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A Positive Observer for Linear Systems

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Abstract: We develop a positive observer for general (i.e. non necessarily positive) linear time varying systems, in both the continuous and discrete time cases. A nice feature of the approach is that no change of coordinates is needed. The observer size is twice the size of the observed system and it is stable whenever the observed system is stable. The design is based on a stable internally positive representation of linear systems that is also an original contribution of the paper. This positive observer can be used to develop interval observers and controllers for systems with several kinds of uncertainties.

1. INTRODUCTION

The problem of designing positive estimates of the state of a dynamical system arises in two different but related areas of investigation. In the first place, this problem has been considered in the context of positive system. In this case the goal is to provide nonnegative estimates of nonnegative states (see for example Ait Rami et al. [2011], Back and Astolfi [2008], Härdin and Van Schuppen [2007], Shu et al. [2008], Van der Hof [1998] and the references therein). The motivation lies in the fact that only nonnegative quantities have physical meaning in the state estimation and control problems of positive systems. A recent area where the problem of positive estimation has also been considered is that of interval observers. A large number of works has been recently devoted to the design of observer that provide an estimate expressed by means of upper and lower bounds to the evolution of the state, in presence of several kinds of uncertainties, for example state or measurement disturbances, parametric uncertainties, unknown delays and so on (see for example Ait Rami et al. [2013], Efimov et al. [2013a,b,c], Gouze et al. [2000], Mazenc and Bernard [2011], Moisan et al. [2009], Raïssi et al. [2012]). In this case, it is interesting to provide an estimate of the state estimate of general (not necessarily positive) systems that satisfies to a positivity constraints on the estimation error, since this allows to express inequalities between the system and the estimates and to exploit the property of trajectory ordering that positive systems possess.

In this paper we consider the problem of providing an estimate of non (necessarily) positive systems by means of an observer with nonnegative state. The estimate, which in general is obviously nonpositive, is obtained as a linear combination of the nonnegative state of the observer. Of course, this approach may solve both problems outlined above: it can be applied to positive systems, providing a nonnegative estimate of nonnegative states, and to non positive systems to exploit the properties of positive systems in the design of interval observers.

The key tool underlying our approach is that of Internally Positive Representation (IPR) of linear systems, first proposed in Germani et al. [2010]. The idea was originally applied to the positive realization problem, when the system must be realized using only positive operations (i.e. no difference is allowed). In this setting, the main problem is to preserve the stability of the original system. In the context of state estimation algorithms, to the contrary, we can design observers that perform any operation as long as the state is nonnegative. In this paper we show that this added flexibility allows to design stable IPRs of any linear system, and in particular to build positive observers of nonpositive systems.

2. INTERNALLY POSITIVE REPRESENTATION OF LTV SYSTEM

2.1 Notation

Let $\mathbb{R}_+$ and $\mathbb{C}_+$ denote the set of nonnegative reals and the set of complex numbers with nonnegative real parts, respectively. $\mathbb{R}_-$ and $\mathbb{C}_-$ denote the sets of nonpositive reals and of complex numbers with nonpositive real parts. $\Re(z)$ and $\Im(z)$ denote the real and imaginary part of a complex z respectively. $\sigma(A)$ denotes the spectrum of a square matrix A. A is a Hurwitz matrix if $\Re(\sigma(A)) \subset (\mathbb{R}_- \setminus \{0\})$ and is a Schur matrix if $|\sigma(A)|$ is inside the unit disk in $\mathbb{C}$. Throughout this paper the inequalities and the min and max operators on vectors and matrices must be understood component-wise: given two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same dimensions, $A \geq B$ is equivalent to $a_{ij} \geq b_{ij}$, and $\max(A, B)$ is the matrix where each entry is $\max(a_{ij}, b_{ij})$. $M \geq 0$ denotes a matrix composed by nonnegative elements ($m_{ij} \geq 0$). $|M|$ denotes the matrix where each entry is $|m_{ij}|$. $A \succ 0$ ($A \succeq 0$) denote that A is positive definite (positive semi-definite). $A \succ B$ ($A \succeq B$) denote that $A - B$ is positive definite (positive semi-definite).

2.2 Positive representation of vectors and matrices

Given a matrix (or vector) M, the symbols M^+ and M^- denote its positive and negative parts, defined as

$$M^+ = \max(M, 0), \; M^- = \max(-M, 0). \qquad (1)$$

M^+ contains the positive entries of M and 0 elsewhere, whereas M^- contains the absolute value of the negative entries of M and 0 elsewhere. As a consequence, $M = M^+ - M^-$ and $|M| = M^+ + M^-$. I_n denotes the identity matrix in $\mathbb{R}^n$. The following matrices will be used throughout the paper:

$$\Delta_n = [I_n \;\; -I_n], \quad \bar{I}_n = \begin{bmatrix} I_n \\ I_n \end{bmatrix}. \tag{2}$$

Note that $\Delta_n \bar{I}_n = 0$. The following definitions were originally introduced in Germani et al. [2010], Cacace et al. [2012a].

Definition 1. A *positive representation* of a vector $x \in \mathbb{R}^n$ is a positive vector $X \in \mathbb{R}_+^{2n}$ such that

$$x = \Delta_n X. \tag{3}$$

The *min-positive representation* of a vector $x \in \mathbb{R}^n$ is the positive vector $\pi(x) \in \mathbb{R}_+^{2n}$ defined as

$$\pi(x) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}. \tag{4}$$

The *min-positive representation* of a matrix $M \in \mathbb{R}^{m\times n}$ is the positive matrix $\widetilde{M} \in \mathbb{R}_+^{2m\times 2n}$ defined as

$$\widetilde{M} = \begin{bmatrix} M^+ & M^- \\ M^- & M^+ \end{bmatrix}. \tag{5}$$

Note that x^+ and x^- are orthogonal, and therefore $\|x\| = \|\pi(x)\|$. Moreover

$$x = \Delta_n \pi(x), \quad \text{and} \quad \Delta_m \widetilde{M} = M\Delta_n = [M, \, -M], \tag{6}$$

for any $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{m\times n}$. From these equalities,

$$\Delta_m \widetilde{M}\, \pi(x) = M\, x, \quad \forall x \in \mathbb{R}^n, \; M \in \mathbb{R}^{m\times n}. \tag{7}$$

For any given $v \in \mathbb{R}_+^n$, the vector $\pi(x) + \bar{I}_n\, v$ is a positive representation of $x \in \mathbb{R}^n$, because (recall that $\Delta_n \bar{I}_n = 0$)

$$x = \Delta_n \big(\pi(x) + \bar{I}_n\, v\big). \tag{8}$$

Definition 2. A matrix M is said to be *Metzler* if all its off-diagonal elements are nonnegative.

Given a square matrix $M \in \mathbb{R}^{n\times n}$ we define its *Metzler representation* $[M] \in \mathbb{R}^{2n\times 2n}$ as

$$[M] = \begin{bmatrix} d^M + (M - d^M)^+ & (M - d^M)^- \\ (M - d^M)^- & d^M + (M - d^M)^+ \end{bmatrix}. \tag{9}$$

where d^M denotes the matrix having the same diagonal as M and 0 elsewhere. It is easy to check that $[M]$ is Metzler and that it enjoys the same property (6) of the positive representation, that is,

$$\Delta_n [M] = M\Delta_n, \quad \text{so that} \quad \Delta_n [M]\pi(x) = M\, x. \tag{10}$$

2.3 Positive systems

Consider a continuous-time linear time variant system Σ_L

$$\Sigma_L : \begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \\ x(0) &= x_0, \end{aligned} \tag{11}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^q$, and therefore $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times p}$, $C(t) \in \mathbb{R}^{q\times n}$.

The system (11) is said to be internally positive if for any given nonnegative initial condition $x_0 \in \mathbb{R}_+$ and input function $u(t) \geq 0$ the corresponding state and output trajectories are nonnegative, i.e. $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^q$, for all $t \geq 0$.

A sufficient condition for Σ_L to be internally positive is the following (see Farina and Rinaldi [2011]).

Theorem 3. If $A(t)$ is Metzler, $B(t) \geq 0$ and $C(t) \geq 0$ for all $t \geq 0$, then the system (11) is internally positive.

Proof. The trajectory $x(t)$ is continuous and initially positive. Its components are bounded to be positive, because if one component becomes null, $x_i(t) = 0$, with $A(t)$ Metzler and nonnegative forcing terms it is necessarily $\dot{x}_i(t) \geq 0$. □

Positive systems have the notable property to be ordered with respect to initial conditions and forcing inputs. The following result is well known.

Theorem 4. Consider a system Σ_L (11) where $A(t)$ is Metzler and $B(t) \geq 0$. Let $x_1(t)$ denote the state trajectory corresponding to the initial state x_{01} and input $u_1(t)$. Similarly, let $x_2(t)$ denote the trajectory corresponding to x_{02} and $u_2(t)$. If $x_{01} \leq x_{02}$ and $u_1(t) \leq u_2(t)$, then $x_1(t) \leq x_2(t)$ $\forall t \geq 0$.

Proof. Consider the differences $\xi(t) = x_2(t) - x_1(t)$ and $\nu(t) = u_2(t) - u_1(t)$. These obey the differential equation $\dot{\xi}(t) = A(t)\xi(t) + B(t)\nu(t)$. Being $\xi(0) \geq 0$ and $\nu(t) \geq 0$ by assumption, according to Theorem 3 we have $\xi(t) \geq 0$ for all $t \geq 0$, i.e. $x_1(t) \leq x_2(t)$. □

2.4 Internally positive representations (IPRs)

Internally Positive Representations (IPRs) of LTI systems have been defined and investigated in Germani et al. [2010], Cacace et al. [2012a] for discrete-time systems, and in Cacace et al. [2012b, 2014] for continuous-time systems. In this section we extend these results to LTV systems.

An IPR of system Σ_L (11) is an internally positive system, endowed with four transformations (forward and backward state transformations T_X^f, T_X^b, input and output transformations T_U and T_Y, see Cacace et al. [2012b]), that provide the same state and output trajectories of the system (11). An IPR of (11) is such that when its positive initial state is computed as $X(0) = T_X^f(x(0))$, and its positive input as $U(t) = T_U(u(t))$, $\forall t \geq 0$, then the state and output trajectories $(X(t), Y(t))$ of the IPR are positive and such that $x(t) = T_X^b\big(X(t)\big)$ and $y(t) = T_Y\big(Y(t)\big)$, $t \geq 0$ (see Cacace et al. [2012b] for more details).

In general, any system admits infinite IPRs. In this paper we use the following IPR.

$$\mathcal{I} : \begin{aligned} &\dot{X}(t) = [A(t)]\, X(t) + \widetilde{B}(t)U(t) \\ &Y(t) = \widetilde{C}(t)X(t) \\ &T_X^f = \pi(x), \; T_U = \pi(u), \\ &T_X^b = \Delta_n X, \; T_Y = \Delta_q Y, \end{aligned} \tag{12}$$

where $X(t) \in \mathbb{R}_+^{2n}$, $U(t) \in \mathbb{R}_+^{2p}$, $Y(t) \in \mathbb{R}_+^{2q}$. $[A(t)] \in \mathbb{R}^{2n\times 2n}$ is the Metzler representation of $A(t)$ defined by (9), $\widetilde{B}(t)$ and $\widetilde{C}(t)$ are the positive representations of $B(t)$ and $C(t)$ defined by (5). The forward state-transformations $T_X^f = \pi(x)$ and the input transformation $T_U = \pi(u)$ in (12) are used to compute the initial state

$X(0) = \pi\big(x(0)\big)$ and the input $U(t) = \pi\big(u(t)\big)$ of the V-IPR, thus ensuring positivity.

Theorem 5. System $\mathcal{I}$ defined in (12) is an IPR of system Σ_L defined in (11).

Proof. To prove the theorem we have to show that $x(t) = \Delta_n X(t)$ and $y(t) = \Delta_q Y(t)$. The first equality implies the second one, since $\Delta_q Y(t) = \Delta_q \widetilde{C}(t)X(t) = C(t)\Delta_n X(t)$, see (6). Let $z(t) = \Delta_n X(t)$, $z(t) \in \mathbb{R}^n$. Therefore

$$\begin{aligned} \dot{z}(t) &= \Delta_n[A(t)]\,X(t) + \Delta_n \widetilde{B}(t)U(t) \\ &= A(t)z(t) + B(t)u(t), \qquad (13) \\ z(0) &= \Delta_n X(0) = x(0), \qquad (14) \end{aligned}$$

and it follows that $z(t) = x(t)$.□

In other words, $\mathcal{I}$ (12) is a positive system with state and output evolving in $\mathbb{R}^{2n}_+$ and $\mathbb{R}^{2q}_+$, respectively, whose projections onto $\mathbb{R}^n$ and $\mathbb{R}^q$, defined by Δ_n and Δ_q, give back the state and output of the original system Σ_L (11), that in general is not positive.

Due to the larger state space, the stability of Σ_L does not imply the stability of $\mathcal{I}$. In the case of LTI systems, when A is Hurwitz, $[A]$ is not necessarily Hurwitz. In Cacace et al. [2014] it is shown how IPRs of size larger than $2n$ can be constructed to attain the same stability properties of the starting system. We do not pursue that approach in this paper. Some results about the stability of the IPR $\mathcal{I}$ of LTI systems via a coordinate change that transforms A in the Real Jordan Form are contained in Cacace et al. [2012b]. When A is not in Real Jordan Form sufficient conditions for the stability of $\mathcal{I}$ in the general case are not known.

3. THE POSITIVE OBSERVER

The idea behind the positive observer is to design a standard observer for the linear system Σ_L. We can subsequently construct an IPR of the kind (12) for this observer. The resulting positive system will behave, when projected onto $\mathbb{R}^n$, as an observer of Σ_L. The approach is quite straightforward, but unfortunately, for the reasons explained in the previous section, the IPR could be unstable even when the original system is stable. In other words, the projection onto $\mathbb{R}^n$ of the IPR is necessarily stable and it yields an estimation error which is asymptotically stable, but the state of the IPR could diverge thus the algorithm cannot be implemented in practice. An original contribution of this work is to provide a modified version of (12) which has the same stability properties as the original system.

Before introducing the positive observer, we therefore provide a stable IPR of a stable Σ_L (11). For the sake of concision we state the result for the autonomous case, but its extension to $u(t) \neq 0$ is immediate. Given $X(t) \in \mathbb{R}^{2n}$, we denote by $X_1(t) = [I_n 0_n]X(t)$, $X_2(t) = [0_n I_n]X(t)$ the first and last n components of $X(t)$.

Theorem 6. Given system Σ_L in (11), suppose that $u(t) \equiv 0$, Σ_L is asymptotically stable and $\|A(t)\| \le c$. Therefore there exists $\kappa \ge 0$ such that the following is an asymptotically stable IPR of Σ_L.

$$\mathcal{I}_s : \begin{aligned} \dot{X}(t) &= [A(t)]\,X(t) - \kappa \bar{I}_n f_X(t) \\ Y(t) &= \widetilde{C}(t)X(t) \end{aligned} \qquad (15)$$

with $X(0) = \pi(x(0))$, $x(t) = \Delta_n X(t)$, $y(t) = \Delta_q Y(t)$, and $f_X(t) = \min(X_1(t), X_2(t))$.

Proof. Notice that the input, state, and output transformations of (15) are the same as in (12), but (15) contains a (nonlinear) forcing term obtained as the component-wise minimum between the first n and the last n components of $X(t)$. Since $\Delta_n \bar{I}_n = 0$, using the same approach as in the proof of Theorem 5 we have $x(t) = \Delta_n X(t)$, thus (15) satisfies one necessary condition to be an IPR of Σ_L. We still have to prove that $X(t)$ is nonnegative and asymptotically stable. Since $x(t) = \Delta_n X(t)$ is equivalent to $x(t) = X_1(t) - X_2(t)$, from

$$\begin{aligned} X_1(t) &\ge f_X(t) \ge X_1(t) - |x(t)| \qquad (16) \\ X_2(t) &\ge f_X(t) \ge X_2(t) - |x(t)|, \qquad (17) \end{aligned}$$

we have $f_X(t) = X_1(t) - |x(t)| + r_1(t)$, $f_X(t) = X_2(t) - |x(t)| + r_2(t)$ with $r_1(t) \ge 0$, $r_1(t) \ge 0$. Moreover, $|x(t)| \ge r_1(t)$ and $|x(t)| \ge r_2(t)$. Consequently, we can write (15) as

$$\dot{X}(t) = \ ([A(t)] - \kappa I_{2n})\ X(t) + \kappa \begin{bmatrix} |x(t)| - r_1(t) \\ |x(t)| - r_2(t) \end{bmatrix}. \qquad (18)$$

It is now easy to conclude that, since $[A(t)] - \kappa I_{2n}$ is Metzler for all $t \ge 0$ and the forcing term is positive, $X(t) \ge 0$, and (15) is an IPR. In order to show that this IPR is also asymptotically stable, since by hypothesis $x(t) \to 0$ and thus the forcing term vanishes, it is sufficient to show that the autonomous equation

$$\dot{X}(t) = ([A(t)] - \kappa I_{2n})\,X(t) \qquad (19)$$

with $A(t)$ bounded in norm and κ arbitrarily chosen is asymptotically stable. $c > \|A(t)\|$ implies that there exists $\bar{c} > \|[A(t)]\|$. Choose the Lyapunov function $V(t) = X^T(t)X(t)$,

$$\begin{aligned} \dot{V}(t) &= 2X^T(t)([A(t)] - \kappa I_{2n})X(t) \\ &\le 2\|[A(t)]\|\|X(t)\|^2 - 2\kappa\|X(t)\|^2 \\ &\le 2(\bar{c} - \kappa)\|X(t)\|^2 \qquad (20) \end{aligned}$$

Therefore, with $\kappa > \bar{c}$ (15) is asymptotically stable.. □

Remark 7. System (15) is actually a nonlinear system due to the presence of the min(.) function, but its projection on $\mathbb{R}^n$ through Δ_n obeys the linear equation in (11). The idea underlying the forcing term added in (15) is to force to 0 the "minimum common" part of $X_1(t)$ and $X_2(t)$, since their difference is already stable. In other words, $X(t)$ in (15) is forced to exponentially tend to the min-positive representation of $x(t)$.

Remark 8. Notice the IPR (15) is in the original coordinate. No coordinate change is needed to make the system positive and stable. This is a remarkable difference with the positive representation approaches proposed so far.

Theorem 5 paves the way to design a positive observer. It is only necessary to use a stabilizing gain for a standard linear observer, which is always possible under the usual observability conditions. A stable IPR (15) can be subsequently derived.

Assumption 1. System Σ_L in (11) is uniformly completely observable (Bucy [1967], Jazwinski [1970]), that is, there exist positive scalars α, β, δ such that its state transition function $\Phi(t_1, t_2)$ satisfies for all $t \ge 0$,

$$\alpha I_n \leq \int_t^{t+\delta} \Phi(\tau, t+\delta)^T C^T(\tau) C(\tau) \Phi(\tau, t+\delta) d\tau \leq \beta I_n. \quad (21)$$

Theorem 9. If system Σ_L in (11) satisfies Assumption 1 and $\|A(t)\| \leq c$ then there exists $\kappa \geq 0$ such that for any $\hat{x}(0)$, the system

$$\Omega_L: \begin{aligned} \dot{X}(t) &= [A(t) - K(t)C(t)]X(t) + \widetilde{B}(t)\pi(u(t)) \\ &\quad + \widetilde{K}(t)\pi(y(t)) - \kappa \bar{I}_n f_X(t), \\ \hat{x}(t) &= \Delta_n X(t), \quad X(0) = \pi(\hat{x}(0)) \\ \dot{P}(t) &= (A(t) - K(t)C(t))\, P(t) \\ &\quad + P(t)\,(A(t) - K(t)C(t))^T + Q(t), \end{aligned} \quad (22)$$

with $K(t) = P(t)C^T(t)$, $f_X(t) = \min(X_1(t), X_2(t))$, $P(0)$ and $Q(t)$ symmetric positive definite, is such that $X(t)$ is nonnegative and bounded when $x(t)$ is bounded. Moreover, $\hat{x}(t)$ is an asymptotic observer for Σ_L, i.e. $\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0$.

Proof. Using (10) and $\hat{x}(t) = \Delta_n X(t)$ it is immediate to see that $\hat{x}(t)$ obeys the equation

$$\begin{aligned} \dot{\hat{x}}(t) =\Delta_n \dot{X}(t) &= (A(t) - K(t)C(t))\,\hat{x}(t) \\ &\quad + B(t)u(t) + K(t)y(t) \\ &= A(t)\hat{x}(t) + B(t)u(t) + K(t)\,(y(t) - \hat{y}(t)), \end{aligned} \quad (23)$$

and it is therefore the standard asymptotic observer of Σ_L. Denoting $A_K(t) = [A(t) - K(t)C(t)]$ the Metzler representation of $A(t) - K(t)C(t)$ and using the same inequalities as in Theorem 6, the equation for $X(t)$ becomes

$$\begin{aligned} \dot{X}(t) = (A_K(t) - \kappa I_{2n})\,X(t) + \kappa\pi(\Delta X(t)) \\ + \widetilde{B}(t)\pi(u(t)) + \widetilde{K}(t)\pi(y(t)) \end{aligned} \quad (24)$$

where we have used the property $X(t) = \bar{I}_n f_X(t) + \pi(\Delta X(t))$, which is easy to verify. Since $A_K(t) - \kappa I_{2n}$ is Metzler and the remaining terms are positive, it follows that $X(t) \geq 0$. In order to prove that (24) is bounded when $x(t)$ is bounded it is sufficient to prove that there is a choice of κ for which the autonomous equation

$$\dot{X}(t) = (A_K(t) - \kappa I_{2n})X(t) + \kappa\pi(\Delta X(t)) \quad (25)$$

is asymptotically stable. With the change of coordinates $Z(t) = MX(t)$,

$$M = \begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix}, \quad M^{-1} = \frac{1}{2}\begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix} \quad (26)$$

(25) becomes

$$\dot{Z}(t) = \begin{bmatrix} |A(t) - K(t)C(t)| - \kappa I_n & \kappa I_n \\ 0 & A(t) - K(t)C(t) \end{bmatrix} Z(t) \quad (27)$$

Due to the uniform observability hypothesis, the second block

$$\dot{Z}_2(t) = \big(A(t) - K(t)C(t)\big) Z_2(t) \quad (28)$$

is asymptotically stable. Moreover, $P(t)$ admits upper and lower bounds of the kind $p_m I_n \leq P(t) \leq p_M I_n$, and $K(t)$ is bounded in norm. The same holds for $A(t) - K(t)C(t)$. It is therefore possible to determine $\bar{c} > \|A(t) - K(t)C(t)\|$ for all t. Following a simple Lyapunov proof as in Theorem 6, it is immediate to conclude that with $\kappa > \bar{c}$ the first block is also asymptotically stable. □

Remark 10. The assumption of uniform complete observability is not restrictive, since it is needed to design a generic (i.e. non positive) observer. Theorem 9 states that, with an appropriate choice of κ, Σ_L and Ω_L have the same stability properties.

Remark 11. Notice that a variable $\kappa(t)$ can be used in Theorem 9 as long as it satisfies the requirement that the solution of (27) is asymptotically stable.

In the LTI case the choice of the forcing gain κ is straightforward.

Theorem 12. Let system Σ_L in (11) be time-invariant, $A(t) = A$, $B(t) = B$, $C(t) = C$, with (A, C) an observable pair. If K is such that $A - KC$ is Hurwitz, and if $\kappa \geq \mu$, where $\mu = \max(\Re(\sigma([A - KC])))$, the system (22) has a solution $X(t) \geq 0$ that is bounded if $x(t)$ is bounded, and such that the error $\epsilon(t) = x(t) - \Delta_n X(t)$ is exponentially stable for any $\hat{x}(0)$.

4. THE CASE OF TIME-DISCRETE SYSTEMS

A positive observer for time-discrete systems can be designed in the same way as in the time-continuous case by using the appropriate IPR. The only difference is that the positive representation of A is used in the place of the Metzler representation. In the time-discrete case it is easier to achieve stable IPRs. Instead of using an additional forcing term, the min-positive representation replaces the state of the IPR at each time step. In this section we provide the equivalent of Theorem 5, 6 and 9 for the time-discrete case.

Consider the linear time-varying system Σ_D

$$\Sigma_D: \begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \\ x(0) &= x_0. \end{aligned} \quad (29)$$

The following IPR of Σ_D was defined in Germani et al. [2010]

$$\mathcal{I}_D: \begin{aligned} X(t+1) &= \widetilde{A}(t)\,X(t) + \widetilde{B}(t)U(t) \\ Y(t) &= \widetilde{C}(t)X(t) \\ T_X^f &= \pi(x),\ T_U = \pi(u), \\ T_X^b &= \Delta_n X,\ T_Y = \Delta_q Y, \end{aligned} \quad (30)$$

Theorem 13. System $\mathcal{I}_D$ defined in (30) is an IPR of system Σ_D defined in (29).

Proof. To prove the theorem we have to show that $x(t) = \Delta_n X(t)$ and $y(t) = \Delta_q Y(t)$. As we know, the first equality implies the second one, see (6). Let $z(t) = \Delta_n X(t)$, $z(t) \in \mathbb{R}^n$. Therefore

$$\begin{aligned} z(t+1) &= \Delta_n \widetilde{A}(t)\,X(t) + \Delta_n \widetilde{B}(t)(t)U(t) \\ &= A(t)z(t) + B(t)u(t), \quad (31) \\ z(0) &= \Delta_n X(0) = x(0), \quad (32) \end{aligned}$$

and it follows that $z(t) = x(t)$.□

In analogy with the time-continuous case, it is easy to show that stability of Σ_D does not imply stability of $\mathcal{I}_D$. The problem is investigated in Germani et al. [2010], Cacace et al. [2012a]. The time-discrete equivalent of Theorem 6 is the following.

Theorem 14. Given system Σ_D in (29), suppose that $u(t) \equiv 0$ and Σ_D is asymptotically stable. Then, the following is an asymptotically stable IPR of Σ_D.

$$\mathcal{I}_s^D : \begin{array}{rcl} \xi(t) &=& \widetilde{A}(t)\, X(t) \\ X(t+1) &=& \pi(\Delta_n \xi(t)) \\ Y(t) &=& \widetilde{C}(t)X(t) \end{array} \tag{33}$$

with $X(0) = \pi(x(0))$, $x(t) = \Delta_n X(t)$, $y(t) = \Delta_q Y(t)$.

Proof. Notice that the input, state, and output transformations of (33) are the same as in (30), but (33) contains a nonlinear equation in that $X(t)$ is obtained as the min-positive representation of $\Delta_n \xi(t)$. From (31) follows that $\Delta_n X(t) = Ax(t-1)$, $t > 0$, thus $x(t) = \Delta_n X(t)$, and (33) satisfies one necessary condition to be an IPR of Σ_D. Moreover, $X(t)$ is a min-positive representation and it is therefore nonnegative. Since $x(t)$ is asymptotically stable, asymptotic stability of $X(t)$ is easily proved by showing that $X(t) = \pi(x(t))$. This is true for $t = 0$, and, by induction,

$$\begin{aligned} X(t+1) &= \pi(\Delta_n \xi(t)) = \pi(\Delta_n \widetilde{A}(t) X(t)) \\ &= \pi(A(t)x(t)) \;=\; \pi(x(t+1)). \;\Box \end{aligned} \tag{34}$$

Assumption 2. System Σ_D in (29) is uniformly completely observable (Bucy [1967], Jazwinski [1970]), that is, there exist positive scalars α, β, N such that its state transition function $\Phi(i,k)$ satisfies for all $i \geq N$,

$$\alpha I_n \;\leq\; \sum_{j=i-N}^{i} \Phi^T(j,i) C^T(j) C(j) \Phi(j,i) \leq\; \beta I_n. \tag{35}$$

Theorem 15. If system Σ_D in (29) satisfies Assumption 2 then for any $\hat{x}(0)$, the system Ω_D,

$$\begin{aligned} \xi(t) &= \widetilde{A_K}(t) X(t) + \widetilde{B}(t)\pi(u(t)) \\ &\quad + \widetilde{K}(t)\pi(y(t)) \\ X(t+1) &= \pi(\Delta_n \xi(t)) \\ \hat{x}(t) &= \Delta_n X(t), \quad X(0) = \pi(\hat{x}(0)) \\ P(t+1) &= A_K(t)P(t)A_K(t)^T \\ &\quad - K(t)C(t)P(t)A_K(t)^T + Q(t), \\ K(t) &= A_K(t)P(t)C(t)^T \left(C(t)P(t)C(t)^T + R(t)\right)^{-1} \end{aligned} \tag{36}$$

with $A_K(t) = A(t) - K(t)C(t)$ and $\widetilde{A_K}(t)$ its positive representation, $Q(t)$, $R(t)$ symmetric positive semidefinite matrices to be chosen, is such that $X(t) \geq 0$ is stable if $x(t)$ is stable and $\hat{x}(t)$ is an asymptotic observer for Σ_D, i.e. $\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0$.

Proof. Using (6) and $\hat{x}(t) = \Delta_n X(t)$ we see that, since $\Delta_n \pi(v) = v$, $\hat{x}(t)$ obeys the equation

$$\begin{aligned} \hat{x}(t+1) =& \Delta_n X(t+1) \;=\; \Delta_n \Big(\widetilde{A_K}(t)X(t) \\ &+ \widetilde{B}(t)\pi(u(t)) + \widetilde{K}(t)\pi(y(t))\Big) \\ &= A(t)\hat{x}(t) + B(t)u(t) + K(t)\left(y(t) - C(t)\hat{x}(t)\right), \end{aligned} \tag{37}$$

and in the hypotheses of the theorem it is therefore an asymptotic observer for Σ_D. $X(t)$ is positive by construction. As in Theorem 14 we have $X(t) = \pi(\hat{x}(t))$, and since $\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0$, $X(t)$ is stable whenever $x(t)$ is stable. $\Box$

5. EXAMPLE

In this section we consider an academic example to illustrate the basic features of the proposed approach. Consider the LTV system Σ_L in (11) of size $n = 2$ where

$$A(t) = \begin{bmatrix} -1/2 + \sin(t/2) & 1 + \sin(t) \\ -1 - \sin(t)/2 & \cos(4t) \end{bmatrix}, \; B(t) = \begin{bmatrix} -6 \\ 1 \end{bmatrix},$$
$$C(t) = [1 + \cos(t) \;\; 2 + \sin(t)], \quad u(t) = \sin(t/2) - 1.$$

This system is stable and its trajectories are shown in Fig. 1 for $x(0) = [10, -20]^T$. In the same figure the plots of $\hat{x}(t)$ from (22) with $Q(t) = I_2$ are plotted, and, as it could be expected, the observer convergence to the true state is very fast. The parameter $\kappa = 5$ has been used in (22) in order to have a stable IPR. The time course of $X_1(t)$ and $X_3(t)$ is shown in Fig. 2 (left). Since $X(t) \in \mathbb{R}^4$, $\hat{x}(t) = \Delta_2 X(t) = [X_1(t) - X_3(t),\, X_2(t) - X_4(t)]$, thus the difference $X_1(t) - X_3(t)$ corresponds to $\hat{x}_1(t)$. Notice that both components are positive and stable. The necessity of stabilizing the IPR is shown in Fig. 2 (right), where κ has been set to 0. In both cases the difference $X_1(t) - X_3(t)$ is $\hat{x}_1(t)$, the estimate of $x_1(t)$. Notice that in the right-hand plot the difference is too small to be visible, but with $\kappa = 0$ the extended state $X(t)$ is not stable.

6. CONCLUSION

In this paper we have extended the IPR proposed in previous papers. The extension introduced is quite straightforward but it allows to to derive a stable positive representation of linear systems. from which positive observers can be designed under the same observability conditions of linear observers. Future work will focus on the use of this positive observer to design interval observer for systems with different kinds of uncertainties.

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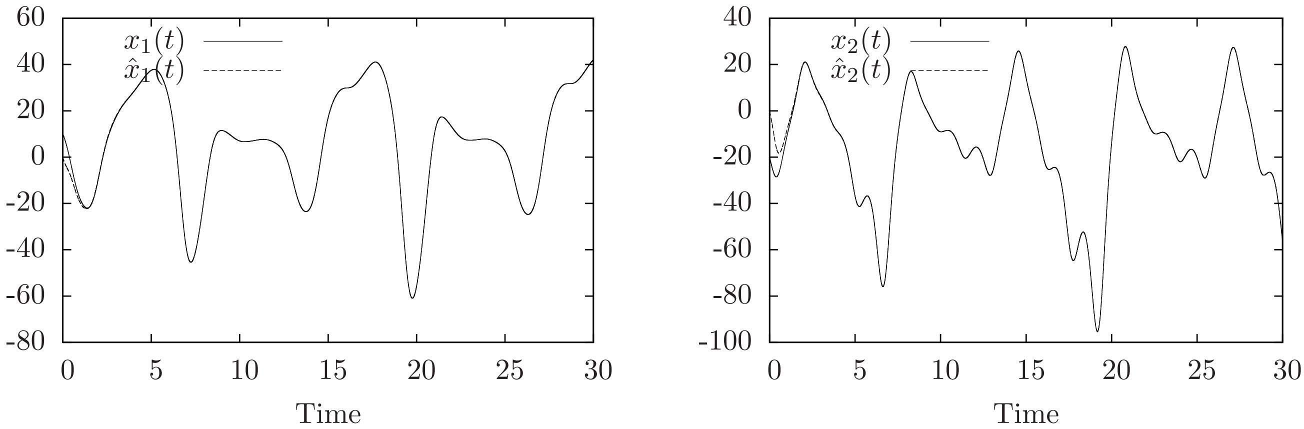

Fig. 1. True and estimated state for the example of Section 5.

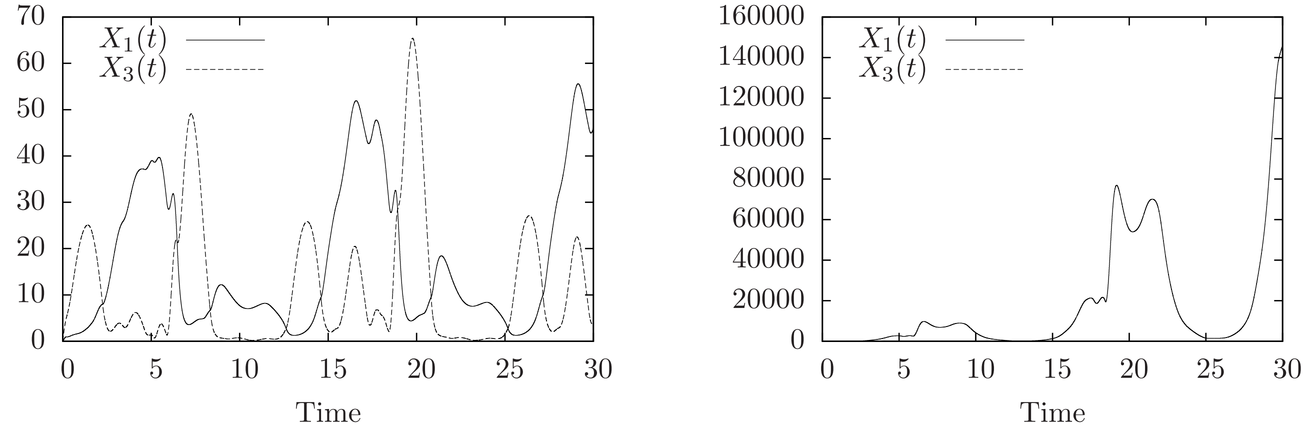

Fig. 2. Plot of of $X_1(t)$ and $X_3(t)$ with $\kappa = 5$ (left) and $\kappa = 0$ (right).

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