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Lyapunov-Based Pursuit Guidance Law with Impact Angle Constraint

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Abstract:
This paper presents Lyapunov-based pursuit guidance law against stationary targets. To design a nonlinear guidance law, Lyapunov candidate function is introduced to reduce the angle between the velocity vector of a missile and the distance vector between the missile and the target. Therefore, the proposed guidance laws have the characteristic of pursuit guidance. To attack a target from a predefined direction, the guidance law should be designed with impact angle constraint at the final phase. To deal with this, the impact angle error is augmented to the Lyapunov candidate function. The proposed guidance laws have simple forms to be implemented easily. Numerical simulations are performed to demonstrate the performance of the proposed guidance laws.

Keywords: Impact Angle, Guidance Systems, Lyapunov Stability, Automated Guided Vehicles

1. INTRODUCTION

Main objective of the missile guidance is to deliver a missile into a target. Proportional Navigation Guidance(PNG) and pursuit guidance have been widely used in missile systems. PNG generates an acceleration command whose value is proportional to the line-of-sight(LOS) rate to the target. Pursuit guidance law constructs an acceleration command that the velocity vector of the vehicle is toward the target.

Many modern control theories have been applied to develop various guidance laws. An adaptive sliding mode control scheme was utilized to design a homing guidance law (Zhou et al. [1999]). A robust guidance law was developed using a variable structure control scheme (Moon et al. [2001]). An aim angle was introduced to design a conceptual nonlinear guidance law based on Lyapunov stability theory (Kim and Kim [2004]). A quadratic Lyapunov candidate function was utilized to design a guidance law that is free of singularities (Lechevin and Rabbath [2004]). A square of the LOS rate was chosen as a term of Lyapunov candidate function to develop a proportional navigation-like guidance law (Yanushevsky and Boord [2005]).

The performance of the missiles can be drastically improved by attacking the vulnerable area of a target. For this reason, many studies have been performed to solve an impact angle control problem. Time-varying biased PNG law considering impact angle constraints was developed (Kim et al. [1998]). State-dependent Riccati-equation(SDRE) was used to obtain a solution of a impact angle constrained guidance problem (Ratnoo and Ghose [2009]). Two-stage PNG-based guidance law was designed to consider impact angle constraint against nonstationary nonmaneuvering target (Ratnoo and Ghose [2010]). Polynomial guidance laws were proposed to consider terminal impact angle and acceleration constraints (Lee et al. [2013a]).

On the other hand, optimal control theory was also used to design a guidance law with impact angle constraint. Minimum energy problems were considered to derive optimal guidance laws with terminal impact angle constraints (Song et al. [1999]), (Ryoo et al. [2005]), and (Ryoo et al. [2006]). Optimal impact angle control guidance(IACG) law for planar engagement was designed to involve a maneuvering target and a time-varying velocity missile (Song et al. [1999]). Closed-form optimal guidance laws considering impact angle constraints for the lag-free and the first-order lag system were investigated, and recursive time-to-go estimation methods were proposed (Ryoo et al. [2005]). Time-varying weighting cost function was adopted to develop optimal guidance law with impact angle constraints (Ryoo et al. [2006]). And, guidance law with impact angle constraint was generalized by optimal control law (Lee et al. [2013b]).

In this study, nonlinear guidance law based on the Lyapunov stability theory is proposed considering impact angle constraint. First, a Lyapunov candidate function to reduce the angle between the velocity vector of a missile and the distance vector between the missile and the target is proposed. Then, the Lyapunov candidate function is modified to consider the impact angle constraint. The stability analysis is performed for the proposed guidance law using the Lyapunov stability theory. The proposed Lyapunov-based pursuit IACG law has a simple form so that the IACG law can be implemented easily.

The remainder of the paper is organized as follows. In Section 2, a two-dimensional geometry of missile-target engagement is presented to formulate the problem. Two nonlinear pursuit guidance laws based on Lyapunov stability theory are proposed in Section 3. The Lyapunov-based pursuit guidance law without impact angle constraints is first presented. Then, the Lyapunov-based pursuit IACG law is proposed to consider the impact angle. The results of numerical simulations are provided to demonstrate the performance of the proposed guidance laws in Section 4. Finally, conclusions are made in Section 5.

2. PROBLEM FORMULATION

A two-dimensional missile-target engagement is considered to design the guidance laws as shown in Fig. 1. In this study, it is assumed that the missile is flying at a constant speed V_M, and the target is not maneuvering. The acceleration vector of the missile a_M is assumed to be perpendicular to the velocity vector of the missile as shown in Fig. 1.

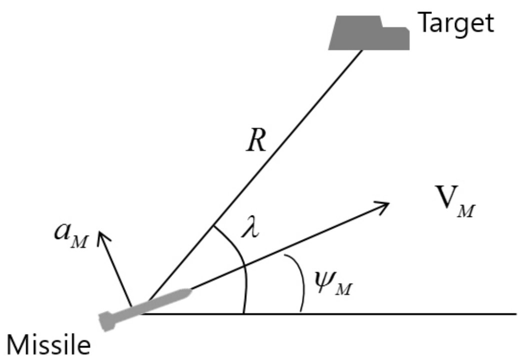

Fig. 1. Two-dimensional geometry of missile-target engagement

Some extra assumptions are required to derive the kinematic equations. First, the missile is assumed to be a point mass. Second, the autopilot and sensor/seeker dynamics are assumed to be faster than the missile dynamics. Third, the angle-of-attack of the missile is very small enough to be neglected. Using these assumptions, the following two-dimensional kinematic equations can be obtained (Kim and Kim [2004]).

$$\dot{R} = -\mathrm{V}_M \cos(\lambda - \psi_M) \tag{1}$$

$$\dot{\lambda} = \frac{\mathrm{V}_M}{R} \sin(\lambda - \psi_M) \tag{2}$$

$$\dot{\psi}_M = a_M/\mathrm{V}_M = a_c \tag{3}$$

where ψ_M denotes the heading angle of the missile, λ denotes the line-of-sight(LOS) angle to the target, R represents the distance between the missile and the target, and a_c represents the normalized acceleration by the speed of the missile.

3. GUIDANCE LAWS DESIGN

In this section, new Lyapunov candidate functions are introduced to reduce the miss distance and the impact angle error. A Lyapunov-based pursuit guidance law without the impact angle constraints is first provided, and then a Lyapunov-based pursuit impact angle control guidance(IACG) law is proposed by augmenting impact angle error to the Lyapunov candidate function.

3.1 Lyapunov-Based Pure Pursuit Guidance Law

In the physical view of the missile-target engagement, the missile head should be toward the target at the final stage. On that account, a following Lyapunov candidate function is considered.

$$V_1 = 2\sin^2\left(\frac{\lambda - \psi_M}{4}\right) \tag{4}$$

The proposed Lyapunov candidate function is chosen to reduce the angle between the velocity vector of the missile and the distance vector between the missile and the target. The condition $\lambda - \psi_M = 0$ means that the missile head is toward the target. The values of the missile's heading angle and LOS angle are determined within $\pm\pi$, and therefore the proposed Lyapunov candidate function (4) has zero value only when the difference between the missile's heading angle and the LOS angle are zero.

The time derivative of the Lyapunov candidate function can be expressed as

$$\dot{V}_1 = \sin\left(\frac{\lambda - \psi_M}{4}\right)\cos\left(\frac{\lambda - \psi_M}{4}\right)\left(\dot{\lambda} - \dot{\psi}_M\right) \tag{5}$$

Substituting (3) into (5) and using the trigonometric double-angle formula yield

$$\dot{V}_1 = \frac{1}{2}\sin\left(\frac{\lambda - \psi_M}{2}\right)\left(\dot{\lambda} - a_c\right) \tag{6}$$

To satisfy $\dot{V}_1 \leq 0$, the following normalized acceleration command is proposed.

$$a_c = \dot{\lambda} + k_1 \sin\left(\frac{\lambda - \psi_M}{2}\right) \tag{7}$$

where k_1 is a positive constant guidance gain. Substituting (7) into (6) yields

$$\begin{aligned}\dot{V}_1 &= -\frac{k_1}{2}\sin^2\left(\frac{\lambda - \psi_M}{2}\right) \\ &\leq 0\end{aligned} \tag{8}$$

The time derivative of the Lyapunov candidate function becomes negative definite, where the proposed guidance command (7) is used. Hence, according to the Lyapunov stability theory, the system is asymptotically stable at the equilibrium point, i.e., $\lambda - \psi_M = 0$.

Note that the guidance gain k_1 is related to the convergence speed to the target. If a large value of the gain k_1 is chosen, then the interception time of the missile will be decreased.

3.2 Lyapunov-Based Pursuit IACG

Let us define impact angle to consider the impact angle constraint

$$e_\lambda \equiv \lambda - \psi_d \tag{9}$$

where ψ_d is a desired impact angle.

Lyapunov candidate function is augmented to control the impact angle as follow

$$V_2 = 2\sin^2\left(\frac{\lambda - \psi_M}{4}\right) + 2k_2\sin^2\frac{e_\lambda}{4} \tag{10}$$

where k_2 is a positive constant coefficient. The time derivative of (10) can be expressed as

$$\begin{aligned}\dot{V}_2 =& \sin\left(\frac{\lambda - \psi_M}{4}\right)\cos\left(\frac{\lambda - \psi_M}{4}\right)\left(\dot{\lambda} - \dot{\psi}_M\right) \\ &+ k_2 \sin\frac{e_\lambda}{4}\cos\frac{e_\lambda}{4}\dot{e}_\lambda\end{aligned} \tag{11}$$

Substituting (3) into (11) and using the trigonometric double-angle formula yield

$$\dot{V}_2 = \frac{1}{2}\sin\left(\frac{\lambda - \psi_M}{2}\right)\left(\dot{\lambda} - a_c\right) + \frac{k_2}{2}\sin\frac{e_\lambda}{2}\dot{\lambda} \tag{12}$$

Note that the desired impact angle is usually pre-specified as a constant value considering the mission. To obtain a guidance commands, let us substitute (2) into the second term of the right hand side of (12).

$$\dot{V}_2 = \frac{1}{2}\sin\left(\frac{\lambda-\psi_M}{2}\right)\left(\dot{\lambda}-a_c\right) + \frac{k_2}{2}\sin\frac{e_\lambda}{2}\frac{V_M}{R}\sin(\lambda-\psi_M) \tag{13}$$

Now, let us propose a following normalized acceleration command.

$$a_c = \dot{\lambda} + k_1\sin\left(\frac{\lambda-\psi_M}{2}\right) + 2k_2\frac{V_M}{R}\sin\frac{e_\lambda}{2}\cos\left(\frac{\lambda-\psi_M}{2}\right) \tag{14}$$

where k_1 is a positive constant guidance gain.

Substituting the proposed normalized acceleration command (14) into (13) yields

$$\begin{aligned}\dot{V}_2 &= -\frac{k_1}{2}\sin^2\left(\frac{\lambda-\psi_M}{2}\right) \\ &\quad -\frac{k_2}{2}\sin\frac{e_\lambda}{2}\frac{V_M}{R}\left(2\sin\left(\frac{\lambda-\psi_M}{2}\right)\cos\left(\frac{\lambda-\psi_M}{2}\right)\right) \\ &\quad +\frac{k_2}{2}\sin\frac{e_\lambda}{2}\frac{V_M}{R}\sin(\lambda-\psi_M) \\ &= -\frac{k_1}{2}\sin^2\left(\frac{\lambda-\psi_M}{2}\right) \\ &\leq 0\end{aligned} \tag{15}$$

The proposed Lyapunov candidate function (10) is a continuously differentiable function, such that for some $r > 0$, $\Omega_r = \{x_L \in \Re^n | V_2(x_L(t)) < r\}$ is bounded, and its first time derivative is negative semi-definite, where $x_L = \{\lambda(t) - \psi_M(t), e_\lambda(t)\}$. Let D be the set of all points to satisfy $\dot{V}_2(t) = 0$ within Ω_r,

$$D = \left\{x_L \in \Re^n | \dot{V}_2(\lambda(t) - \psi_M(t)) = 0\right\} \tag{16}$$

and M is the largest invariant set in D.

$$M = \{x_L \in \Re^n | \lambda(t) - \psi_M(t) = 0, e_\lambda(t) = 0\} \tag{17}$$

Let us assume that M contains a point with $e_\lambda(t) \neq 0$.

$$\begin{aligned}\dot{\lambda}(t) - \dot{\psi}_M(t) &= \dot{\lambda} - a_c \\ &= -k_1\sin\left(\frac{\lambda-\psi_M}{2}\right) \\ &\quad -2k_2\frac{V_M}{R}\sin\frac{e_\lambda}{2}\cos\left(\frac{\lambda-\psi_M}{2}\right) \\ &= -2k_2\frac{V_M}{R}\sin\frac{e_\lambda}{2} \\ &\neq 0\end{aligned} \tag{18}$$

Hence, if the trajectory moves out of D, then it will also move out of M. This is the contradiction to the definition of the invariant set M. Therefore, according to LaSalle's invariance theorem, the proposed normalized acceleration command makes the system be asymptotically stable at the equilibrium point, i.e., $\lambda(t) - \psi_M(t) = 0, e_\lambda(t) = 0$.

Note that the proposed IACG law has a form of biased proportional navigation guidance(PNG) law. The positive constant coefficient k_2 has a role which is related to the satisfaction of the impact angle constraint, while the gain k_1 is related to the interception time of the missile.

4. NUMERICAL SIMULATIONS

Numerical simulations are performed to demonstrate the performance of the proposed guidance law. The result of the Lyapunov-based pure pursuit guidance law is first shown, and then the result of the Lyapunov-based pursuit IACG law is provided.

4.1 Lyapunov-Based Pure Pursuit Guidance Law

In this simulation, three missiles fired at different locations are considered. The speed of each missile is set as 250 m/s. The initial position of each missile is (0, 0), (10,000, 0), and (-1,000, 5,000), respectively, and the position of the stationary target is located at (6,000, 6,000). The initial heading angle of each missile is set as $\lambda_0 - \pi/4$. The limit of the acceleration command is set as $10g$, where g is a gravitational acceleration. The guidance gain k_1 is chosen as 3 for each missile.

Figure 2 shows the planar trajectories of three missiles, and Fig. 3 shows the time history of miss distance. The trajectories are generated smoothly, and all missiles intercept the target successfully as shown in Figs. 2 and 3. Figure 4 shows the time histories of normalized acceleration commands. The smooth guidance commands are generated as shown in Fig. 4.

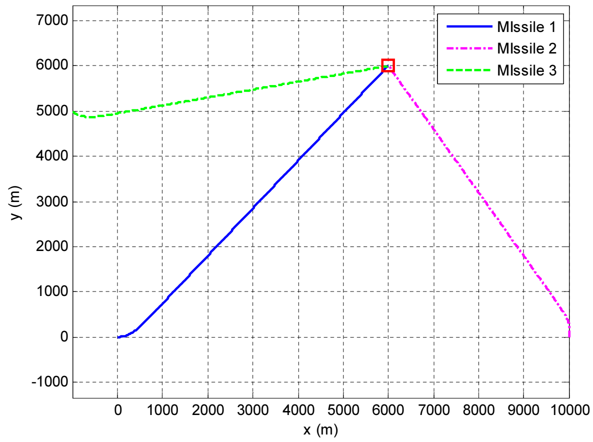

Fig. 2. Two-dimensional trajectories of three missile

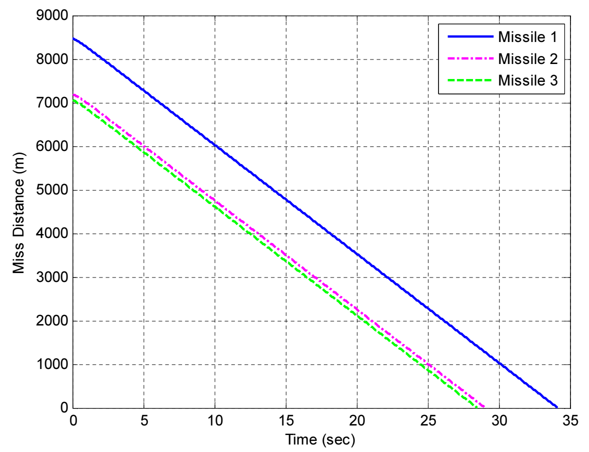

Fig. 3. Time history of miss distance

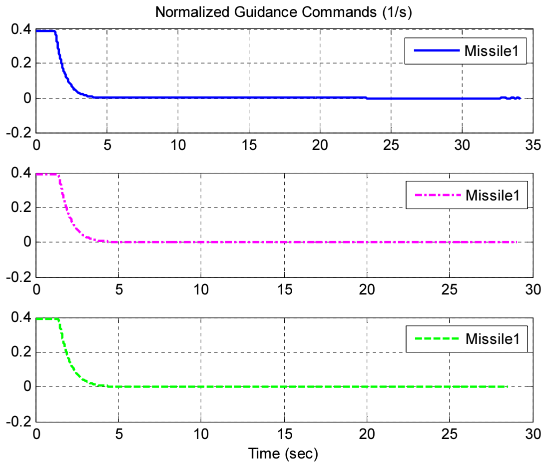

Fig. 4. Time history of normalized acceleration commands

The PNG law is used to compare with the performance of the proposed guidance law. The following normalized PNG command is used.

$$a_{PNG} = N\dot{\lambda} \tag{19}$$

where PNG gain N is chosen as 3.

The initial positions of the three missiles are same as (0, 0), but the initial heading angle of each missile is differently set as $\lambda_0 - \pi/2$, $\lambda_0 - 3\pi/4$, and $\lambda_0 - \pi$, respectively. The following transfer functions of time delay and actuator dynamics are considered in the simulation.

$$\frac{a_m}{a_{m_d}} = \frac{1}{\tau s + 1} \tag{20}$$

$$\frac{a_{m_d}}{a_{m_c}} = \frac{\omega_n^2}{s^2 + 2\omega_n \zeta s + \omega_n^2} \tag{21}$$

where $\tau = 0.2\,(sec)$, $\zeta = 0.7$, and $\omega_m = 100\,(rad/s)$. Other simulation environments are same as those in the previous simulation.

Figure 5 shows the two-dimensional trajectories of the missiles. Solid lines stand for the trajectories of the missiles using the proposed guidance law, and dashed lines denote the trajectories of the missiles using the PNG law. All missiles using the proposed Lyapunov-based pursuit guidance law intercept the target successfully as shown in Fig. 5. However, the missile using the PNG law whose initial heading angle is set as $\lambda_0 - \pi$ fails the interception. This is because the PNG command depends on $\lambda - \psi_M$ of LOS rate, while the proposed guidance command is determined by the value of $(\lambda - \psi_M)/2$. Figure 6 shows the time history of the normalized acceleration commands.

4.2 Lyapunov-based Pursuit IACG

In this simulation, the initial positions of the three missiles are equally set as (0, 0), and the initial heading angles of the missiles are also equally set as $\lambda_0 - \pi/4$. The position of the stationary target is located at (6,000, 0). The desired impact angles are differently chosen as 0, $3\pi/4$, and $-3\pi/4$, respectively. The guidance gains of each missile are chosen as ($k_1 = 3.0$, $k_2 = 16.4$), ($k_1 = 20$, $k_2 = 89.5$), and ($k_1 = 20$, $k_2 = 89.5$), respectively. Other conditions including time delay and actuator dynamics are same as those in the previous simulation.

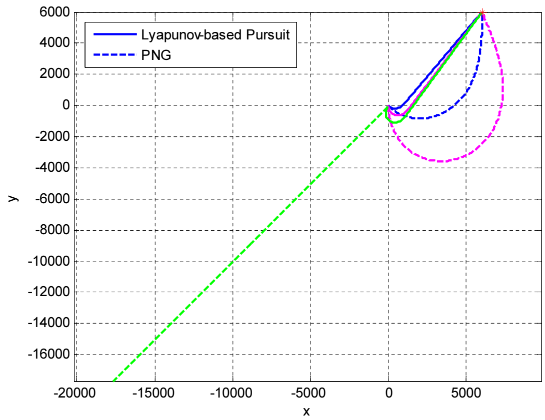

Fig. 5. Two-dimensional trajectories of three missile: Comparison with PNG

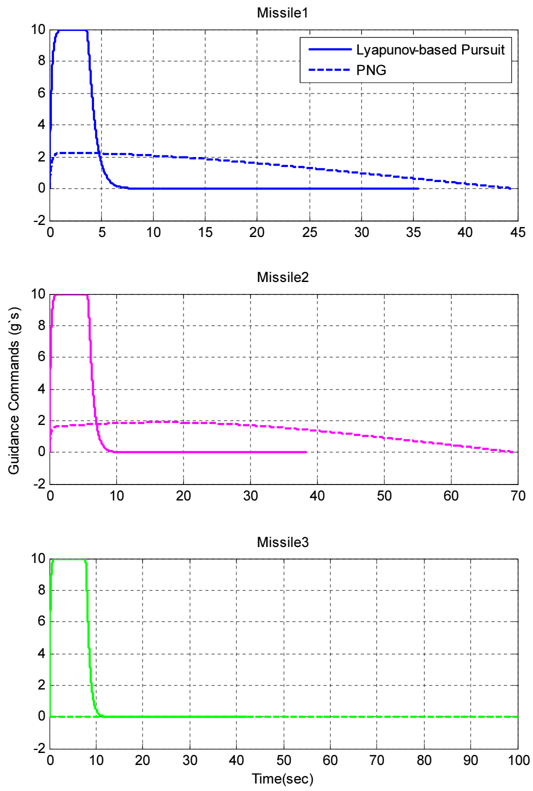

Fig. 6. Time history of normalized acceleration commands: Comparison with PNG

To compare the performance of the proposed Lyapunov-based pursuit IACG Law, two different IACG laws, i.e. the trajectory shaping guidance(TSG) law (Zarchan [2007]) and the optimal guidance law(OGL) with the impact angle constraint (Ryoo et al. [2005]), are considered as

$$a_{TSG} = 4\dot{\lambda} + 2\frac{e_\lambda}{\hat{t}_{go}} \tag{22}$$

$$a_{OGL} = \frac{1}{\hat{t}_{go}}\left[6\lambda - 4\psi_M + 2\psi_d\right] \tag{23}$$

where $\hat{t}_{go}$ is the time-to-go estimate (Jeon et al. [2006]).

$$\hat{t}_{go} = \left[1 + \frac{(\lambda - \psi_M)^2}{10}\right] \frac{R}{V_M} \tag{24}$$

Figure 7 shows the trajectories of the missiles. Solid lines represent the trajectories of the missiles using the proposed Lyapunov-based pursuit IACG law, dashed lines show the trajectories of the missiles using the TSG law with the impact angle constraints, and dashed-dot lines show the trajectories of the missiles using the OGL with the impact angle constraints. All missiles have similar trajectories while satisfying the impact angle constraints. Figure 8 shows the time history of the normalized acceleration commands.

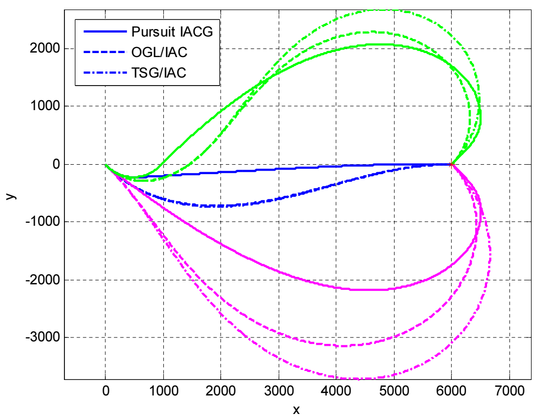

Fig. 7. Two-dimensional trajectories of three missile: IACG Laws

To compare the performance of the guidance laws, the final impact angle error $\Delta\psi_f$, fuel consumption performance index J_F, and energy consumption performance index J_E are considered. Table 1 summarizes the final impact angle errors, Table 2 summarizes the fuel consumption, and Table 3 summarizes the energy consumption, respectively. The final impact angle errors of three methods are similar as shown in Table 1. The fuel consumption of the proposed IACG is the smallest among the three guidance laws as shown in Table 2. This is because the proposed Lyapunov-based pursuit IACG law generates the command that the missile intercepts the target faster than other guidance laws. The TSG and OGL were derived by solving the minimum energy problem, thus the energy consumptions of the TSG and OGL are less than that of the proposed IACG law as shown in Table 3. Note that in the physical point of view, the fuel consumption is more important that the energy consumption.

$$\Delta\psi_f = \left|\psi_{M_f} - \psi_d\right| \tag{25}$$

$$J_F = \int_{t_0}^{t_f} |a_M| dt \tag{26}$$

$$J_E = \int_{t_0}^{t_f} a_M^2 dt \tag{27}$$

Lastly, various time constants are considered to analyze the sensitivity of the proposed guidance law. The case that the desired impact angle is $3\pi/4$ is considered in the simulation. Other conditions including actuator dynamics and guidance gains are same as those in the previous simulation.

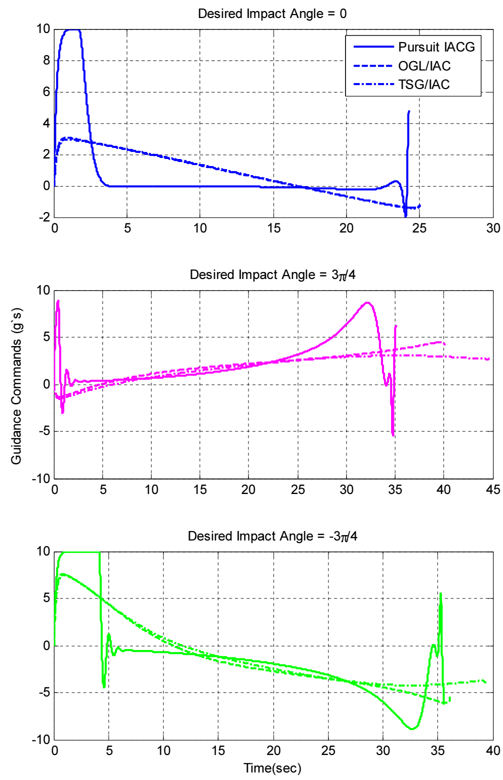

Fig. 8. Time history of normalized acceleration commands: IACG Laws

Table 1. Final impact angle errors (deg)

Desired impact angle	Proposed IACG	OGL/IAC	TSG/IAC
0	0.002	0.068	0.066
135	0.051	-0.145	-0.081
-135	-0.012	0.203	0.138

Table 2. Fuel consumption

Desired impact angle	Proposed IACG	OGL/IAC	TSG/IAC
0	2.375×10^4	3.276×10^4	3.301×10^4
135	8.201×10^4	8.537×10^4	8.783×10^4
-135	1.214×10^5	1.225×10^5	1.245×10^5

Table 3. Energy consumption

Desired impact angle	Proposed IACG	OGL/IAC	TSG/IAC
0	1.787×10^6	6.146×10^5	6.147×10^5
135	3.719×10^6	2.357×10^6	2.126×10^6
-135	7.503×10^6	5.360×10^6	5.065×10^6

Figure 9 shows the trajectories of the missiles. The missile can intercept the target in the cases that the time constant is 0.1, 0.2, 0.3, and 0.4sec, but the missile cannot intercept the target for the cases that the time constant is over 0.5sec, as shown in Fig. 9. Note that the time constant of missile systems is usually less than 0.1sec. Therefore, the proposed guidance law can be used in the missile system.

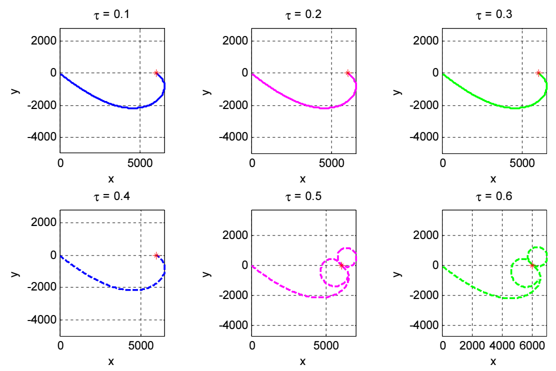

Fig. 9. Two-dimensional flight trajectories: various time constants

5. CONCLUSION

The Lyapunov-based pursuit guidance law with impact angle constraints was proposed. The Lyapunov stability theory was used to develop nonlinear guidance law and to analyze the stability. The proposed guidance law has a simple form so that the guidance law can be implemented easily. Numerical simulations were performed to investigate the performance of the proposed guidance law, and were compared with those of the trajectory shaping guidance law and the optimal guidance law. Sensitivity analysis was also performed with respect to the time-delay. The proposed Lyapunov-based pursuit guidance law can be used in the case that the initial heading angle of a missile is opposite to the target unlike the widely used proportional navigation guidance law.

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