated switching action can lead to an unstable behavior. On the other hand, if the switching rule is conveniently designed, the stability of the overall switched system can be assured even if all subsystems are unstable. Moreover, the consistency property defined in the recent reference Geromel et al. [2013] makes clear the importance of the switching rule design, since it can enhance the overall performance when compared to that of each isolated subsystem. The stabilization results have been generalized to cope with state feedback control Geromel & Deaecto [2009], Ji et al. [2005], Skafidas et al. [1999] and output feedback control Deaecto et al. [2011], Geromel et al. [2008].

Due to the success obtained in the field of switched linear systems, the interest in the study of switched nonlinear systems is increasing, as reveal the recent references Aleksandrov et al. [2011], Colaneri & Geromel [2008], Long & Zhao [2011], Moulay et al. [2007], Sun & Wang [2013], Wang et al. [2009], Wu [2009], Yang et al. [2009], and Zhao & Hill [2008]. More specifically, Colaneri & Geromel [2008], Long & Zhao [2011], Wang et al. [2009], Yang et al. [2009] treat the stability analysis of general switched nonlinear systems, in which Long & Zhao [2011] and Wu [2009] consider the case where the switching rule is arbitrary and Wang et al. [2009], Yang et al. [2009] deal with the design of a stabilizing one. In Colaneri & Geromel [2008] the two classes already mentioned of the switching rule are treated. For switched nonlinear systems an important subclass is the one composed by the Lur'e-type switched systems. They are characterized by presenting a feedback connection of a switched linear system and a nonlinearity bounded by a sector. For time invariant Lur'e-type systems the celebrated Popov criterion is an important issue, in which the stability analysis is based on a condition formulated in the frequency domain. However, to the best of

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our knowledge, there is no stability test in the frequency domain allowing us to determine a stabilizing switching rule for Lur'e-type switched nonlinear systems. It is important to mention that finding an interpretation in the frequency domain is far from being trivial since switched systems are time-varying and, in principle, they do not admit a frequency domain representation. In other words, the determination of a stabilizing switching function, in general, can not be done on the basis of the subsystem transfer functions. Hence, the problem of finding a transfer function that not only represents the switched system but also allows us to obtain a globally stabilizing switching function opens the possibility to treat several control design problems of the literature as, for instance, the generalization of the Popov criterion to cope with switched systems, that we have just discussed.

In this paper, we focus on determination of a stabilizing state-input switching function based on a transfer function approach. More specifically, our results are derived from the existence conditions of the Lyapunov-Metzler inequalities firstly introduced in Geromel & Colaneri [2006]. A new class of switching function depending on the state and on the exogenous input variables is the key point that allow us to accomplish the goal of reducing the design conditions to the search of an adequate convex combination of subsystems state space matrices. A performance measured in terms similar to $\mathcal{H}_\infty$ norm is addressed. With respect to this particular class of performance index we go beyond the previous existing results available in the literature as for instance Zhai [2012]. As a natural generalization, passivity of switched systems is also treated. The theory is illustrated by means of an academical example.

The notations are standard. For real matrix A or vectors, A' indicates transpose of A. For symmetric matrices, the symbol $(\bullet)$ denotes each of its symmetric blocks. The convex combination of matrices $\{J_1, \cdots, J_N\}$ with the same dimensions is denoted by $J_\lambda = \sum_{j=1}^N \lambda_j J_j$ where $\lambda = [\lambda_1 \cdots \lambda_N]' \in \mathbb{R}^N$ belongs to the unitary simplex Λ composed by all nonnegative vectors $\lambda \in \mathbb{R}^N$ such that $\sum_{j=1}^N \lambda_j = 1$. The set $\mathcal{M}_c$ is composed by all Metzler matrices $\Pi = \{\pi_{ji}\} \in \mathbb{R}^{N\times N}$ with nonnegative off diagonal elements satisfying $\sum_{j\in\mathbb{K}} \pi_{ji} = 0$ for all $i \in \mathbb{K}$. The norm of a trajectory defined for all $t \geq 0$ is given by $\|w\|_2^2 = \int_0^\infty w(t)'w(t)dt$ and $\mathcal{L}_2$ denotes the set of all trajectories with finite norm, that is $\|w\|_2 < \infty$. A square matrix is called Hurwitz stable if all eigenvalues are located in the open left part of the complex plane.

2. PRELIMINARIES

Consider a switched linear system with the following state space representation

$$\dot{x} = A_\sigma x + H_\sigma w \tag{1}$$

$$z = E_\sigma x + G_\sigma w \tag{2}$$

evolving from the initial condition $x(0) = 0$. The vectors $x(\cdot) \in \mathbb{R}^n$, $w(\cdot) \in \mathbb{R}^m$ and $z(\cdot) \in \mathbb{R}^r$ are the state, the exogenous input and the controlled output, respectively. The switching function to be designed denoted by $\sigma(\cdot)$ selects at each instant of time $t \geq 0$ a subsystem among those belonging to the set $\mathbb{K} = \{1, \cdots, N\}$. In the sequel, we analyze some relevant aspects that arise in the design of a suitable state-input switching function of the form $\sigma(x, w)$ whose stability and performance design conditions are expressed in terms of a convex combination of certain affine matrix functions. For the moment, in order to establish the mentioned conditions for global stability and performance optimization, we consider the Lyapunov function candidate

$$v(x) = \min_{i\in\mathbb{K}} x'P_i x \tag{3}$$

with symmetric matrices $0 < P_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{K}$ to be determined. Moreover, whenever associated to this min-type Lyapunov function, the switching strategy

$$\sigma(x) = \arg\min_{i\in\mathbb{K}} x'P_i x \tag{4}$$

preserves global asymptotical stability and performance of the closed-loop switched linear system under consideration as it can be viewed in several papers included as references. Notice that the design of this kind of switching function depends exclusively on the determination of positive definite matrices P_i, $\forall i \in \mathbb{K}$ yielding the well known Lyapunov and Riccati-Metzler inequalities, see Geromel & Colaneri [2006], Geromel et al. [2008], Deaecto et al. [2011]. The results provided in this paper are obtained from the adoption of the same Lyapunov function (3) but with

$$\sigma(x, w) = \arg\min_{i\in\mathbb{K}} \begin{bmatrix} x \\ w \end{bmatrix}' \mathcal{R}_i \begin{bmatrix} x \\ w \end{bmatrix} \tag{5}$$

as the associated switching function. Of course, matrices $\mathcal{R}_i$ for all $i \in \mathbb{K}$ of compatible dimensions can be constrained in an obvious way such that this state-input dependent switching strategy collapses to the state dependent one (4). As it will be clear in the sequel, this more general switching function has an important impact as far as performance quality of the closed-loop switched system is concerned but, clearly, for implementation it needs the online measurement of the state and the exogenous input of system (1)-(2).

The switched linear system (1)-(2) is composed by N subsystems with transfer functions $\mathcal{S}_i(s) = E_i(sI - A_i)^{-1}H_i + G_i$, $\forall i \in \mathbb{K}$. If the matrix A_i is Hurwitz then we can determine its $\mathcal{H}_\infty$ norm as being $\|\mathcal{S}_i\|_\infty$ or verify if it is strictly positive real, that is if $\mathcal{S}_i(-j\omega)' + \mathcal{S}_i(j\omega) > 0$ for all $\omega \in \mathbb{R}$. In this paper, we will show that a switching function of the form (5) can be designed from the determination of $\lambda \in \Lambda$ such that the transfer function $\mathcal{S}_\lambda(s) = E_\lambda(sI - A_\lambda)^{-1}H_\lambda + G_\lambda$ reaches a pre-specified property as $\mathcal{H}_\infty$ or positive realness.

3. STABILITY

In this section, we analyze global asymptotical stability of the switched linear system (1) with zero input $w = 0$ and arbitrary initial condition $x(0) = x_0 \in \mathbb{R}^n$. The following theorem, which is a consequence of several well known results, is central for the existence of a stabilizing switching function of type (4). It states that a stabilizing strategy exists provided that the set of matrices $\{A_i\}_{i\in\mathbb{K}}$ admits a Hurwitz stable convex combination.

Theorem 1. Suppose there exist $0 < P \in \mathbb{R}^{n\times n}$ and $\lambda \in \Lambda$ such that

$$\sum_{i\in\mathbb{K}} \lambda_i \mathcal{L}_i(P) < 0 \tag{6}$$

where $\mathcal{L}_i(P) = A_i'P + PA_i, \ \forall i \in \mathbb{K}$. There exists a switching function of the form (4) such that the continuous-time switched linear system $\dot{x}(t) = A_{\sigma(x(t))}x(t)$ is globally asymptotically stable.

Proof: Assume that (6) holds for some $\lambda \in \Lambda$ and $P > 0$. Define $-R_i = A_i'P + PA_i$ for all $i \in \mathbb{K}$ and notice that $\sum_{i\in\mathbb{K}} \lambda_i R_i > 0$. Furthermore, $\Pi = -I + \lambda e' \in \mathcal{M}_c$ where $e = [1 \cdots 1]' \in \mathbb{R}^N$. Setting $W_i = W_N + (R_N - R_i), \ i \in \mathbb{K}$ with W_N arbitrary we have for each $i \in \mathbb{K}$

$$\begin{aligned}\sum_{j\in\mathbb{K}} \pi_{ji} W_j &= \sum_{j\in\mathbb{K}} \lambda_j W_j - W_i \\ &= -\sum_{j\in\mathbb{K}} \lambda_j R_j + R_i \\ &< R_i \end{aligned} \tag{7}$$

Now, choosing $W_N > 0$ big enough we can compute $W_i > 0, \ i \in \mathbb{K}$ satisfying the matrix inequalities

$$A_i'P + PA_i + \sum_{j\in\mathbb{K}} \pi_{ji} W_j < 0, \ i \in \mathbb{K} \tag{8}$$

and, consequently, taking $\mu > 0$ large enough we can add the quantity $\mu^{-1}(A_i'W_i + W_iA_i)$ on the left hand side of inequality (8) to obtain

$$A_i'P_i + P_iA_i + \sum_{j\in\mathbb{K}} \pi_{ji} W_j < 0, \ i \in \mathbb{K} \tag{9}$$

where $P_i = P + \mu^{-1}W_i > 0, \ i \in \mathbb{K}$. Finally, defining $\overline{\Pi} = \mu\Pi \in \mathcal{M}_c$ and observing that the following equality $\sum_{j\in\mathbb{K}} \pi_{ji} W_j = \sum_{j\in\mathbb{K}} \overline{\pi}_{ji} P_i$ holds for all $i \in \mathbb{K}$, it is seen that

$$A_i'P_i + P_iA_i + \sum_{j\in\mathbb{K}} \overline{\pi}_{ji} P_j < 0, \ i \in \mathbb{K} \tag{10}$$

hold which means that we have found matrices $P_i > 0, \ i \in \mathbb{K}$ satisfying the Lyapunov-Metzler inequalities, see Geromel & Colaneri [2006]. From this fact, the same reference ensures that the switching function (4) is globally stabilizing and the proof is concluded. □

The existence of a Hurwitz stable convex combination of matrices $\{A_i\}_{i\in\mathbb{K}}$ assures the existence of a solution to the Lyapunov-Metzler inequalities is a known fact already pointed out in Geromel & Colaneri [2006], Geromel et al. [2008]. However, in the present framework the novelty is that matrices $P_i, \ i \in \mathbb{K}$ do not need to be determined for the switching function implementation. Indeed, from the proof of Theorem 1, it can be verified that

$$\begin{aligned}\arg\min_{i\in\mathbb{K}} x'P_ix &= \arg\min_{i\in\mathbb{K}} x'W_ix \\ &= \arg\max_{i\in\mathbb{K}} x'R_ix \\ &= \arg\min_{i\in\mathbb{K}} x'\mathcal{L}_i(P)x \end{aligned} \tag{11}$$

which depends on the matrix $P > 0$ satisfying (6), exclusively. This result makes clear that a global stabilizing switching function exists and can be determined whenever there exists $P > 0$ such that $A_\lambda'P + PA_\lambda < 0$ for some $\lambda \in \Lambda$. The determination of a feasible pair (P, λ), if any, is not a simple task but can be simplified if we search directly $\lambda \in \Lambda$ such that A_λ is Hurwitz. This numerical aspect will be discussed in the sequel.

4. PERFORMANCE

We consider now the switched linear system (1)-(2). Although any performance index can, in principle, be adopted we focus our attention to two different performance indexes that have important consequences in global stabilization of robust and switched nonlinear systems. The main goal is to search a min-type switching strategy of the form

$$\sigma(x, w) = \arg\min_{i\in\mathbb{K}} \begin{bmatrix} x \\ w \end{bmatrix}' \mathcal{R}_i \begin{bmatrix} x \\ w \end{bmatrix} \tag{12}$$

where the augmented matrices $\mathcal{R}_i \in \mathbb{R}^{(n+m)\times(n+m)}$ for all $i \in \mathbb{K}$ are symmetric and have to be determined in such a way that a pre-specified level of the performance index under consideration is attained by the closed-loop switched system.

4.1 $\mathcal{H}_\infty$ Performance

The $\mathcal{H}_\infty$ performance index is defined for any asymptotical stabilizing switching strategy denoted $\sigma \in \mathcal{A}$, as being

$$J_\infty(\sigma) = \sup_{0\neq w\in\mathcal{L}_2} \frac{\|z\|_2^2}{\|w\|_2^2} \tag{13}$$

whose rationale stems on the fact that it equals the $\mathcal{H}_\infty$ squared norm of the i-th subsystem transfer function whenever $\sigma(t) = i \in \mathcal{A}$ is kept constant for all $t \geq 0$. Ideally, we want to determine a minimum guaranteed cost associated to the optimal control problem $\inf_{\sigma\in\mathcal{A}} J_\infty(\sigma)$. However, as we know, the optimal solution of this problem is virtually impossible to be calculated due to the discontinuous nature of the switching function. Hence, we focus on a sub-optimal solution by searching a switching function of the form (12). The next theorem puts in evidence the conditions for the existence of a switching strategy that imposes to the closed-loop switched system a pre-specified guaranteed $\mathcal{H}_\infty$ performance level associated to $J_\infty(\sigma)$ defined in (13).

Theorem 2. Assume the output matrices satisfy $(E_i, G_i) = (E, G)$ for all $i \in \mathbb{K}$ and suppose there exist $0 < P \in \mathbb{R}^{n\times n}$, $\rho > 0$ and $\lambda \in \Lambda$ such that

$$\sum_{i\in\mathbb{K}} \lambda_i \mathcal{L}_i(P, \rho) < 0 \tag{14}$$

where

$$\mathcal{L}_i(P, \rho) = \begin{bmatrix} A_i'P + PA_i & PH_i \\ \bullet & -\rho I \end{bmatrix} + \begin{bmatrix} E' \\ G' \end{bmatrix} \begin{bmatrix} E' \\ G' \end{bmatrix}', \ i \in \mathbb{K} \tag{15}$$

There exists a switching function of the form (12) with $\mathcal{R}_i = \mathcal{L}_i(P, \rho)$ such that the continuous-time switched linear system (1)-(2) is globally asymptotically stable and $J_\infty(\sigma) < \rho$.

Proof: We prove directly the claim from the adoption of the quadratic Lyapunov function $v(x) = x'Px$ where $P > 0$ has to be adequately determined. To this end, calculating the time derivative along a trajectory of the continuous-time switched linear system (1)-(2), after simple algebraic manipulations at an arbitrary instant of time $t > 0$ we have

$$\dot{v}(x) + z'z - \rho w'w = \begin{bmatrix} x \\ w \end{bmatrix}' \mathcal{L}_{\sigma(x,w)}(P, \rho) \begin{bmatrix} x \\ w \end{bmatrix} \tag{16}$$

Hence, choosing matrices $\mathcal{R}_i = \mathcal{L}_i(P,\rho)$ for all $i \in \mathbb{K}$, the switching function (12) and the existence of a pair (P,ρ) satisfying (14) allow us to rewrite equality (16) as

$$\begin{aligned} \dot{v}(x) + z'z - \rho w'w &= \min_{i\in\mathbb{K}} \begin{bmatrix} x \\ w \end{bmatrix}' \mathcal{L}_i(P,\rho) \begin{bmatrix} x \\ w \end{bmatrix} \\ &= \min_{\lambda\in\Lambda} \begin{bmatrix} x \\ w \end{bmatrix}' \left(\sum_{i\in\mathbb{K}} \lambda_i \mathcal{L}_i(P,\rho) \right) \begin{bmatrix} x \\ w \end{bmatrix} \\ &< 0 \ \forall (x,w) \neq 0 \end{aligned} \tag{17}$$

from which the claim follows, because setting $w = 0$ it is seen that $\dot{v}(x) < 0, \forall x \neq 0$ implies global asymptotical stability which together with $x(0) = 0$ enforces $\|z\|_2^2 < \rho \|w\|_2^2$ for all $w \neq 0$, consequently, $J_\infty(\sigma) < \rho$ and the proof is concluded. □

At this point it is important to make clear that the choice of the switching rule of the form (12) is crucial to get the result of Theorem 2. Indeed, if instead of (12) we consider a pure state dependent switching function of the form $\sigma(x) = \arg\min_{i\in\mathbb{K}} x' R_i x$, then inequality (16) reduces to

$$\begin{aligned} \dot{v}(x) + z'z - \rho w'w &= \begin{bmatrix} x \\ w \end{bmatrix}' \mathcal{L}_{\sigma(x)}(P,\rho) \begin{bmatrix} x \\ w \end{bmatrix} \\ &\leq x' \mathcal{N}_{\sigma(x)}(P,\rho) x \end{aligned} \tag{18}$$

which holds from the determination of the worst input perturbation depending on the state variable x and the switching function σ, yielding

$$x' \mathcal{N}_{\sigma(x)}(P,\rho) x = \sup_{w} \begin{bmatrix} x \\ w \end{bmatrix}' \mathcal{L}_{\sigma(x)}(P,\rho) \begin{bmatrix} x \\ w \end{bmatrix} \tag{19}$$

where

$$\begin{aligned} \mathcal{N}_i(P,\rho) = & A_i'P + PA_i + E'E + \\ & +(PH_i - E'G)(\rho I - G'G)^{-1}(PH_i - E'G)' \end{aligned}$$

provided that $\rho I > G'G$. Hence, the result of Theorem 2 remains valid if we set $R_i = \mathcal{N}_i(P,\rho),\ i \in \mathbb{K}$ and assume the existence of $\lambda \in \Lambda$ such that $\sum_{i\in\mathbb{K}} \lambda_i \mathcal{N}_i(P,\rho) < 0$. As we can see, unfortunately, this last inequality is a convex combination of N quadratic matrix functions for which it can be verified that

$$\mathcal{N}_\lambda(P,\rho) \leq \sum_{i\in\mathbb{K}} \lambda_i \mathcal{N}_i(P,\rho) \ \forall \lambda \in \Lambda \tag{20}$$

holds. Consequently, in this case, the only way to translate the result of Theorem 2 in terms of a convex combination of the state space matrices is to assume that the input matrices do not depend on the switching strategy, that is $H_i = H, \forall i \in \mathbb{K}$, which implies that $\mathcal{N}_\lambda(P,\rho) = \sum_{i\in\mathbb{K}} \lambda_i \mathcal{N}_i(P,\rho)$ for all $\lambda \in \Lambda$. This is exactly the result reported in Zhai [2012] which we have generalized by adopting a more general class of state and input dependent switching functions. It is important to stress that, for the same reason, the result of Theorem 2 has the same limitation if just one of the output matrices depend on the switching function. In this case, from (15), in a similar way, we have

$$\mathcal{L}_\lambda(P,\rho) \leq \sum_{i\in\mathbb{K}} \lambda_i \mathcal{L}_i(P,\rho) \ \forall \lambda \in \Lambda \tag{21}$$

which allows us to conclude that the convex combination does not enforce any performance upper bound to the closed-loop system. The determination of $\mathcal{H}_\infty$ design conditions for general switched linear systems whose state space realization matrices depend on the switching rule remains an open problem of great interest. This is also true for the whole class of discrete-time switched linear systems.

For the class of continuous-time switched linear systems (1)-(2) characterized by the fact that the output matrices do not depend on the switching function, then $\mathcal{L}_i(P,\rho)$ defined in (15) is linear with respect to the pair of remaining matrices (A_i, H_i) which implies that inequality (14) holds if and only if there exist $P > 0$ and $\rho > 0$ such that

$$\begin{bmatrix} A_\lambda'P + PA_\lambda & PH_\lambda \\ \bullet & -\rho I \end{bmatrix} + \begin{bmatrix} E' \\ G' \end{bmatrix} \begin{bmatrix} E' \\ G' \end{bmatrix}' < 0 \tag{22}$$

for some $\lambda \in \Lambda$. Clearly, this is equivalent to say that A_λ is Hurwitz and $\|\mathcal{S}_\lambda\|_\infty^2 < \rho$ which indicates that we have to determine the optimal convex combination from the solution of the nonconvex problem

$$\min_{\lambda\in\Lambda} \|\mathcal{S}_\lambda\|_\infty^2 \tag{23}$$

which is not easy to solve mainly due to the intricate dependence of the $\mathcal{H}_\infty$ norm on the elements of the matrices that define the objective function to be minimized.

This aspect is relevant for several reasons, in particular, as far as consistency is concerned, see Geromel et al. [2013] for details. Indeed, a switching strategy $\sigma(\cdot)$ is said *strictly consistent* if the switched linear system has better performance than the performance of each isolated subsystem. Assuming that A_i for some $i \in \mathbb{K}$ is Hurwitz, because otherwise strict consistency follows trivially since $\|\mathcal{S}_i(s)\|_\infty^2$ is unbounded then, under this assumption, strict consistency holds whenever

$$\min_{i\in\mathbb{K}} \|\mathcal{S}_i\|_\infty^2 - \min_{\lambda\in\Lambda} \|\mathcal{S}_\lambda\|_\infty^2 > 0 \tag{24}$$

and we conclude that the optimal solution of problem (23) provides a consistent solution which, in general, is strictly consistent if it belongs to the strict interior of the unitary simplex Λ. This performance gain is due exclusively to the min-type switching strategy (12) that we have designed.

4.2 Passivity

The concept of passivity applied to the switched linear system (1)-(2) follows from the consideration of the following cost associated to any stabilizing switching strategy $\sigma \in \mathcal{A}$

$$J_+(\sigma) = \sup_{w\in\mathcal{L}_2} -\int_0^\infty z(t)'w(t)dt \tag{25}$$

and requires that the dimensions of the input and output vectors be the same, that is $m = r$. Our main purpose is to determine a switching strategy of the form (12) such that $J_+(\sigma) = 0$. In this case the closed-loop system is said *passive*, see Geromel et al. [2012] for details. It is interesting to observe that if we set $\sigma(t) = i \in \mathbb{K}$ for all $t \geq 0$ and assume that such strategy belongs to $\mathcal{A}$ then passivity of the i-th subsystem is equivalent to strict positive realness of the transfer function $\mathcal{S}_i(s)$, a property that can be tested by $\mathcal{S}_i(-j\omega)' + \mathcal{S}_i(j\omega) > 0, \forall \omega \in \mathbb{R}$.

Theorem 3. Suppose there exist $0 < P \in \mathbb{R}^{n\times n}$ and $\lambda \in \Lambda$ such that

$$\sum_{i\in\mathbb{K}} \lambda_i \mathcal{L}_i(P) < 0 \tag{26}$$

where

$$\mathcal{L}_i(P) = \begin{bmatrix} A_i'P + PA_i & PH_i - E_i' \\ \bullet & -G_i - G_i' \end{bmatrix}, \; i \in \mathbb{K} \tag{27}$$

There exists a switching function of the form (12) with $\mathcal{R}_i = \mathcal{L}_i(P)$ such that the continuous-time switched linear system (1)-(2) is globally asymptotically stable and $J_+(\sigma) = 0$.

Proof: As in the proof of Theorem 2, we prove directly the claim from the adoption of the quadratic Lyapunov function $v(x) = x'Px$ where $P > 0$ has to be adequately determined. To this end, calculating the time derivative along a trajectory of the continuous-time switched linear system (1)-(2), at an arbitrary instant of time $t > 0$ we have

$$\dot{v}(x) - z'w - w'z = \begin{bmatrix} x \\ w \end{bmatrix}' \mathcal{L}_{\sigma(x,w)}(P) \begin{bmatrix} x \\ w \end{bmatrix} \tag{28}$$

Hence, choosing matrices $\mathcal{R}_i = \mathcal{L}_i(P)$ for all $i \in \mathbb{K}$, the switching function (12) and the existence of $P > 0$ satisfying (26) allow us to write

$$\begin{aligned} \dot{v}(x) - z'w - w'z &= \min_{i\in\mathbb{K}} \begin{bmatrix} x \\ w \end{bmatrix}' \mathcal{L}_i(P) \begin{bmatrix} x \\ w \end{bmatrix} \\ &= \min_{\lambda\in\Lambda} \begin{bmatrix} x \\ w \end{bmatrix}' \left(\sum_{i\in\mathbb{K}} \lambda_i \mathcal{L}_i(P) \right) \begin{bmatrix} x \\ w \end{bmatrix} \\ &< 0 \; \forall (x,w) \neq 0 \end{aligned} \tag{29}$$

from which the claim follows because setting $w = 0$ it is seen that $\dot{v}(x) < 0, \forall x \neq 0$ implies global asymptotical stability which together with $x(0) = 0$ enforces

$$-\int_0^\infty z(t)'w(t)dt < 0 \; \forall (x,w) \neq 0 \tag{30}$$

and the supremum is clearly attained at $w = 0$ which is the claim. □

From the result of Theorem 3, it is evident that inequality (26) holds if and only if there exists $P > 0$ such that $\mathcal{L}_\lambda(P) < 0$ for some $\lambda \in \Lambda$ which by its turn implies that this is true if and only if the transfer function $\mathcal{S}_\lambda(s)$ is positive real for some $\lambda \in \Lambda$. Hence, Theorem 3 puts in evidence the quality of the state-input switching strategy proposed in this paper. Indeed, if the switching strategy is constrained to be only state dependent then the same result remains valid only for a restrictive subclass of switched linear systems characterized by having only matrix A_σ switching dependent.

Remark 1. The proposed switching strategy is clearly *consistent* in the sense that it may render passive a switched linear system composed by non-passive subsystems, exclusively. This is an important aspect of the proposed result.

Remark 2. The determination of $\lambda \in \Lambda$ such that $\mathcal{S}_\lambda(s)$ is positive real can be faced in the frequency domain by solving the nonconvex programming problem

$$\sup_{\mu,\lambda\in\Lambda_a} \{\mu : \mathcal{S}_\lambda(-j\omega)' + \mathcal{S}_\lambda(j\omega) > \mu I, \forall \omega \in \mathbb{R}\} \tag{31}$$

where $\Lambda_a \subset \Lambda$ is the set of all $\lambda \in \Lambda$ such that A_λ is Hurwitz stable and verifying if at the optimal solution it provides $\mu_{opt} > 0$.

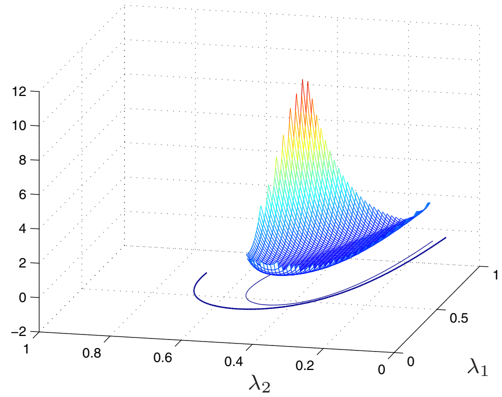

Fig. 1. Stability domain and J_∞ index.

5. EXAMPLE

This section is entirely devoted to present and discuss an academical example to illustrate the results provided in this paper. To this end, we consider a switched linear system of the form (1)-(2) composed by $N = 3$ forth order, unstable, SISO subsystems given by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -4 & -9 & 0 \end{bmatrix}, H_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -6 & -6 \end{bmatrix}, H_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & -7 & -3 & 0 \end{bmatrix}, H_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and the output matrices $E_i = E = [1\ 0\ 0\ 0]$ and $G_i = G = 1$ which are the same for all $i \in \mathbb{K}$.

5.1 Stability and $\mathcal{H}_\infty$ performance

Figure 1 shows the external curve with inside all points $\lambda \in \Lambda$ such that A_λ is Hurwitz and the internal curve with inside all points such that the eigenvalues of A_λ satisfy $\mathrm{Re}(s) < -0.1$. For each point of this last region, the same figure provides the value of the squared norm $\|\mathcal{S}_\lambda\|_\infty^2$. Finally, an exhaustive search gives the optimal value of problem (23) as being $\lambda_{opt} \approx [0.29\ 0.33\ 0.38]'$ and $\|\mathcal{S}_{\lambda_{opt}}\|_\infty^2 \approx 1.1608$.

5.2 Time simulation

It is interesting to know that there exists a suitable switching strategy such that when applied to this switched system, composed by three unstable subsystems, the closed-loop system presents the remarkable performance $J_\infty(\sigma) < \|\mathcal{S}_{\lambda_{opt}}\|_\infty^2 \approx 1.1608$. This is an immediate consequence of Theorem 2. Indeed, setting $\lambda = \lambda_{opt}$ we are able to determine $(P_{opt} > 0, \rho_{opt} > 0)$ which minimizes ρ subject to the LMI (14). Doing this we obtain the augmented

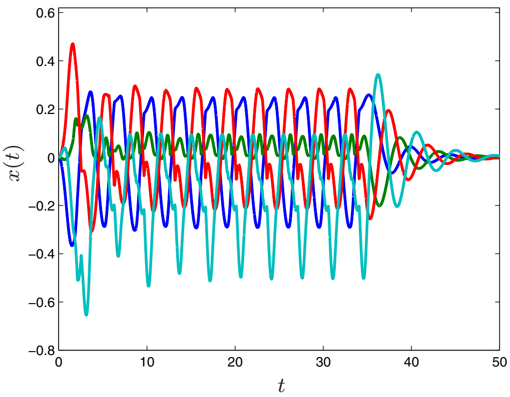

Fig. 2. Time simulation.

matrices $\mathcal{R}_i = \mathcal{L}_i(P_{opt}, \rho_{opt})$ for all $i \in \mathbb{K}$ that are used to implement the desired switched rule (12).

Figure 2 shows the time simulation of the closed-loop switched system with input $w(t) = \sin(\omega_{opt} t)$ for all $0 \leq t \leq 10(2\pi/\omega_{opt})$ and $w(t) = 0$ for all $t > 10(2\pi/\omega_{opt})$ where $\omega_{opt} = 1.8042$ rad/s has been determined from $\omega_{opt} = \arg\max_{\omega\in\mathbb{R}} |\mathcal{S}_{\lambda_{opt}}(j\omega)|$. Numerically, we have determined $\|w\|_2^2 \approx 17.41$ and $\|z\|_2^2 \approx 15.85$ which leads to the lower bound $J_\infty(\sigma) > 0.91$. Even though all subsystems are unstable, the switching rule brings all the states to zero whenever the input vanishes.

6. CONCLUSION

This paper is entirely devoted to the design of a min-type switching strategy based on the convex combination of the subsystems transfer functions. The novelty is the proposition of a new class of state-input dependent switching functions that allows to consider a wider class os switched linear systems. Performance indexes similar to $\mathcal{H}_\infty$ norm and passivity that are usual for LTI systems are discussed and generalized. An illustrative example puts in evidence the usefulness of the proposed methodology.

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