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LMI results for robust control design of observer-based controllers, the discrete-time case with polytopic uncertainties

D. Peaucelle *,** **Y. Ebihara** ***

* *CNRS, LAAS, 7 avenue du colonel Roche, F-31400 Toulouse, France*
** *Univ de Toulouse, LAAS, F-31400 Toulouse, France*
*** *Department of Electrical Engineering, Kyoto University, Kyotodaigaku-Katsura, Nishikyo-ku, Kyoto 615-8510, Japan*

Abstract: Design of robust observers is considered in the context of linear discrete-time, time invariant systems. Robustness is achieved with respect to polytopic type uncertainties that affect the dynamics of the plant. At the difference with the uncertainty-free situation, state-feedback / observer separation principle does not hold. Therefore, the observer design has to take into account the state-feedback gain. Results are derived with linear matrix inequality formalism and involve up-to-date slack-variables approach. A numerical example illustrates the results. Limitations of the method are discussed and prospective work for improving these is exposed.

Keywords: Robust control, State observers, Linear systems, Convex optimization

1. INTRODUCTION

The goal of the paper is to investigate a state-feedback/observer design strategy for LTI discrete-time systems affected by polytopic uncertainties. This problem is not new, not quite solved and results we provide do not claim to be a final answer. Nevertheless, they provide some systematic procedure involving up-to-date LMI convex optimization based methods and bring some new insight to the robust observer design issue.

Controllers of state-feedback/observer form are one way for searching for dynamic output feedback controllers. That general issue has convex LMI-based solutions as long as the systems are not affected by uncertainties, see Scherer et al. [1997], Arzelier et al. [2006], to cite just a few. As soon as the systems are affected by uncertainties, the problem becomes more complex. If there is just one non-structured norm-bounded-like uncertainty in the model, some results are available, see Peaucelle and Arzelier [1998] as well as Lien [2004], Lien and Yu [2008] for the state-feedback/observer structure case. Unfortunately for structured uncertainties, as it is in the polytopic case, results boil down to solving non-convex bilinear matrix inequality constraints, see for example Kanev et al. [2004], Geromel et al. [2007]. As suggested in these papers, some heuristics can be used to solve the BMIs. The strategy we propose in this paper can be considered as one of such.

A classical alternative to the one-shot design of output-feedback controllers is to design separately a state-feedback static gain and associate to it some state-observer. That strategy is well known and taught in all control-oriented classes. It is most effective thanks to the separation principle that, among other properties, guarantees stability of the closed-loop as soon as the state-feedback and observer gains are individually chosen to be stabilizing. Unfortunately, the separation principle no more holds in the case of systems with uncertainties. The contribution of the present paper is a methodology to approach approximately the separation principle for systems with uncertainties.

Results take advantage of existing methods. There are for example many results for LMI-based design of robust state-feedback controllers. See Boyd et al. [1994], Oliveira et al. [1999] for example. We shall not discuss these in details. A robust state-feedback gain is assumed to be designed separately from the observer according to some closed-loop performance requirements.

For the robust observer problem, results from the literature are diverse. Two types of results can be distinguished. A first category tackles the observer problem as part of the more general issue of output filtering. Geromel and Oliveira [2001] gives for that problem an LMI formulation in the case of systems with structured polytopic uncertainties. As discussed in section VI of that paper, the significant feature of robust filtering (which is usually considered as dual with respect to state-feedback) is that it needs to optimize over more decision variables than just one gain. At the difference of state-feedback where only a gain K is designed, the methodology of Geromel and Oliveira [2001] illustrates that robust filtering involves the design of both an observer-like gain L but also of the state dynamics matrix A_o of the filter. The same conclusions hold for results of Scherer and Köse [2008] in the context of IQCs.

The second category of results tackles directly the observer design problem. These, at the difference of filter-design results, have the main advantage not to assume open-loop stability of the plant. Only the error between the plant states and the observer states is required to be asymptotically stable. Surprisingly, robust observer results do not consider the upper formulated issue about having the matrix of dynamics A_o as a design variable. See for example resents results of Abbaszadeh and Marquez [2009], Mondal et al. [2010] in which only the L matrix is designed and A_o is chosen a priori to be the one of the nominal system. In Lien and Yu [2008] the assumption of a fixed A_o is alleviated, but results are restricted to unstructured norm bounded uncertainty. Our result considers A_o as a free to design matrix for the case of structured polytopic uncertainties on the plant A matrix. As discussed in Polyak et al. [2004] this problem is a difficult one, and we do not claim to provide a final answer.

The outline of the paper is as follows. Section II is dedicated to some preliminaries about state-feedback design and for the analysis of that ideal closed-loop control. Then a section is devoted to the main contribution in terms of robust observer design. This design is done assuming the observer has the task of practical implementation of the previously designed state-feedback. A fourth section is dedicated to the robustness analysis of the resulting state-feedback/observer control loop. Section 5 illustrates the results on an academic example. Finally a section is dedicated to some comments about the methods applied to derive the results and to conclusions about the advantages and drawback of the proposed design procedure.

Notation: I stands for the identity matrix. A^T is the transpose of the matrix A. $\{A\}^{\mathcal{S}}$ stands for the symmetric matrix $\{A\}^{\mathcal{S}} = A + A^T$. $A \prec B$ is the matrix inequality stating that $A - B$ is negative definite. $\Xi_{\bar{v}} = \{\theta_{v=1\dots\bar{v}} \geq 0,\ \sum_{v=1}^{\bar{v}} \theta_v = 1\}$ is the unit simplex in $\mathbb{R}^{\bar{v}}$. $\|v\|_2^2 = \sum_{k=0}^{\infty} v_k^T v_k$ is the squared l_2 norm of the signal v and $\|v\|_{2,\bar{k}}^2 = \sum_{k=0}^{\bar{k}} v_k^T v_k$ stands for the truncated squared norm. $\|v\|_p = \max_{k\geq 0}(v_k^T v_k)^{1/2}$ denotes the peak of the euclidian norm over time.

2. PRELIMINARIES

The systems to be considered are discrete-time linear time invariant:

$$x_{k+1} = A(\theta)x_k + Bu_k \ ,\ \ y_k = Cx_k \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$ are respectively the vector state, the control input vector and the measured output vector at time $k \in \mathbb{N}$. The matrix $A(\theta)$ is assumed to be affine in a vector of uncertainties $\theta \in \Xi_{\bar{v}}$:

$$A(\theta) = \sum_{v=1}^{\bar{v}} \theta_v A^{[v]}. \tag{2}$$

$A^{[v=1\dots\bar{v}]}$ are given vertex matrices. The system defined in this way is said to be affine polytopic. The important key feature of this uncertain model is that $A(\theta)$ lies for all $\theta \in \Xi_{\bar{v}}$ in the convex hull of the finite number of vertices. This is the key feature that allows to build testable robust stability results involving finite number of LMI constraints on the vertices rather than infinitely many constraints over all possibles realizations of the uncertainties.

2.1 State-feedback design

We recall here some known results for robust state feedback design. We aim not at being exhaustive and hence concentrate on the mono-objective robust H_∞ design. Multi-objective design can be performed in the same framework, see for example Peaucelle [2000], Ebihara and Hagiwara [2004] and the freely distributed RoMulOC toolbox Peaucelle [2005].

Consider the following system with performance inputs w and outputs z:

$$x_{k+1} = A(\theta)x_k + Bu_k + B_w w_k \ ,\ \ z_k = C_z(\theta)x_k \tag{3}$$

The H_∞ problem is to design a controller $u_k = Kx_k$ that guarantees $\|T(z,\theta,K)\|_\infty \leq \mu_\infty, \forall\theta \in \Xi_{\bar{v}}$ where $T(z,\theta,K)$ is the transfer matrix of the closed-loop system. The design of such state-feedback gain can be done by solving an LMI problem with the following constraints

$$\begin{bmatrix} P_1^{[v]} & 0 & 0 \\ 0 & B_w^T B_w - P_1^{[v]} & 0 \\ 0 & 0 & -\mu_\infty^2 I \end{bmatrix} \prec \left\{ \begin{bmatrix} F_1 \\ -(A^{[v]}F_1 + B\hat{K}) \\ -C_z^{[v]}F_1 \end{bmatrix} [\, I\ 0\ 0 \,] \right\}^{\mathcal{S}}. \tag{4}$$

Theorem 1. Let $P_1^{[v=1\dots\bar{v}]} \succ 0$, F_1 and $\hat{K}$ be feasible solutions to the LMI constraints (4) for all $v = 1\dots\bar{v}$, then F_1 is non-singular, $K = \hat{K}F_1^{-1}$ is robustly stabilizing and $\|T(z,\theta,K)\|_\infty \leq \mu_\infty, \forall\theta \in \Xi_{\bar{v}}$.

The proof is similar to the ones that follow. Since the result is not new, the proof is omitted for space limitation reasons.

2.2 Analysis of a given state-feedback

Lemma 2. Assume K is obtained by means of Theorem 1 then there exists $P_2^{[v=1\dots\bar{v}]} \succ 0$ and G_2 solution to the LMI conditions

$$\begin{bmatrix} P_2^{[v]} & 0 \\ 0 & -P_2^{[v]} \end{bmatrix} \prec \left\{ G_2 \left[\, I \ -(A^{[v]} + BK) \,\right] \right\}^{\mathcal{S}} \tag{5}$$

for all $v = 1\dots\bar{v}$.

Proof Replace $\hat{K}$ by its value KF_1 in (4) and then pre and post-multiply the resulting constraints by $\begin{bmatrix} 0 & F_1^{-T} & 0 \\ F_1^{-T} & 0 & 0 \end{bmatrix}$ and its transpose respectively. The obtained conditions are exactly

$$\begin{bmatrix} P_2^{[v]} + R & 0 \\ 0 & -P_2^{[v]} \end{bmatrix} \prec \left\{ G_2 \left[\, I \ -(A^{[v]} + BK) \,\right] \right\}^{\mathcal{S}}$$

where $P_2^{[v]} = F_1^{-1\mathcal{S}} - F_1^{-T}P_1^{[v]}F_1^{-1}$, $R = F_1^{-T}B_w^T B_w F_1^{-1}$ and $G_2^T = \left[\, F_1^{-1}\ 0 \,\right]$. Since $R \succeq 0$ this in turn implies (5). ■

The LMIs (5) happen to be sufficient conditions for proving robust stability of the closed-loop system with controller K (see Peaucelle et al. [2000]). For the following we assume some robustly stabilizing state-feedback gain $u_k = Kx_k$ has been designed and we assume it is such that the LMI condition (5) is feasible. Examples of such controllers are those that can be designed using the LMI method of Theorem 1.

Before designing an observer, we shall first analyze this state-feedback in terms of influence of additive input perturbations on the system states. It is done by optimization over the following constraints:

$$\begin{bmatrix} P_3^{[v]} & 0 & 0 \\ 0 & Q - P_3^{[v]} & 0 \\ 0 & 0 & -I \end{bmatrix} \prec \left\{ G_3 \left[\, I \quad -(A^{[v]} + BK) \quad B \,\right] \right\}^{\mathcal{S}}. \tag{6}$$

Theorem 3. Let $P_3^{[v=1\dots\bar{v}]} \succ 0$, $Q \succ 0$ and G_3 be feasible solutions to the LMI constraints (6) for all $v = 1\dots\bar{v}$, then the trajectories of the state-feedback closed-loop

$$x_{k+1} = (A(\theta) + BK)x_k - B\hat{u}_k. \tag{7}$$

satisfy $\|Wx\|_2 \leq \|\hat{u}\|_2$ for all zero initial condition and all bounded inputs $\hat{u}$, where $W = Q^{1/2}$.

Proof By convexity, if (6) hold for all vertices, then the inequalities also hold for all $\theta \in \Xi_{\bar{v}}$:

$$\begin{bmatrix} P_3(\theta) & 0 & 0 \\ 0 & Q - P_3(\theta) & 0 \\ 0 & 0 & -I \end{bmatrix} \prec \{G_3 \left[\, I \quad -(A(\theta) + BK) \quad B \,\right]\}^{\mathcal{S}}$$

where $P_3(\theta) = \sum_{v=1}^{\bar{v}} \theta_v P_3^{[v]}$. Pre and post multiply this inequality by $\left(x_{k+1}^T \; x_k^T \; \hat{u}_k^T \right)$ and its transpose respectively. Along trajectories of (7) the right-hand side terms are zero and remains

$$x_{k+1}^T P_3(\theta) x_{k+1} - x_k^T P_3(\theta) x_k + x_k^T Q x_k \leq \hat{u}_k^T \hat{u}_k.$$

Assuming zero initial conditions, the sum of these inequalities from $k = 0$ to $\bar{k}$ gives:

$$x_{\bar{k}+1}^T P_3(\theta) x_{\bar{k}+1} + \|Wx\|_{2,\bar{k}}^2 \leq \|\hat{u}\|_{2,\bar{k}}^2.$$

As $\bar{k} \to \infty$, since the system is assumed stable, the left-hand side term goes to zero and remains $\|Wx\|_2 \leq \|\hat{u}\|_2$. ■

Assume two matrices W_1 and W_2 solution to the LMIs (6) and satisfying $W_1 \prec W_2$ then one gets for the same $\hat{u}$ the following inequalities $\|W_1 x\|_2 \leq \|W_2 x\|_2 \leq \|\hat{u}\|_2$. It is clear from these that the matrix W_2 provides a tighter information in terms of the effect of $\hat{u}$ on the state x. To characterize worst case effects of perturbations $\hat{u}$ on the state trajectories it is hence natural to "maximize" W. For the examples it is done in the sense of the maximization of the trace of Q.

3. MAIN RESULTS

3.1 State-feedback dependent robust observer design

The aim of this paper is to design some state observer with output $\hat{x}_k$ in order to replace the state-feedback law by $u_k = K\hat{x}_k$. The goal of this observer design is to have a closed-loop behavior as resembling as possible to the ideal state-feedback. Such problem has been intensively studied in the literature for example in papers such as Aldhaheri and Khalil [1996], Mahmoud [2002] where uncertainties and non-linearities are on the inputs and output of the system. In our case the uncertainties are on the $A(\theta)$ matrix.

We shall search for a full-order observer with the following Luenberger like form:

$$\hat{x}_{k+1} = A_o \hat{x}_k + B u_k + L(C\hat{x}_k - y_k) \tag{8}$$

where the parameters to design are A_o and the gain L.

Let the error $e_k = x_k - \hat{x}_k$. The overall dynamics of the system combined to the observation error are driven by

$$\begin{pmatrix} x_{k+1} \\ e_{k+1} \end{pmatrix} = \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_o & A_o + LC \end{bmatrix} \begin{pmatrix} x_k \\ e_k \end{pmatrix}.$$

In the case of systems without uncertainties ($A(\theta) = A$), the classical choice of $A_o = A$ leads to the separation principle. That is, one can design separately K and L such that $A + BK$ and $A + LC$ are stable. Any such choice makes the overall system stable.

In the considered case of systems with uncertainties, the separation principle no more holds and we suggest to search for A_o and L such that the dynamics of x_k and e_k be the most decoupled possible. Decoupling will be almost achieved if BKe_k is small, which is obtained when Ke_k is small. e_k can be large, but Ke_k should be small. The question here is what norm to use to measure Ke_k. One classical norm would be the l_2 norm $\|Ke\|_2$. Unfortunately, such norm that measures the total energy over all time samples could be small but with high peak values. Typically it can give a very fast converging observers generating large, irrelevant, spikes on Ke_k at the first initial values $k = 1, 2, 3$ etc. See Khalil [2008] for discussions about this fact.

To avoid such phenomena the considered observer design is the search for A_o and L that minimizes a compromise between the l_2 norm $\|Ke\|$ and the peak $\|Ke\|_p$ where e is driven by:

$$e_{k+1} = (A_o + LC)e_k + (A(\theta) - A_o)x_k. \tag{9}$$

This should not be done whatever x but for those state trajectories that are expected to occur in the closed-loop system. Based on the upper analysis of the state-feedback law, such expected trajectories are defined by $\|Wx\| \leq \alpha$ where α is some scalar. The observer design problem is based on the following LMIs:

$$\begin{bmatrix} P_{42}^{[v]} & 0 & 0 \\ 0 & K^T K - P_{42}^{[v]} & 0 \\ 0 & 0 & -\gamma_2^2 Q \end{bmatrix} \prec \left\{ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} M_2^{[v]} \right\}^{\mathcal{S}} \tag{10}$$

where $M_2^{[v]} = \left[\, F_4 \quad -(\hat{A}_o + \hat{L}C) \quad \hat{A}_o - F_4 A^{[v]} \,\right]$,

$$\begin{bmatrix} P_{4p}^{[v]} & 0 & 0 \\ 0 & -P_{4p}^{[v]} & 0 \\ 0 & 0 & -\gamma_p^2 Q \end{bmatrix} \prec \left\{ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} M_p^{[v]} \right\}^{\mathcal{S}} \tag{11}$$

where $M_p^{[v]} = \left[\, F_4 \quad -(\hat{A}_o + \hat{L}C) \quad \hat{A}_o - F_4 A^{[v]} \,\right]$ and

$$K^T K \preceq P_{4\infty}^{[v]}. \tag{12}$$

Theorem 4. Let $P_{42}^{[v=1\ldots\bar{v}]} \succ 0$, $P_{4p}^{[v=1\ldots\bar{v}]}$, F_4, $\hat{A}_o$, $\hat{L}$, γ_2 and γ_p be feasible solutions to the LMIs (10), (11), (12) for all $v = 1 \ldots \bar{v}$, then F_4 is non singular and $A_o = F_4^{-1}\hat{A}_o$, $L = F_4^{-1}\hat{L}$ are matrices of the observer that guarantee $\|Ke\|_2 \leq \gamma_2 \|Wx\|_2$ and $\|Ke\|_p \leq \gamma_p \|Wx\|_2$ whatever the uncertainty $\theta \in \Xi_{\bar{v}}$ and whatever bounded input x.

Proof The upper-right block of (10) is exactly $P_{42}^{[v]} \prec F_4 + F_4^T$. Since $P_{42}^{[v]} \succ 0$ it implies that F_4 is non-singular.

Pre and post multiply (10) by $\left[\, (A_o + LC)^T \quad I \quad 0 \,\right]$ and its transpose respectively. The result is exactly

$$(A_o + LC)^T P_{42}^{[v]} (A_o + LC) - P_{42}^{[v]} \prec -K^T K \preceq 0.$$

Since $P_{42}^{[v]} \succ 0$ it proves asymptotic stability of the observation error model (9).

By convexity, if the conditions (10) hold on vertices they also hold for all $\theta \in \Xi_{\bar{v}}$. Recalling the definition of A_o and L this reads as

$$\begin{bmatrix} P_{42}(\theta) & 0 & 0 \\ 0 & K^T K - P_{42}(\theta) & 0 \\ 0 & 0 & -\gamma_2^2 Q \end{bmatrix} \prec \left\{ \begin{bmatrix} F_4 \\ 0 \\ 0 \end{bmatrix} N_2(\theta) \right\}^{\mathcal{S}}$$

where $N_2(\theta) = [\, I \quad -(A_o + LC) \quad A_o - A(\theta) \,]$ and $P_{42}(\theta) = \sum_{v=1}^{\bar{v}} \theta_v P_{42}^{[v]}$. Pre and post multiply this matrix inequality by $\left(e_{k+1}^T \; e_k^T \; x_k^T \right)$ and its transpose respectively. Along trajectories of (9) the right-hand side terms are zero and remains

$$e_{k+1}^T P_{42}(\theta) e_{k+1} - e_k^T P_{42}(\theta) e_k + e_k^T K^T K e_k \leq \gamma_2^2 x_k^T Q x_k.$$

For zero initial error $e_0 = 0$ and taking the sum for $k = 0$ to $k = \bar{k} - 1$ one gets:

$$e_{\bar{k}}^T P_{42}(\theta) e_{\bar{k}} + \|Ke\|_{2,\bar{k}-1}^2 \leq \gamma_2^2 \|Wx\|_{2,\bar{k}-1}^2$$

As $\bar{k} \to \infty$, since the observation error model is stable, the left-hand side term goes to zero and remains $\|Ke\|_2 \leq \gamma_2^2 \|Wx\|_2$.

Following the same lines starting from (11) one gets for all $\bar{k}$ and all $\theta \in \Xi_{\bar{v}}$ along the trajectories of (9)

$$e_{\bar{k}}^T P_{4p}(\theta) e_{\bar{k}} \leq \gamma_p^2 \|Wx\|_{2,\bar{k}-1}^2$$

with $P_{4p}(\theta) = \sum_{v=1}^{\bar{v}} \theta_v P_{4p}^{[v]}$. Similarily, starting from (12) one gets for all $\bar{k}$: $e_{\bar{k}}^T K^T K e_{\bar{k}} \preceq e_{\bar{k}}^T P_{4p}(\theta) e_{\bar{k}}$. Combining the two gives

$$\|Ke\|_p \leq \gamma_p \|Wx\|_{2,\bar{k}-1} \leq \gamma_p \|Wx\|_2 .$$

when $\bar{k} = \arg\max_{k\geq 0} (e_k^T K^T K e_k)^{1/2}$. ∎

In practice one is interested in minimizing both γ_2 and γ_p, or at least a tradeoff of these two. In the examples we have minimized the weighted sum $\beta_2\gamma_2 + \beta_p\gamma_p$ for a priori chosen values of β_2 and β_p.

3.2 Robust analysis of the closed-loop

An important feature at this point is that the small gain theorem guarantees robust stability of the overall loop:

Theorem 5. If $\gamma_2 < 1$, the closed-loop composed of (1) and

$$\hat{x}_{k+1} = (A_o + BK + LC)\hat{x}_k - Ly_k \ , \ u_k = K\hat{x}_k \qquad (13)$$

is robustly stable for all $\theta \in \Xi_{\bar{v}}$.

Proof Introducing again the error signal $e_k = x_k - \hat{x}_k$ the closed-loop system writes as well as the feedback interconnection of the two following systems

$$x_{k+1} = (A(\theta) + BK)x_k - B\hat{u}_k \ , \ \hat{y}_k = Wx_k$$

$$e_{k+1} = (A_o + LC)e_k + (A(\theta) - A_o)W^{-1}\hat{y}_k \ , \ \hat{u}_k = Ke_k$$

Theorem 3 guarantees that the l_2-induced norm of the first system is less than 1 while Theorem 4 guarantees the l_2-induced norm of the second is less than γ_2. By small gain theorem the closed-loop is hence stable if $\gamma_2 < 1$. Since the upper bounds are valid for all uncertainties $\theta \in \Xi_{\bar{v}}$, stability is robust. ∎

An other test is to perform a closed-loop analysis of the system with state-feedback/observer control. Assuming the original state-feedback controller has been designed to ensure some H_∞ performance (as considered in subsection 2.1), the closed-loop analysis can be done with the LMI constraints:

$$\begin{bmatrix} P_6^{[v]} & 0 & 0 \\ 0 & \begin{bmatrix} C_z^{[v]T} \\ 0 \end{bmatrix} \begin{bmatrix} C_z^{[v]} & 0 \end{bmatrix} - P_6^{[v]} & 0 \\ 0 & 0 & -\nu_\infty^2 I \end{bmatrix} \prec \left\{ G_6 \left[\begin{array}{cc|cc|c} I & 0 & -A^{[v]} & -BK & -B_w \\ 0 & I & LC & -A_o - BK - LC & 0 \end{array} \right] \right\}^{\mathcal{S}} . \qquad (14)$$

Theorem 6. Let $P_6^{[v=1\ldots\bar{v}]}$ and G_6 be feasible solutions to (14) for all $v = 1\ldots\bar{v}$, then the closed-loop composed of (3) and (13) is robustly stable and such that its transfer function satisfies

$$\|T_{cl}(z,\theta,K)\|_\infty \leq \nu_\infty, \forall\theta \in \Xi_{\bar{v}}.$$

Proof See results in Ebihara et al. [2005] applied to the closed-loop system. ∎

4. NUMERICAL EXAMPLE

For illustration purpose we consider the following simple toy example:

$$x_{k+1} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} w_k, \quad \begin{array}{l} y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k \\ z_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k \end{array}$$

with two independent scalar uncertainties $a \in [\,0.9,\ 1.1\,]$ and $b \in [\,0.9,\ 1.1\,]$ (that is 10% discrepancy around nominal values of 1). This system is trivially a polytopic model with $\bar{v} = 4$ vertices obtained by taking combinations of extremal values of the uncertainties. None of the vertices are stable.

First we apply Theorem 1 with $\mu_\infty = 1$ and get the following robust state-feedback gain $K = [\,-1.0633\ \ -1.0324\,]$.

For that fixed value of K we minimize the trace of Q under conditions of Theorem 3. As guaranteed by Lemma 2 the LMIs are feasible and we get

$$Q = \begin{bmatrix} 0.1239 & 0.0527 \\ 0.0527 & 0.5730 \end{bmatrix}.$$

Next we minimize $\beta_2\gamma_2 + \beta_p\gamma_p$ under conditions of Theorem 4. This is done for $\beta_p = 1$ and three different choices of β_2. Results are as follows:

- For $\beta_2 = 10^4$ we get $\gamma_2 = \gamma_p = 0.8696$ and
$$A_o = \begin{bmatrix} 0.9945 & 0.9805 \\ 0.9945 & -0.0195 \end{bmatrix}, \ L = \begin{bmatrix} -2.3931 \\ -1.3931 \end{bmatrix}.$$
- For $\beta_2 = 1$ we get $\gamma_2 = 0.8797$, $\gamma_p = 0.8575$ and
$$A_o = \begin{bmatrix} 0.9946 & 0.9807 \\ 0.9946 & -0.0191 \end{bmatrix}, \ L = \begin{bmatrix} -2.3637 \\ -1.3565 \end{bmatrix}.$$
- For $\beta_2 = 10^{-4}$ we get $\gamma_2 = 1.5896$, $\gamma_p = 0.8040$ and
$$A_o = \begin{bmatrix} 0.9937 & 0.9853 \\ 0.9927 & -0.0136 \end{bmatrix}, \ L = \begin{bmatrix} -2.0081 \\ -0.9979 \end{bmatrix}.$$

Results show that the A_o matrices are close to the nominal A matrix (for which $a = 1$, $b = 1$). It is as expected since in the ideal case without uncertainties $A_o = A$ would have been the optimal choice ensuring $\gamma_2 = 0$.

As expected also, when β_2 is decreased, the costs γ_2 grow and observers are found with reduced upper bounds γ_p on peaks.

Theorem 5 allows to conclude directly that the first two observers ensure robust stability of the closed-loop system. To have further information about the closed-loops, we apply Theorem 6 while minimizing ν_∞. Results for the three different state-feedback/observer controllers are:

- For 1st observer ($\beta_2 = 10^4$) we get $\nu_\infty = 1.1139$.
- For 2nd observer ($\beta_2 = 1$) we get $\nu_\infty = 1.0268$.
- For 3rd observer ($\beta_2 = 10^{-4}$) we get $\nu_\infty = 1.7164$.

All three output-feedback controllers robustly stabilize the plant. The second one is the best one in terms of keeping the closed-loop close to the initially requested H_∞ norm of 1.

All LMI problems have been coded in Matlab using YALMIP parser (Löfberg [2001]) and solved with SDPT3 (Toh et al. [1999]). For Theorems 1 and 6 we have used the pre-coded LMIs from RoMulOC (Peaucelle [2005]). For this very simple example all LMIs are solved in less than one second.

To further illustrate the results some time-domain simulations are performed. A first simulation gives the impulse response of the plant assuming state-feedback. The impulse is applied on the w input and the plotted output is z. Impulse responses for ten random values of the uncertainties (a, b) are plotted in Figure 1. Responses are close to the optimal state-feedback for the nominal model (transforms the system into a two sample delay).

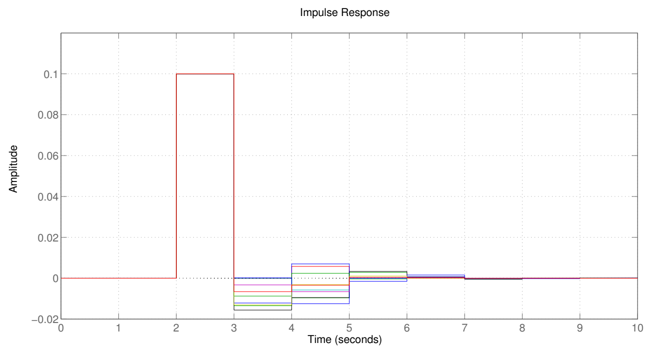

Fig. 1. Impulse responses (with state-feedback)

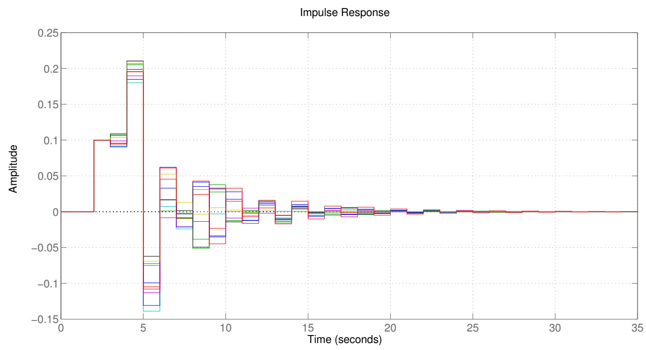

Fig. 2. Impulse responses (with observer-based control)

Impulse responses of the closed-loop with 2nd observer-based output feedback (the one computed with $\beta_2 = 1$) are plotted in Figure 2. Here also several values of the uncertainties are tested. All responses converge, thus illustrating robustness. Performance is degraded as expected compared to the state-feedback case.

In order to illustrate the difference between the computed observer-based control loops, we plot the control input $u = K\hat{x}$ for two different cases. The control signals for the 2nd controller (computed with $\beta_2 = 1$) are plotted in Figure 3 while figure 4 gives the time histories of the control signal for the 3rd controller (computed with $\beta_2 = 10^4$). In both cases the time histories are for impulse inputs on w and several simulations are done for random values of uncertainties (a, b). The 3rd control law ensures a reduced peak on the control signal, as expected by the indicator γ_p at the observer design stage. This is at the expense of degraded time response in terms of convergence (and hence of the l_2 norm indicator γ_2).

5. COMMENTS AND CONCLUSIONS

5.1 Comments about the applied slack-variable approach

All theorems and lemma exposed in the paper involve so-called "slack variables", SVs for short. Proofs of stability are obtained thanks to Lyapunov matrices $P_\bullet(\theta)$. But the conditions involve additional variables $F_\bullet$ or $G_\bullet$. As seen in the proofs, these matrices vanish as soon as the trajectories of the system are taken into account. These SVs are related to Finsler's lemma and are used for decoupling the system data from the Lyapunov matrices. This allows the search for parameter-dependent $P_\bullet(\theta)$. The SVs have first been introduced in Oliveira et al. [1999] for

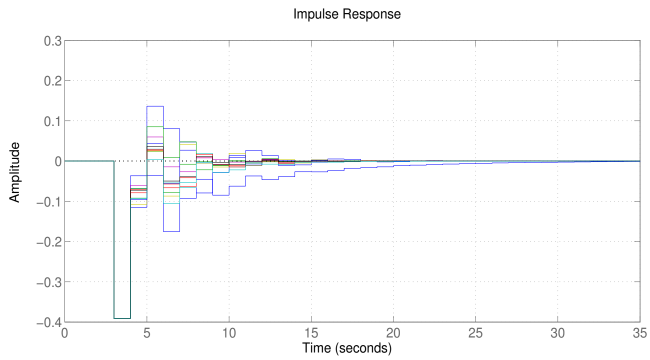

Fig. 3. Control input with 2nd observer-based control, $\beta_2 = 1$

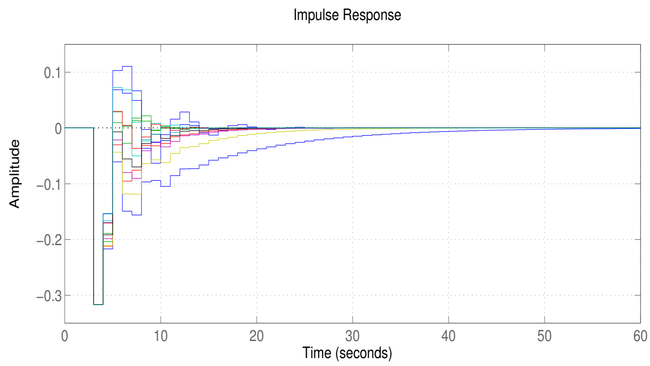

Fig. 4. Control input with 3rd observer-based control, $\beta_2 = 10^4$

state-feedback design and results were extended for improved robust analysis in Peaucelle et al. [2000]. See as well Pipeleers et al. [2009] for an independent point of view on the technique.

Notice that the SVs involved in the analysis Theorems 3 and 6 as well as in Lemma 2 are tall full matrices $G_\bullet$, while in the design Theorems 1 and 4 the corresponding SVs have the following structure $G_\bullet = \begin{bmatrix} F_\bullet^T & 0 & 0 \end{bmatrix}^T$. This structure is conservative, but useful because makes the constraints linear and hence convex in the decision variables. As discussed in Oliveira et al. [1999], Peaucelle and Gouaisbaut [2005], Farges et al. [2007] other choices are possible. This one has the advantage of guaranteeing the results to be less conservative than those build with a unique Lyapunov matrix for all uncertainties. It also makes possible a result such as Lemma 2 which is at our knowledge new.

5.2 Perspectives

Future work will be devoted to testing the method on more realistic examples. It is unfortunately expected that the methodology can fail even on output-feedback robustly stabilizable plants. This may be difficult to demonstrate since robust output-feedback stabilization is an open problem.

A trivial extension of the method is to consider the reverse problem of state-feedback gain K design for a given observer. Based on such, iterative K / (A_o, L) design procedures can be constructed. They are expected to provide enhanced closed-loop performances.

Further derivations will consider the design of interval observers providing upper and lower estimates of the states as in Efimov et al. [2013]. Adaptive observers as in Efimov and Fradkov [2006] may as well be considered for improved robustness properties.

5.3 Conclusions

A separation principle like methodology is described in the paper. It is at our knowledge new in terms of observer design with robustness properties guaranteed for structured uncertainties on the dynamics of the plant. The result is clearly a heuristic, yet it does apply on a simple example. Further work is needed both for testing the method on more realistic examples and for reducing conservatism.

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